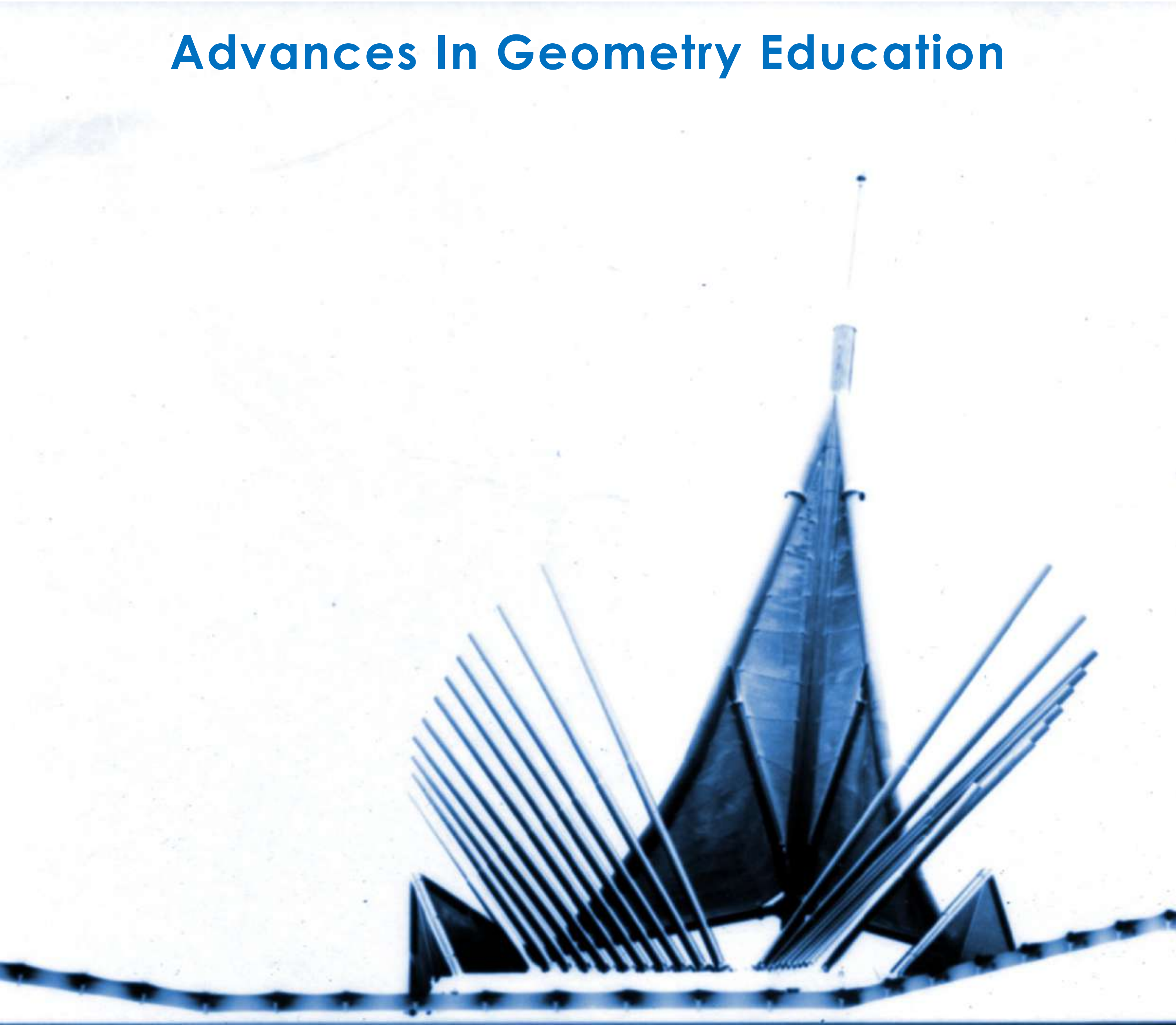
Pre-Proceedings Of The Twenty-Sixth ICMI Study

Advances In Geometry Education



Reims, France,
INSPÉ de l'académie de Reims,
University of Reims Champagne-Ardenne
April 23 – 26, 2024

The Twenty-Sixth ICMI Study

Advances in Geometry Education

CONFERENCE PRE-PROCEEDINGS

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April 23 – 26, 2024

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	Assemblée des Directeurs d'IREM - Réseau National des IREM (ADIREM) (France)
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	Université de Reims Champagne-Ardenne (France)
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We also express our deep appreciation to Lena Koch (IMU Secretariat) who assisted us in solving many administrative matters promptly and efficiently.

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The Discusion Document

PREFACE

Our engagement with this ICMI Study Conference began when we accepted invitations from the ICMI President, Frederick Leung, to co-chair the *Study 26: Advances in Geometry Education*. Although we have both worked internationally in geometry education for decades, we had not previously met one another. A briefing meeting with the President, and the Secretary General, Jean-Luc Dorier, provided context and scope for the Study. As with all ICMI studies, the central aims are to consider the topic (in our case geometry education) with respect to theory, research, practice, and policy; and to suggest new directions of research that take into account contextual, cultural, national, and political dimensions of the field. We then worked collaboratively with Professors Leung, Dorier, and other members of the ICMI Executive, to establish an International Program Committee (IPC) with expertise in geometry education. The IPC members have invested substantial time and effort to be part of the IPC, over an extended period. We are grateful for their collective wisdom and technical expertise in framing the core work of the Study. We are especially appreciative of Professor Fabien Emprin's commitment to the Study by taking on the responsibility of Conference Convenor. Fabien's leadership and organisational skills became on integral part of the Study's success. We would also like to acknowledge the effort that some participants from economically disadvantaged countries have made to attend the conference, and the economic support we have received from ICMI and Trent University (Canada) to support their attendance.

The IPC met in Valencia in February 2023 to develop the ICMI Study 26 Discussion Document, included at the end of the Proceedings, which was finalized and distributed internationally in June 2023. We express our gratitude to the University of Valencia for providing working spaces and facilities during the three-day meeting. The Study Document served as a call for papers for the ICMI Study 26 Conference, hosted by the Université de Reims Champagne Ardenne, France, during April, 2024.

An ICMI Study Conference is quite different to a traditional conference. A majority of the time is devoted to the engagement of Working Groups organized around the topics and led by IPC members. During these Working Groups brief presentations by the participants, based on their papers, serve as a catalyst for in-depth exploration of the topics and sub-topics —with the primary purpose being the preparation and initial development of a published volume.

The activity of the Study is organized around four focused topics, aimed to provide complementary perspectives and approaches to the teaching and learning of geometry. Contributions to the topics were organized around sets of related sub-topics, each sub-topic focusing on a specific issue and stating a set of questions aimed to lead discussions, as defined in the Discussion Document. The four topics are:

A. Theoretical perspectives;

B. Curricular and methodological approaches;

C. Resources for teaching and learning geometry; and

Lowrie, T., Gutiérrez, A., Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. ix-x). ICMI.

D. Multidisciplinary perspectives.

These topics and related sub-topics provided the basis for inviting papers. Each submission was reviewed by at least two IPC members, and authors of accepted papers were invited to participate in the Study Conference. Accepted papers were then revised (when necessary) by the authors, before being published in these electronic conference proceedings. The conference proceedings include 60 papers from around the world. The countries represented in the Proceedings include: Argentina, Australia, Brazil, Canada, Chile, China, Colombia, Costa Rica, Croatia, France, Germany, Greece, Hong Kong, India, Iran, Israel, Italy, Japan, Malawi, Mexico, New Zealand, Portugal, Serbia, South Africa, Spain, Sweden, Switzerland, United Kingdom, and the United States.

In addition, we invited two esteemed scholars to present plenary lectures and another five to participate in a plenary panel on geometry practices within diverse cultural settings. To ensure that teachers' voices were well-represented at the Study Conference, we invited several practitioners actively involved in collaborative geometry projects to participate in a plenary panel that focused on French perspectives of teaching geometry.

We would like to thank the Local Organizing Committee members and their support staff and students of the l'IREM de Reims at the Université de Reims Champagne-Ardenne for all their work in hosting a successful ICMI Study 26 Conference. Preparation of the ICMI Study 26 volume, begun during the Study Conference, will continue in four virtual collaborative groups, during 2024-25. We aim for the Study Volume to be published in 2026 by Springer Nature in the New ICMI Studies Series.

An ICMI Study requires sustained commitment on three major projects (the Discussion Document, Study Conference and Study Volume) throughout its focused program of work. Along with the unwavering support and guidance we have received from Frederick Leung and Jean-Luc Dorier, we would like to acknowledge the help we have received from Merrilyn Goos (Vice President of ICMI), and Lena Koch (IMU Secretariat) at critical times during the development of the work program.

Angel Gutiérrez and Thomas Lowrie

26th ICMI Study co-chairs

PLENARY LECTURES

RESEARCH ON TEACHING KNOWLEDGE IN GEOMETRY: THE CASE OF PROOF IN THE UNITED STATES

Patricio Herbst

University of Michigan

High school geometry courses, at least in the United States, have historically been places where students could encounter important elements of mathematical practice–proofs in particular. But a quick look at this practice also attests to a cleavage between expert mathematical practice and school mathematical practice. For instructional improvement to bridge that cleavage (improving the curriculum or improving teachers), our understanding of existing instructional practice needs to be informed by more than a quick look. How can we describe the teaching knowledge implicit in the instructional practices of proof in geometry and how can this knowledge assist in the design and study of improvements in instruction?

In this lecture, I will describe a program of descriptive research that seeks to understand what we call the practical rationality of mathematics teaching and how this rationality can make room for instructional resources and practices that seek to improve instruction. The program of research on the practical rationality of mathematics teaching has had research on the teaching of mathematical proof in geometry as one of its core cases for empirical investigation. Hence, this presentation goes back and forth between relatively general theoretical ideas and specific research and development projects involving geometric proof.

I argue for a conceptualization of teaching knowledge that includes both explicit and tacit knowledge, knowledge that can be located in individuals as well as knowledge that is held collectively in the practice of teaching. I illustrate the first type of knowledge with examples of mathematical knowledge for teaching proof in geometry that include knowledge needed to manage instructional situations in geometry and knowledge needed to enact mathematical tasks of teaching. I illustrate the second type of knowledge with recognition of the norms of the situation of doing proofs in geometry. I will describe, in particular, how we measure teachers' recognition of situational norms in geometry and how this recognition relates to measures of explicit mathematical knowledge for teaching geometry. The main results of this line of research are theoretical and methodological: We have been able to demonstrate that the practice of teaching geometry affords particular categories of perception that play a role in the teaching of proof in geometry and that teachers' reactions to scenarios can be used to measure the extent to which teachers recognize those categories. Our work has also contributed to instructional improvement by uncovering possibilities for developing instructional resources and practices that support an enhanced role for proof in the study of geometry.

TEACHING AND LEARNING GEOMETRY IN EARLY GRADES WITH TECHNOLOGY

Nathalie Sinclair

Simon Fraser University, Canada

This plenary talk will be divided into three parts. In the first, I will provide an overview of the main findings of my research on the teaching and learning of geometry using of dynamic geometry software (specifically, The Geometer's Sketchpad) at the primary school level. This will include studies focussing on a number of different curricular concepts such as symmetry, angles, triangles and area involving students from 6 to 12 years old. I will discuss the specific affordances of Sketchpad that were relevant to these findings, as well as task design principles employed, and the specific theoretical constructs that were used to generate insights into the teaching and learning of geometry. In the second part, I will discuss this research in light of more contemporary perspectives. Indeed, since this research was carried out, for the most part, a decade ago, I am now in a position to reflect on how it could be conjugated with some new directions in the research literature, including recent work on spatial reasoning and the duo of artefact approach. Finally, in the third part of the talk, I will discuss more recent exploratory work involving new digital technologies (including 3d pens and the multitouch application The Griddler), and less Euclidean-based concepts (including 3d geometry, modelling and perspective geometry). I will use this work to propose some productive directions for future research in the teaching and learning of geometry using technology at the primary school level. I will link my proposals to some recent theorising about the role of geometry in contemporary mathematics, about alternative curriculum approaches that challenge both Western and Piagetian progressions and about pedagogical approaches that centre making and acting.

PLENARY PANEL 1

ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

FRAMING GEOMETRIC REPRESENTATIONS AND PRACTICES IN CULTURALLY DIVERSE SETTINGS

Tom Lowrie
University of Canberra

Our plenary panel will draw upon the experiences of mathematics educators from five countries, spanning four continents. The focus of the panel is to consider new directions for geometry education research framed with the political, cultural, and contextual dimensions of practice.

INTRODUCTION

One way to better understand the complexities of classroom practices in diverse contextual, cultural, and political contexts is to position mathematics education as a practice. Mathematics education practice differs across classrooms within a school but is likely more diverse across countries and continents. This plenary panel is initially framed within a theoretical stance that acknowledges that practice is informed by the social–political (relatings), cultural-discursive (sayings), and material-economic (doings) conditions of practice both from within the school community and across the broader professional context. These arrangements of practice (relatings, sayings and doings) are a mechanism for understanding how educational practices take place and are influenced by policies, the discourse of school communities, and the economic conditions of place and work (Practice Architectures, Kemmis, 2019; Kemmis & Grootenboer, 2008).

Relatings are shaped by the prefigured existing social orders and norms of schools and systems. It is acknowledged that the practice of education is conducted in and through relationships with others (teacher/student, teacher/teacher, teacher/principal, teacher/student's family). Sayings are shaped by culturally-discursive conditions that emerge from the 'living out' of traditions and theory. In a digital age, access to various forms of communication influence the specialized language of the practice of 'education' including how policies, rules, and curricula are reported. Doings are shaped by the economies that enable and constrain education, the materials available, and the settings where the practice of education takes place. In terms of doings, much of the work that takes place is influenced by the traditions of the settings and context.

Framing geometry education in diverse settings

It could be argued that the school mathematics curriculum should reflect the political, cultural, and contextual contexts of the students for which it is planned. Political influences include the way power is achieved or used in a society. Increasingly, political influences involve the way politicians and the media provide personal views on how curricula should be shaped. Cultural practices include the shared values, attitudes, beliefs, and arrangements of a community. These practices can influence the language used in instruction or the degree to which informal and formal geometric knowledge is valued and represented.

Geometric representations and sense-making exist in all cultural groups and their meaning may differ from one culture to another.

Making sense of practice(s) across settings

Cultural artifacts are objects created by members of distinct cultural groups that convey cultural meanings and information about their creators and users. Geometric ideas, procedures, techniques, and (indeed) practices are related to the development of cultural artifacts that are socio-culturally situated as well as distributed among these members from generation to generation. This approach also includes embodied cognition in which cognitive activities use symbols as external resources that assist these members to develop their mental representations and manipulation of objects.

THEORETICAL AND METHODOLICAL CONSIEDERATIONS

A Practice Architecture Framework provides a theoretical lens for understanding small-scale classroom and community initiatives as well as large-scale national agendas and curriculum reform (Lowrie, 2014). Practice architectures (Schatzki, 2002) are based on the premise that many practices are designed, especially the meta-practices associated with educational policy making and administration, curriculum development, and teaching and learning. Consequently, the relative fixed nature of facilities, equipment, resources, policy mandates, and curricula associated with specific subject areas all play a part in how practices are constructed, enabled, developed, and repurposed (Lowrie, 2014). To this point, practices are influenced by the context, culture, and infrastructure of the architecture that has been designed (see Figure 1). Consequently, practices are situated and shaped by the circumstances and conditions of the physical location in which they occur, and of course the time, and space of the circumstances.

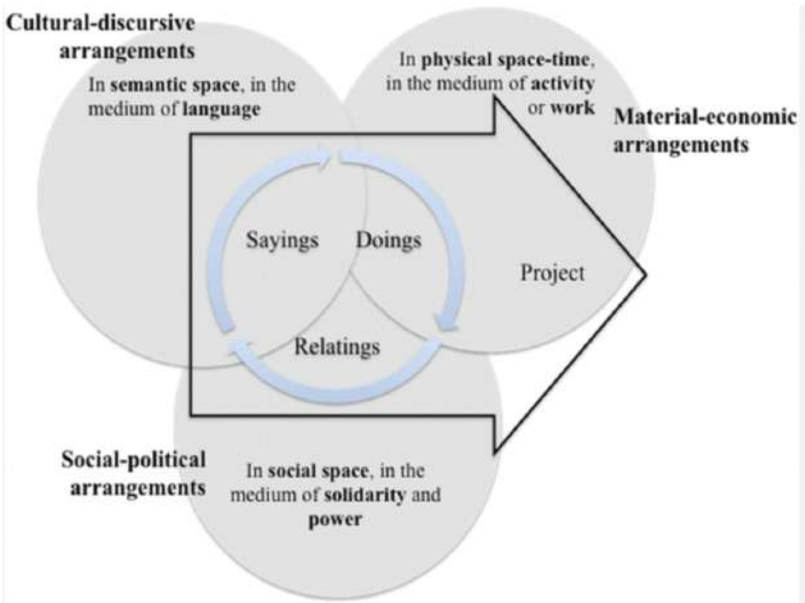

Figure 1: Geometry performance by spatial level and strategy dominance

This panel provides an opportunity to explore the diverse circumstances and conditions of spaces in classrooms and systems in Australasia, Africa, Asia, and South America.

An ethnomathematics methodology (Rosa & Orey, 2016) will be used to examine and make sense of the mathematical and geometrical knowledge (ideas, processes, and practices) that originate from diverse cultural contexts through history. The ethnomathematics methodology is multi-dimensional and provides a broad scope of affordances, namely: cognitive; conceptual; educational; epistemological; historical; and political. According to Rosa and Orey (2016) these dimensions are interrelated and aim to analyze sociocultural roots of mathematical knowledge. Moreover, the theoretical underpinnings of practice architectures, and the methodological tools established within ethnomathematics, align well and provide synergies for a rich panel discussion. Ethnomathematics is also a program that studies geometric concepts and theories developed locally as well as it is related to dynamic pedagogical actions that respond to the environmental, social, cultural, political, and economic needs of the students, which enables them to develop their imagination and creativity.

Questions to be consider by the panelists

- How can socio-political and cultural contexts of mathematics curricula help to recognize and respect the history, tradition, and mathematical thinking developed by members of distinct cultural groups through the development of geometric thinking?
- What is the significance of the influence on the curriculum of different political and cultural contexts for researchers, policy makers, and teachers?
- What cultural traits contribute to the construction of local geometric knowledge?
- What role does ethnomathematics play in the understanding of diverse indigenous ways of doing geometry?
- What are the sociocultural influences of the use of this knowledge in distinct cultural groups?

DISCUSSION

The forthcoming papers describe the complex interactions between practice arrangements within distinct contexts in Africa (Malawi), Asia (Iran), and South America (Argentina and Brazil). The cultural traditions, political and social constructs, and the economic arrangements of these countries differ dramatically, even with continents. Nevertheless, some common tensions endure across these cases, particularly in relation to tensions that exist between formal (represented) school geometry and meaningful geometric practices that play out in lived contexts. Geometry education has a sustained and important history in these contexts, however, culturally-meaningful practices are not necessarily valued.

It may be the case that that the global (and dominant) understanding of geometry education is too narrow, and consequently, it undoes the relationship between knowing and practicing. The following four cases propose that geometry practices should value cultural and contextual traditions in order to flourish and sustain communities.

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ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

CONTEXT AND LANGUAGE IN MALAWAI CLASSROOMS

Lisnet Mwadzaangati
University of Malawi

THE CONTEXT OF MATHEMATICS EDUCATION IN MALAWI

Malawi gained independence from a British colony to an authoritarian political regime in 1964 and later to a democratic regime in 1994. The difference on type and quality of education prior to and during the authoritarian political regime was very little as there were no spaces for negotiating education reform (Mwakapenda, 2002). From 1994, the major reforms in education began to take place, including the introduction of free primary education and a curriculum review by the Malawi government through the Ministry of Education (MoE). The introduction of free primary education led to a massive increase in enrolment at primary school, which resulted in an increase in secondary education to an extent that the Malawi government converted Malawi College of Distance Education (MCDE) centres into community day secondary schools (CDSSs) in 1998 (Malawi Government, 2008). The increase in access to primary and secondary education exerted enormous pressure on the education infrastructure, learning materials, and education human resources. In primary school, there was a problem of teaching resources, infrastructure, and lack of teachers because many experienced teachers were deployed to teach at the CDSSs. In the secondary schools, the challenge was a lack of teaching resources and lack of qualified teachers in quality and quantity. The challenge of quality rose because the teachers who were deployed into CDSSs were not trained thoroughly while the challenge of quantity rose because the government did not deploy enough teachers to the CDSSs.

Despite making strides in achieving education access through introduction of free primary education, education quality remains a challenge as the Malawi education system continues to grapple with the effects of introduction of free primary education as well as the rapid increase in population growth (MoE, 2023). The deteriorating of education quality perpetuates learners' poor performance at both national and international assessments in literacy and numeracy at both local and international primary education level assessments (Milner et al., 2011), which in return led to students' low performance in mathematics at secondary level national examinations (Ministry of Education, Science and Technoloy [MoEST], 2020).

Several education reforms have taken place in all subjects and at all education levels since 1994 with an aim of improving education quality and democratising education (Mwakapenda, 2002). In 2006, the Malawi government shifted from objective education model (OEM) to outcome-based education model (OBE) with underlying argument that OEM was teacher centred, hence teachers played a more active role in achieving learning objectives than the learners (MoE, 2006). The assumption was that OBE enhances education quality and students' performance by promoting students' active participation during instruction and sustaining independent learning after instruction (Chang & Salalahi, 2017). Although the curriculum review was accompanied by several other interventions to improve education quality such as training of more primary and secondary school teachers and building more classrooms, the teaching and learning of mathematics (especially geometry) remains a challenge (MoEST, 2020). The main contributing factor being the lack of qualified teachers. The Malawi education statistics reports that only 10 percent of the secondary school teachers are qualified to teach mathematics (MoE, 2023). This implies persistence of two worrisome challenges on development of mathematics education in Malawi; (1) access to secondary mathematics education is limited in Malawi, and (2) low quality of mathematics education in Malawi, hence limiting students. This is in contradiction with Malawi's aspirations that the quality of education has to be at par with international standards in an increasingly knowledge based world (MoE, 2023). This means that mathematics education will continue to be a challenge if there are no concerted and consolidated efforts amongst the education stakeholders to address the teacher knowledge issues.

To achieve learner centred education and education quality in Malawi, teachers continue to be urged to move from traditional teaching methods, which are characterized as 'chalk and talk', to more inquiry-based learning characterized as 'student-centred' (MoEST, 2020). However, due to the challenges of teacher knowledge, large class sizes, and very few resources for schools, most classrooms in Malawi continue to be dominated by teacher talk, hence learners find the learning of mathematics hard (Phiri, 2018). Both Malawian primary and secondary school class sizes can average 100 learners especially in CDSSs. Nevertheless, the teachers do try to use different student centred strategies including discussions in small groups where students learn from and support each other using carefully designed hands-on teaching activities. This is evidenced by Malemya (2018) who reports that Malawian mathematics classroom instruction includes the following characteristics:

> combines both learner-centred and teacher centred approaches where not much is done to help learners deeply understand the concepts to be able to independently develop thinking capabilities, relies on a textbook, focuses on developing a mathematical skill, devotes most available time to practising routine procedures, features isolated tasks, there is more listening from a teacher talking, relatively large classes. (p. 171)

These characteristics point to the challenge of knowledge of mathematical language use in Malawian classrooms, also called language responsive teaching as evidenced by some researchers in secondary geometry teaching in Malawi (Adler, 2022; Planas et al., 2022).

LANGUAGE AND THE TEACHING OF GEOMETRY IN MALAWI

Some of the reforms that have been made in the education sector since Malawi got her independence in 1964 have been in the area of language education. The education system in Malawi comprises 8 years of primary education, (Standard 1 to Standard 8), 4 years of secondary education (Form 1 to 4) and 4 years of university education year 1 to 4). There are over 15 local languages in Malawi, but Chichewa remains the Malawi's national language, which is used as a mandatory medium of instruction from standards (grade) 1 to 4 in all public primary schools. Since 1989, all pupils' books (except those for English studies) for standards 1 to 4 are written in Chichewa (Kunje, 2009; Malemya, 2018). Although the language policy changed in 1996, instructing that the language of instruction from standards 1 to 4 be the pupils' mother tongue, classroom evidence reveal that Chichewa remains the language that children learn mathematics even in areas where it is not dominantly spoken, probably lack of support to teachers in the form of mother tongue training and development of mother tongue instructional material (Kunje et al., 2009). This is of great worry because learning in unfamiliar language especially during early years implies learning the language itself and the content of the subject being studied at the same time.

From standard 5, the language policy dictates that instruction should be in English, which remains Malawi's official language even in the postcolonial period. As such, all curriculum materials from standard 5 up to secondary level mathematics are in English, including the syllabus and all textbooks. Unlike in other developed countries where the language of instruction is the language of the majority, in Malawi, the majority of learners speak a different language than the language of instruction. There is rich literature that argues for the benefit of engaging learners in rich mathematical discourse practices especially for learners in multilingual classrooms to enable them to access the mathematics register, ways of speaking, reading and writing mathematics, which emphasizes mathematical reasoning and explanations (e.g., Prediger, 2019).

Both the Malawian primary and secondary geometry entails a relatively formal, static, and deductive approaches. The routine approach to secondary geometry starts from definitions to illustrations of geometric concepts, and then examples for practicing and applying the geometric concepts in usually what is called dialogic instruction. For some teachers who embrace learner-centred education related values, and are concerned with learner involvement, there is use of everyday examples, concrete objects and diagrams for empirical measurement tasks, which are suggested in curriculum materials such as the syllabus and textbooks (Mwadzaangati et al., 2022). However, there remains a challenge of engaging learners in discussing and exploring of meanings even in classrooms where these learner-centred approaches are practiced (Adler et al., 2022). Consequently, the deductive approach prolongs the well-known student difficulties in geometry and obscures them from working simultaneously and in connected ways with the visual and verbal representations of geometric objects (See Bansilal & Ubah, 2019).

Seago et al. (2014) has argued that although presenting of both static approach and the dynamic approaches (in their case the transformations-based approach to the teaching of similarity) to the teaching of geometry are mathematically precise, the dynamic approach provides learners with robust learning opportunities because it offers more clarity to geometric concepts than the static approach. In Mwadzaangati (2019), I have argued that while dynamic approaches to geometry are considered to be effective in enhancing learners understanding of geometric concepts, its conceptualization is not possible in developing countries like Malawi, which are characterized by dire lack of teaching and learning resources. As such the Malawian curriculum recourses such as textbooks are designed in a way that can enable teachers and learners to use readily accessible resources to them such as pencil and paper. Furthermore, there are attempts in some of the textbooks to include empirical exploration tasks that can serve as attempts to address limitations of pencil and paper tasks, hence the instructional approach used in the textbook, which present some proof problems as experimental problems and then as a formal proof, might help the learners to both discuss and explore inductive and deductive reasoning geometric process (Mwadzaangati, 2019). This means that the challenges for mathematics researchers, educators, and even policy makers in addressing the teachers' challenges in geometry teaching in resource constrained contexts like Malawi require strategic actions of dealing with how to teach geometry effectively using static approaches. Thus, how can we help the teachers to promote learners discussions and explorations using the readily available recourses (e,g., paper and pencil)?

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FRAMING GEOMETRIC REPRESENTATIONS AND PRACTICES IN CULTURALLY-DIVERSE SETTINGS

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UNR-CONICET, Argentina

INTRODUCTION

In Argentina, compulsory education covers children and young people from 4 to 18 years of age. In the province of Santa Fe, these 14 years are divided into 2 years of Initial Education, 7 years of Primary Education and 5 years of Secondary Education (6 years in technical-vocational schools). At the curricular level, the mathematics area is present in all these years (and even in numerous higher-level courses: university and tertiary). With respect to geometry education, compulsory education deals with geometric, linear, plane, and spatial forms, their characteristics and properties, providing a connection with aspects of daily life while training in observation, visualization, representation, estimation and deduction, among other mental operations. Measurements of different magnitudes are also developed, with their systems of units, conversions, and procedures.

The Ministry of Education of the Nation is responsible for guaranteeing comprehensive, permanent, and quality education for all the inhabitants of the country, setting the educational policy and controlling its compliance. The Ministry of Education of the Province of Santa Fe, in concert and concurrently with the first one, is responsible for planning, organization, supervision and financing, and guaranteeing access.

For the coordinated planning and academic organization of educational actions, and in harmony with the Priority Learning Nuclei at the national level, the provincial Ministry prepares Curriculum Designs in which it establishes general guidelines for teaching, as well as specifications of topics to be developed in each area along with methodological considerations. Focusing on Primary and Secondary Education, the topics are organized into four axes: Numbers and Operations; Geometry and Measurement; Statistics and Probability; Algebra and Functions.

Geometry and Measurement in Argentina's Curricula

The Geometry component includes the study of geometric figures (non-empty sets whose elements are points) in different dimensions—including points, lines, segments, curves, planes, angles, polygons, circles, polyhedra, and round bodies. In addition, students study geometric places, conic sections, and vectors. Initial understandings involve synthetic approaches associated with the *geometry of shapes* in primary school and the basic cycle of secondary school, where some simple demonstrations can be carried out, but very formative in reasoning, to be integrated with the analytical approach *geometry of coordinates* in the last section of secondary school. Classifications of geometric figures are made, taking into account their constituent elements, shapes, and properties, and geometric constructions are carried out. For example, drawings of figures with ruler and compass, which consolidates the importance of geometric invariants and criteria to consider. Relationships between

Lowrie, T., Gutiérrez, A., Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (p. 13-17). ICMI.

geometric "objects" are also analyzed—parallelism and perpendicularity of lines; tangency between curves; adjacency of angles; and congruence and similarity of figures.

The Measurement part could be a separate axis since it has its own identity and sufficient relevance due to its direct applicability in numerous situations of daily life. Measurement systems are studied and students learn to measure time, angles, length, area, volume, capacity, both directly using some unit of measurement to measure objects; and indirectly, through the use of formulas and arithmetic operations. It is therefore an interesting space for integrating topics. For example, geometry with the field of numbers (to calculate the volume of a cylinder, two geometric elements must be identified that define its shape and size—the radius of the base *r* and the height *h*—and its measurements for the calculation $\text{Volume}=\pi*r^2*h$). The notion of independence of the measurements of perimeter, area and volume is worked on, analyzing concrete and simple situations of non-linearity (if the area of one figure is equal to twice the area of another, the same relationship between its perimeters does not have to be replicated), to strengthen the meaning of each of these concepts. Trigonometry fits here, conceptually based on the similarity between triangles, an extremely useful topic in the measurement of numerous practical situations -in surveying, astronomy, satellite navigation, it can be used to calculate inaccessible distances through the measurements of angles and measurable sides of appropriate triangles. Furthermore, its subsequent amplification to the field of trigonometric functions constitutes a basic area for the modeling of physical phenomena, particularly electrical ones.

THE ROLE OF CONTEXT

This type of teaching organization is metaphorically designated "helical", which is the shape of a surface like that of a spiral staircase, since when climbing it you pass over a point already traveled, but with another height, representing the idea that over time the degree of formalization and complexity with which a previously discussed concept is worked on increases. A risk that is presented and we must try to avoid is that the helical strategy of approaching the issues becomes a "carousel", which always rotates and rotates at the same height level.

In accordance with the global context, references to information and communication technologies have been growing, transversally to the curriculum, as well as to interdisciplinarity. Regarding the latter, in particular in Santa Fe, there are proposals to address teaching through Interdisciplinary Content Nuclei in terms of bringing school disciplines into dialogue to promote the appropriation of socially significant knowledge by students. In effect, they suggest taking social problems, understood as events, as a starting point; among others: "problematic substance use", "food", "energy", "the challenges of democracy", "dengue", "cultures", "climate change", "the universe", "technology in the digital age". This provides a concrete and conducive way to give voice to cultural and contextual issues to break them down from mathematics education in each singular classroom from situated institutional frameworks.

Likewise, in primary school classrooms, work on the Numbers and Operations axis is usually prioritized, and from a purely procedural level. This unbalances the other axes, such as Geometry and Measurement, which are sometimes addressed very superficially, leaving many relevant concepts or properties untreated.

Sometimes the teacher feels that he cannot "move on" to another topic because what is related to numbers and their operations has not yet been well learned by all the students. But you must keep in mind that this topic can most likely be used in another subject through integration, such as in the calculation of volumes.

In the case of geometry specifically, many topics and aspects are not taught because they are unknown, due to the teacher's own school experiences. Typically, teachers have not been provided with appropriate formative understandings of this branch of mathematics. As a result, they generally view it with fear and distrust. Regarding measurement, those relating to volume and capacity are usually neglected, in line with the little or no treatment of the notion and calculation of volumes of bodies.

A similar phenomenon occurs in many Secondary Education classrooms, when the emphasis is on the simplification of long arithmetic or algebraic expressions, with combined operations, with multiple parentheses, brackets and keys, and strictly memorized cases of factoring, without prior motivations or later applications. The somewhat more formal approach to new topics in geometry such as congruence and similarity of figures and trigonometry seems to have no chance of being deployed. Among the arguments are those of the type “students need to master operations with algebraic expressions for higher studies”, “they cannot understand the basics, how we are going to advance towards other things” and so on. These naturalized statements denote deep-rooted epistemological positions that condition and configure the teaching practices of mathematics.

However, at the prescribed level it is expected that all the axes in which school mathematics is organized will be developed. Furthermore, activities that integrate topics from different axes are promoted; for example, work with algebraic expressions can be combined with geometric interpretations, as well as with situations that can be modeled using such expressions. And even with historical-socio-cultural practices of the spatio-temporal context of the vast national territory, which remains a pending matter. In this framework, a student to be a teacher is trained in the specific aspects of the discipline/s that he/she will teach, placing himself at the educational level in which he will do so (Specific Training Field), learning the general humanistic and pedagogical foundations that will support his work as an educator (Field of General/Pedagogical Training) and having the possibility of deploying and rehearsing own abilities of teaching in making classroom decisions (Field of Professional Practice).

In the provincial proposal for the Teacher Training at the Initial Level of Education (2752 hours), the distribution of the minimum suggested hours for each Training Field is summarized as: Specific (1365, 50%), General (768 hours, 28%) and Practical Professional (619 hours, 22%). Punctually, mathematical disciplinary training is specified only in some portions of three curricular units, which also include aspects of didactics: Problem Solving and Creativity in the first year (42.7 h), Mathematics and its Didactics I in the second year (85.3 h), and Mathematics and its Didactics II in the third year (85.3 h). Thus, training in Mathematics is less than 213.3 h, which represents 16% of the Specific Training Field and, in turn, 8% of the total career proposal.

Likewise, within this scarcity, relevance is given to geometric issues, with two of the three blocks assigned to them: Number; Space and Geometry; Measurement. In addition, among the proposals for optional reduced formats, seminars on Ethnomathematics, geometry and movement, as well as plastic-visual language and Mathematics are included.

In the Teacher Training at the Primary Level of Education (2816 h) the Specific Training Field is made up of the same spaces mentioned for the Initial Education Teaching Staff. The workloads by Field are similar: Specific (1429, 51%), General (768 hours, 27%) and Professional Practice (619 hours, 22%). Regarding training in Mathematics, an analogous proportion is allocated to the previous level (Initial), even slightly lower (15% of the Specific Training Field; 7.5% of the total Plan). How could a teacher, qualified to teach Mathematics many hours a week, from first to seventh grade (children from 6 to 12 years old), have acquired tools for effective professional development in such a short training time? In turn, the proposal for the area is arranged in six blocks, two of which are: Space and Geometry; Measurement.

In Argentina, the training of teachers to work in Secondary Education is already by discipline (mathematics in our case) and two types of institutions coexist: provincial higher education institutes and national universities. To provide context, currently in the field of state management, there are around 30 Mathematics Professorships (universities) and 14 at the provincial level in Santa Fe. The study plans that govern them and the institutional cultures are typical of each one (each university is autonomous and each province too). In these Teacher Training Courses, the Specific Training Fields comprise at least 60% of the total hourly load of the Study Plan. Among the branches that make up the mathematics area, there is a greater emphasis on calculus, with geometry along with algebra being the other two classic branches of development. Others mentioned are statistics and probability, mathematical modeling, discrete mathematics, and applications.

To this point, the relative weight of Geometry within the Curriculum is between 10 and 11%, and is developed gradually throughout the four years of the degree: Synthetic Geometry, Analytical Geometry, Dynamic Geometry, Euclidean geometry, Fractal geometry, Non-Euclidean geometries, Language in Geometry, Euclid's axiomatic system, Construction to conceptualization, Invariant properties, Transformations, Hilbert axiomatics, Contributions of Descartes and Klein, Erlangen Program, Incursion into Topology. In spaces of specific didactics and professional practice, disciplinary content is integrated with relevant theoretical currents and contexts located where future teachers carry out their immersion in real educational institutions. This is with the purpose of substantiating from initial training a specialized knowledge of the content (particular mathematical domain expected of a teacher). For example, calculating the perimeter of a rectangle requires activating knowledge different from that necessary to analyze a student's conjecture about a possible relationship between perimeter and area. The first requires only knowing ways to calculate the perimeter of the rectangle (common knowledge), while the second requires an ability to think flexibly about the notion of perimeter in order to analyze a statement, correct or not, of another person (knowledge specialized).

In general terms, it can be noted that, from the analysis of the curricular designs for teacher training in force, it emerges that in the case of Primary Education Teachers it is difficult to develop common knowledge of the content corresponding to that section, due to the scarcity of the time allocated to specific Mathematics topics (or geometry specifically). Hence the need for a review and readjustment, because a teacher cannot teach what he does not know, or even manage at the same level at which he must teach it. Even fewer will be able to articulate it in terms of specialized knowledge and/or integrate it with cultural practices in situated contexts.

DISCUSSION

In its origins, the province of Santa Fe was a territory inhabited by groups of natives, organized into tribes (Tobas, Guaraníes, Mocovíes, Pilagás, Guaycurúes, Querandíes, Abipones, Timbúes, Quiloazas, Mocoretás, and Corondas). Inquiring into their worldviews, in connection with nature and about how geometry has been intrinsically present in their practices also presents fertile, unexplored territory. As an example, cultural diversity is celebrated on October 12 each year—it is a very emotional moment of reflection that can be recovered through school projects that bring disciplines, or branches of these, into dialogue, such as geometry. Some possibilities may be to explore geometric shapes in cultural legends, develop geometric art inspired by cultures, study monuments and iconic structures of various cultures, recognize the presence of geometry in Nature, or build geometric models.

At the national and provincial level, in particular, agricultural and industrial-based activity is influential, proving interesting activities that can support geometry with transversal themes such as environment, gender, technologies, and human rights. This is on the agenda, with initial developments in the daily classroom. Starting with Teacher Training, initial and continuous, is considered a favorable path for this; punctually, from socialization of examples of similar practices in relatively close fields.

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ADVANCES IN GEOMETRY EDUCATION
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A GLIMPSE OF THE PRACTICES OF GEOMETRY EDUCATION IN IRAN

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INTRODUCTION

The purpose of this paper is to provide a glimpse of the evolution of geometry education from the foundation of the first modern education institution in the late 18th century to the establishment of the formal education system in 1925. From inception until 2011, geometry education was divided into three different periods. This paper will give a brief account of the characteristics of each period in terms of the geometry curricula/textbooks in Iran.

Geometry is part of the Iranian scientific history and the contribution of Iranian mathematicians to the advancement of geometry has been remarkable in the history of mathematics. Scholars include Abu al-wafa Buzjani (10th century), al-Karaji (11th century), Khayyam (11th to 12th century) and Nasir al-Din al-Tusi (12th century), to name a few. Furthermore, geometry has been part of the Iranian culture, artistry, architecture, and handcrafts. Nevertheless, geometry education in Iran, did not take the soul of this cultural wealth and in contrast, was influenced by abstract aspects of geometry adapted from French geometry curricula. In the following, I will take a glimpse at the geometry education from Dar ul-Funun[1] to the new century.

Geometry Education since the first modern institution

Dar ul-Funun was the first modern education institution in Iran, founded in Tehran in 1851. The tradition of geometry education in Dar ul-Funun was drawn from French traditions. Teachers taught in their own language and translators were in classes to translate them into Farsi for students. Gradually several of the translators started to write geometry textbooks in Farsi with an emphasis on abstraction and rigor as presented in Euclid's Elements. This synthetic approach had almost no relation with traditional geometry that was developed in Iran in the past.

After the establishment of formal secondary education in Iran in 1925, the senior high school started with two streams: namely, "science" and "literature". Geometry was one of the main subjects of the science stream. The approach to geometry education continued the tradition of Dar ul-Funun and focused on deductive reasoning and proof. However, in the beginning of formal secondary education, the number of students was low and those choosing the science stream for the senior high school were limited. Typically, elite students entered the science stream and for most of them, the abstract level of deductive practice of geometry was attractive and manageable. With the expansion of secondary education, this approach prevented many students selecting the "science stream" and the abstract approach to mathematics education, acted as "gatekeeper" for general students.

From the creation of the secondary education in 1925, three distinct periods of practice took place.

[1] polytechnic

Lowrie, T., Gutiérrez, A., Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (p. 19-22). ICMI.

First period from 1942 to 1970: In this period, the Ministry of Education allowed different publishers to produce textbooks for each subject from different perspectives, including geometry. However, the approach to geometry education continued to focus on deductive reasoning and proof. In this period, various changes took place at the senior secondary science stream and later, the "mathematics-physics" stream, but geometry subjects autonomously. There were various school subjects under the geometry umbrella such as "descriptive geometry", "spatial geometry", "analytical geometry" and "conic sections", all faithful to an axiomatic, Euclidian approach.

Second period from 1970 to 1995: In early 1970's the school mathematics curriculum in Iran adapted the "new math", except for geometry education and geometry textbooks. Geometry endured intact, despite Dieudonné's desire to replace Euclidean approaches with transformational geometry based on vectors and vector space in two dimensions (Dieudonné, 1972). In fact, Euclidian approaches were mandated until 1995 despite not being aligned with the traditional Iranian geometry. European influences prevailed with the expansion of formal education and the rapid increase of high school students. Geometry textbooks were written to teach proof, using deductive reasoning.

After the revolution of 1979 and the end of the eight-year Iran-Iraq war in 1988, the secondary education system underwent change in 1991 and the curriculum and textbooks for all subjects changed accordingly2. Regarding mathematics curriculum, the "new math" subject including the emphasis on formal logic was eliminated and new textbooks were written towards moderate reform and as usual, two geometry textbooks for grades nine and ten, were written with the same axiomatic Euclidean tradition. However, in contrast to the past, the number of students entering secondary education and willing to choose mathematics-physics and science streams increased rapidly ensuring geometry's audience was much broader. The consequence was that in less than one year, and after the report of the national assessment of progress in geometry (Kiamanesh, 1994), the Ministry decided to re-author two geometry textbooks. I was invited with several others to develop the geometry curriculum and write Geometry I and Geometry II textbooks3. From my perspective, this was an opportunity to take a different stance on geometry education, allowing secondary students to enjoy learning geometry. To start the new endeavor, we first reviewed the historical development of geometry education from Dar ul-Funun to two periods of the foundation of formal education system in Iran and the influence of new math describing the third period.

Third period from 1995 to 2014: To begin the work, we believed that cultural practices of geometry in Iran, its tradition and its place in the Iranian heritage should be our first step towards developing the curriculum. As well, the historical trend of modern and formal education should be considered seriously. In the meanwhile, I participated at PME 18 in Lisbon in 1994 when the study document for the 14th ICMI Study titled "Perspectives on the teaching of geometry for the 21st century" was released. By reviewing the related documents at the national and international level, we concluded that school geometry should encourage students to formulate conjectures and proofs, through activities that were accessible to as many students as possible. Relations between intuition, inductive

[2] Iran education system is centralized and there is one national textbook for every school subject.

[3] The details of this development have been presented in various conferences and the team and myself, have written several papers to describe that new trend. The last paper written by Gholamazad, Gooya & Zangeneh is ready for submission.

and deductive proofs, age of students at which proofs can be introduced, and different levels of rigor and abstraction needed to be accounted for.

Geometry I started with simple and computational/algebraic examples and used axioms implicitly that were based on student experiences. For example, for the similarity of two triangles, we relied on what students knew from junior high school related to the concept of rectangle and area rather than focusing on computational examples. We used inductive and deductive reasoning across the book. Students calculated the volume of spheres by first calculating the area of circles.

Geometry 2 encouraged students to use inequality, deductive reasoning, and indirect proof as tools and methods for direct proof. Analytic approaches included considering transformational geometry with coordinated geometry. A serious difficulty for university graduates was that they could not use their higher mathematics knowledge fluently to teach axiomatic Euclidian high school geometry. However, the new approach gave them confidence to use their new knowledge for teaching geometry. For the development of Geometry I and Geometry II textbooks, the following principles were considered:

1. Guessing (conjecturing) using inductive reasoning and proving using mathematical induction is a great opportunity for students to develop logical thinking.
2. Coordination and unification of different parts of mathematics as a legacy of Golden Age of Iranian-Islamic era and its potential that was the opposite of new math in which various mathematics branches, topics or even concepts dealt with differently.
3. Teaching theorems are important, but not in a way that in traditional or new math approach, teaching a theorem and its proof with all its details at once. On the contrary, we broke famous theorems such as Thales into several parts and through suitable activities, students engaged in doing the proof with other students in small groups and via whole class discussions and contributed to the development of proof. In this manner, each part acted as a scaffolder to build the other part and at last, the textbook brought all pieces in one big picture as "proving the theorem".

To give an example, the following activity is taken from Geometry 1 (1995):

This regular dodecahedron has been divided into six parts. Cut the pieces and put them together in a way that makes a square. Note that one of the pieces is an equilateral triangle. In the first step, students should solve it by manipulating the pieces. But in the next step they were asked to show it is a square. In other word they should prove that the new figure is a square.

Figure 1. Puzzle posed by Abu al-Wafa al-Buzjani (11th century)

Discussion

Geometry has strongly rooted in Iranian culture and thus, in every period of schooling, has had its own dominant place in school mathematics curricula. Since the foundation of Dar ul-Funun and later, the establishment of formal education system in Iran, geometry has always been "of its own merit" and not necessarily regarded as part of school mathematics. Reviewing the history of geometry in Iran, shows that great mathematicians, philosophers, and poets created the most sophisticated geometry that have boosted mathematics and science throughout the history of humankind and had a certain role in the development of the western renaissance. This cultural and traditional geometry is heavily based on practices and real-life problems and its move has been from intuition and reality to abstraction. Geometry is strongly rooted in Iranian culture, and it is the soul and base of many artifacts, buildings, handcrafts, and carpet weavings and is part of our cultural heritage. However, since the modern and formal education was imported from Europe and particularly France, the approach to its education as school subject, was purely focused on the Euclidean tradition. This resulted in two independent and strong culture practices in geometry education in Iran. Before the reform of 1995, the formal geometry education in Iran was disconnected from the real world, and the other mathematics subjects as well. Its Euclidean axiomatic approach, prepared the scene for memorization of facts, proofs and procedures, and the language used was unfamiliar for students.

CONCLUSION

However, the beauty and flexibility of geometry is partly due to its capacity for developing intuition, visualization, imagination, and application in the real world that is not limited to an axiomatic framework. In our view, this is the central reason for geometry having a self-contained place in school mathematics since no other parts of mathematics have the potential of geometry to include both extremes of far intuition and far abstraction within one frame! In addition, geometry is closely connected with culture and cultural artifacts and people in different regions around the world feel that they are part of this endeavor and thus could contribute to its development. Rather than using artificial cultural symbols, it is more appropriate to implement the findings of various ethnomathematics to help students to see the relation between mathematics and culture (Gooya & Gholamazad, 2021).

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ETHNOMODELS OF LANDLESS PEOPLES' MOVEMENT: MEASURING LAND – BRAZIL

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INTRODUCTION

In the struggle for agrarian reform, the access to a plot of land and to live and produce on it makes the practices of measuring the land to be a central activity of the members of the *Landless Peoples' Movement (Movimento dos Sem Terra - MST*), in Southern Brazil, mainly because of the importance placed on sustainability and planning of agricultural production. In the study conducted by Knijnik (1993) it was proposed that the elaboration of curricular mathematics activities related to the demarcation of land with participants of this movement. These activities were related to the method of *cubação of land*, which is a traditional mathematical practice applied by participants of this specific cultural group to measure and determine the area of the land in their settlements (occupation sites).

The daily necessities of the MST members caused them to capture the procedures of these techniques, showing that, despite their low level of schooling, they were able to apprehend and apply knowledge related to the methods of cubação of land, which is one of the tools used to solve problems related to the measurement of land with irregular shapes by applying distinct methods to determine this area. According to Knijnik (1996), this method met the specific needs of the members of this movement because they applied it to determine land areas related to the delimitation of planting sectors as well to demarcate the plot of land of each family in the settlement.

They established production goals related to their own logistic possibilities such as storage and drying, bagging, transport, and selling products at local markets. The land worked by MST members was prepared according to the type of farm and quantity of the product these members harvested and commercialized. The emic knowledge related to the development of these methods was orally transmitted and diffused to MST family members by their ancestors across generations. Thus, the mathematical knowledge involved in these local methods is also related to productive activities that members of this specific cultural group performed in their daily routines. For example, the need for the development of *cubação of land* with irregular shapes was in accordance with its accessibility depending on its topology and the quality of desired agricultural products.

This method is used to calculate the total area of a land in order to calculate the amount of money needed to be paid or received for the cleaning work of the property, or for the preparation of the land for planting as well the demarcation of areas to be cultivated, to plan and to delimitate areas for the construction of houses and shelters for animals. It is also used to make payments for work done by the members of a settlement in the state of Bahia, in Brazil, according to the land *frames* or shapes. For example, there is job related to land with *two corners*, *three corners*, or *four corners* in accordance with to the shape of the cultivated area (Silva, 2012). According to D'Ambrosio (1999), the validation of these methods within agricultural communities and settlements results from the development of informal agreements of signification that results from a long cumulative process of generation, intellectual organization, social organization, and diffusion of this knowledge.

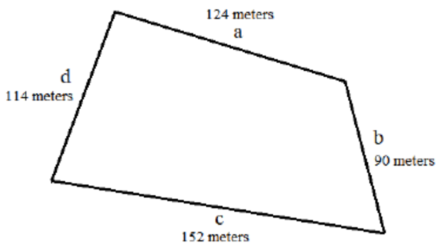
CUBAÇÃO WITH QUADRILATERAL SHAPES: CALCULATION OF LAND AREA WITH FOUR CORNERS

Mathematical practices investigated in the study conducted by Knijnik (1993) consisted of two methods that were called by her students in the classroom as *Adão's Method* and *Jorge's Method*. These two students who were members of the *Landless Peoples' Movement* (*MST*) presented, explained, and taught these specific ways of measuring their land in their settlements to the other learners in the classroom. The investigation of these two methods discusses the interrelations between local (emic) and academic (etic) mathematical knowledge concerning the upper bound estimation of the area of a tract of land with irregular shapes.

Transforming the Shape of an Irregular Quadrilateral into a Rectangle

The first method is called the *Adão's Method* that transforms the shape of an irregular quadrilateral into a rectangle. In this context, Adão explained how to determine his method, which we consider as an emic ethnomodel:

> Well folks, this is the most common formula that is used on the countryside, up there on the farm, right? And, let's assume that I am the owner of a crop and I lent this *frame* here to a friend *to mow* and I told him that I will pay three thousand by the *fourth*. Then, he mowed this land and he even *passed the rope* himself to find its area. Then, he measured this *wall* here, 90 meters, the other, 152 meters, 114 meters, 124 meters. Did you notice that there is no *wall*, no base, and no height that has the same measure, right? The two landmarks that are *lying down* are the *bases* and the *heights* are those that are *standing up*. Ok. So, I did the following here, right: I added the two bases and divided the sum by 2. I found 138. So, the base is 138 here and 138 there, understood? So, I have here the two heights, 114 plus 90. I found 204 and divided it by 2, 102, right? So, now we just need to multiply the base times height, Ok? I think the answer is 14076 square meters, right? This is the area that he *mowed*.



It is important to state that, during his narrative, Adão used expressions such as:

a) *Walls* (paredes) that mean the landmarks of the land.

b) *Frame* (quadro) that means the area of a land with a quadrilateral shape.

c) *To mow* (carpir) means to clean or to prepare the land for planting.

d) *Fourth* (quarta) that means an area measurement used in the Brazilian rural context that is equivalent to a quarter of a Paulista bushel that is used in the state of São Paulo, Brazil, which measures 24200 square meters.

e) *Pass the rope* (passar a corda) means to measure the land by using a rope.

These terms are the vocabularies (jargons) used by the members of this distinct cultural group to describe the procedures of the development of their local mathematical practices. Table 1 shows Adão's method of estimating an area of a land with irregular shape.

Adão's explanation (Emic/Local knowledge)	**Academic explanation (Etic/Global Knowledge)**
This is a piece of land with four walls	This is a convex quadrilateral
First, we add two of the opposite walls and divide them by two	First, we find the average of two opposite sides
Second, we add the other two opposite sides and also divide them by two	Second, we find the average of the other two opposite sides
Third, we multiply the first obtained number by the second one	Third, we determine the product of the two average numbers previously determined
That is the *cubação* of the land	This is the area of the rectangle whose sides are the average of the two pairs of opposite sides of the convex quadrilateral

Table 1: Adão's method of estimating an area of a land with irregular shape. Source: Adapted from Knijnik (1993, p. 24)

Thus, this emic (local) mathematical knowledge can be represented by an etic (global) ethnomodel that transforms the shape of the given land into a rectangle of 138 metres x 102 metres with an area of 14076 square meters.

$$Area = \left(\frac{a+c}{2}\right) x \left(\frac{b+d}{2}\right)$$

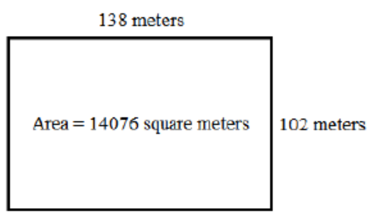
$$Area = \left(\frac{124+152}{2}\right) + \left(\frac{90+114}{2}\right)$$

$$Area = \left(\frac{276}{2}\right) x \left(\frac{204}{2}\right)$$

$$Area = 138x102$$

$$Area = 14076\ square\ meters$$

The representation of this mathematical practice can be explained by the following etic ethnomodel procedures: a) transform the shape of the irregular quadrilateral in a rectangle whose area can be determined through the application of the area formula, b) determine the dimensions of the rectangle by calculating the average of the two opposite sides of the irregular quadrilateral, and c) determine the area of the rectangle by applying the formula: $A = b\ x\ h$.



It is important, indeed relevant here to state that there is historical evidence that the method of *cubação* in which a quadrilateral is transformed into a rectangle was used with the purpose of land taxation in Ptolemaic and Roman periods, as well in ancient Egypt (Peet, 1970). This method is also used in Chile and Nepal, and in the Brazilian states of Bahia, Pernambuco, Rio Grande do Norte, Rio Grande do Sul, São Paulo e Sergipe (Silva, 2012).

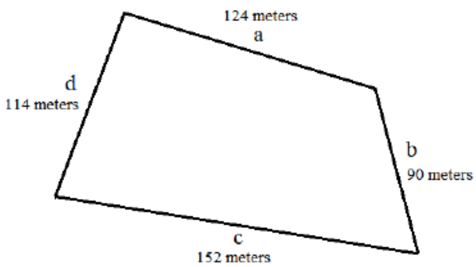
Transforming the Shape of an Irregular Quadrilateral into a Square

The second approach is called *Jorge's Method* that is related to *how to square the land*, which means to transform the initial quadrilateral into a square with the same perimeter. Table 2 shows Adão's method of estimating an area of a land with irregular shape.

Adão's explanation (Emic knowledge)	Academic explanation (Etic knowledge)
Here is a piece of land with four walls	This is a convex quadrilateral
First, we add all the walls	First, we determine the perimeter of this convex quadrilateral
Second, we divide the sum by four	Second, we divide the perimeter by four
Third, we multiply the obtained number by itself	Third, we determine the area of the square whose side is given by diving its perimeter by four
This is the *cubação* of this land	This is the area of the square obtained from the perimeter of the convex quadrilateral

Table 2: Adão's method of estimating an area of a land with irregular shape. Source: Adapted from Knijnik (1993, p. 24)

In this context, Jorge explained that "Since the land has four different sides [irregular shape], I added all four sides: 90, 124, 114, and 152 and the result is 480. Now, I divide this result by 4, which gives 120. Then, I multiply 120 by 120, which gives me 14400". Thus, the quadrilateral is transformed into a square whose side is the fourth part of the perimeter of the original polygon.



Thus, their emic mathematical knowledge can be represented by an etic ethnomodel that transforms the shape of the given land into a square of 120 metres each side.

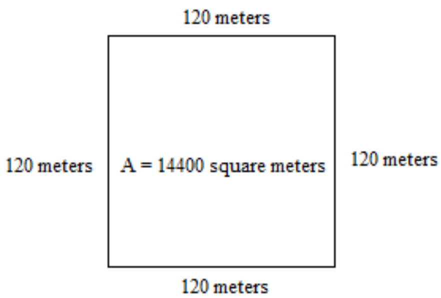
$$Area = \left(\frac{a+b+c+d}{4}\right)$$

$$Area = \left(\frac{124+90+152+114}{4}\right)$$

$$Area = \left(\frac{480}{4}\right)$$

$$Area = 120 \; square \; meters$$

The representation of this mathematical practice can be explained by the following ethnomodel: a) transform the shape of the irregular quadrilateral in a rectangle whose area can be easily determined through the application of the area formula, b) determine the dimensions of the rectangle by calculating the average of the two opposite sides of the irregular quadrilateral, and c) determine the area of the square by applying the formula $A=a \; x \; a = a^2$.



In this context, Knijnik (1996) has affirmed that the methods used by Adão and Jorge are mathematical practices that rural workers in southern Brazil employ in order to transform irregular figures into regular ones. For example, in the *Adão's Method*, there is a reduction of the area of the land in a rectangular shape while in the *Jorge's Method* this area is reduced into a quadrangular shape. Nevertheless, it is important to state that the method applied by Jorge shows that there is an increase in area in relation to the method used by Adam because among all the quadrilaterals with the same perimeter, the square has the largest area.

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PLENARY PANEL 2

ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

TEACHING GEOMETRY IN FRANCE

Fabien Emprin[1] (chair of the panel), Isabelle Audra, Mélanie Binet, Bernadette Da Motta, Marie-Paul Foy, Aurélie Marche, Christine Trouillet

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It is traditional for ICMI Studies to enable participants to visit classes in the host country. This 26th ICMI Study took place in France during the school vacations, so we were able to organize a plenary session of discussion with French teachers. They will present their school context, their practices, and their questions to the group of researchers.

AN OVERVIEW OF THE FRENCH EDUCATIONAL SYSTEM

The main principles

The French educational system is guided by five principles, namely: academic freedom, free provision, neutrality, secularism, and compulsory education.

The freedom of education and relations with private education defines that parents can choose between public schools at no cost or private school subjected to state control, which can benefit from state aid [in return for a contract signed with the state] ("Debré Law" n°59–1557, December 31, 1959). Only the government is authorized to issue university diplomas and educational degrees. Teachers and students are expected to be philosophically and politically neutral. Neither religious instruction nor proselytizing are allowed. Education is compulsory for all children from 3 to 16 years old residing in France (the compulsoriness of school for the three-year-old child is recent, dating from September 2019). Regardless of their nationality, students must be educated, whether at school or at home (with a prior declaration).

Student With Special Needs

French public schools aim at welcoming all children of all intellectual and physical abilities from the age of three in normal classes (disabled students; students with language and learning difficulties; students with long-term educational difficulties; gifted students). This schooling in mainstream education can be with or without specific support or arrangements according to the need of the student. A local service, the Departmental House for Disabled People (MDPH in French) can work out a personalized education plan for those children. Depending on the nature of a child's disability, they can either be homeschooled, taught in the local school within an educational inclusion unit or hosted in a specialized educational unit or at the hospital.

Organization of the School Year

In metropolitan France, the school year runs from early September to early July. The school calendar is standardized but with variations within the regions (3 areas of vacation: for example, students of Créteil, or Toulouse are at work during ICMI Study, but Reims' students are not). Typically schools cycle around 7 weeks of work and 2 weeks of vacation. Summer vacation lasts approximately 2 months (July and August).

Most schools are open 4 days a week (Monday, Tuesday, Thursday, Friday). Some schools are open on Wednesday evening and secondary school often work on Wednesday and/or Saturday evening.

Organization of scholarship

French school is divided into preschool, elementary school, middle school, high school and upper education including university, preparatory class for the *"grandes écoles",* University Institute of Technology.

Age	French name	French name/English name
3	Petite section	Maternelle / Preschool
4	Moyenne section	
5	Grande sections	
6	Cours préparatoire — CP	Élémentaire / Primary school
7	Cours élémentaire première année — CE1	
8	Cours élémentaire seconde année — CE2	(École maternelle + élémentaire = école primaire)
9	Cours moyen première année — CM1	
10	Cours moyen seconde année — CM2	
11	6e — sixième	Collège / Secondary school
12	5e — cinquième	
13	4e — quatrième	
14	3e — troisième	
15	2de — seconde	Lycée / High school
16	1ère — première	(Collège + lycée = enseignement secondaire)
17	Terminale	
	Baccalauréat	Examen / exam
	Licence (3 years)	University grades
	Master (2 years)	University post-grades
	Doctorat (3 years)	

Table 1. French school system by student age

Teachers' education and recruitment

In France, teachers are public servants. They must pass a competitive entrance examination. There are three possible examinations : CAPE (certificate of aptitude to teach in preschool or primary school), CAPES (certificate of aptitude to teach in secondary school) and Agrégation which also allows teaching in secondary school (but also to some extent in preparatory classes and university). All teachers must have a master's degree.

Students with a bachelor's degree can enroll in a master's degree in teaching, education, and training (MEEF) at an Institut Supérieur du Professorat et de l'Education (INSPÉ). At the INSPÉ they prepare for a master's degree and for the competitive examination.

Some Peculiarities of the French System

Schools do not recruit teachers. The system for assigning teachers to their posts is governed by a national and regional system based on the allocation of points according to seniority, career, and family situation.

The *agrégation* allows teachers to teach at the same level as the CAPES, but an *"agrégé"* teacher teaches fewer hours per week (15h vs. 18h) and at a higher salary.

In higher education, there are several stages in the process of obtaining a research position. First, you need to obtain a PhD, and then apply to a section of the national university committee (CNU). This involves putting together a dossier outlining how your research fits into a particular disciplinary field. There are two CNU sections for math education researchers: Applied Mathematics and Application of Mathematics (26th section) and Education and Training Sciences (70th CNU section). Only PhDs who have obtained this national qualification are eligible to take part in competitive examinations for positions as lecturers (associate professor).

To become a full professor, you need to obtain a second diploma, the *"habilitation à diriger des recherches*" (habilitation to supervise research), and then apply for a position. Qualification has existed in the same way as for lecturers but is no longer compulsory for associate professors working at a French university.

THE FRENCH SYSTEM IN FIGURES

Average salary (in July 2021, source Ministry of Education):

- Certified teachers: 2490 euros net per month. Time in front of pupils: 18 h per week
- Primary school teachers: 2407 euros net per month. Time in front of pupils: 24h per week
- Agrégé : 3719 euros net per month. Time in front of students: 15 h per week
- Lecturer: 2805 euros net per month. Time in front of students: 192 h/year
- University professor: 4200 euros net per month. Time in front of students: 192 h/year

Pupils:

- 11,997,900 schoolchildren (public and private, forecast for the start of the 2023 school year)
- 6,349,600 primary school pupils
- 5,648,300 secondary school students, including 3,397,300 middle school students and 2,251,000 high school students (including 627,100 vocational high school students)

Schools:

- 58,910 public and private secondary schools under contract (back to 2022) [48,220 schools; 6,980 collèges]
- 3,710 lycées and Erea (établissement régional d'enseignement adapté), including 2,080 lycées for vocational high school students

Teachers:

- 853,700 national education employees teach 1st and 2nd grade students (public and private under contract, back to 2022)

Class sizes (back to 2022)

Public and private primary education

- 22.4 pupils per class, average number of pupils in pre-elementary classes
- 21.6 pupils per class, average number of pupils in elementary schools

Secondary public and private

- 25.9 students per class, average number of students in college training (excluding Segpa)
- 17.9 students per class, average number of students in vocational high school courses
- 30.3 students per class, average number of students in general and technological high school courses

Diplomas (session 223)

- 89.1 % Diplôme national du brevet (DNB) pass rate (after 3e)
- 90.9 % Baccalauréat pass rate (after terminale)
- 79.3% Proportion of baccalaureate holders in a generation

Average expenditure per pupil per year (calendar year 2018, provisional data)

- €7,440 per primary school pupil
- €9,150 per middle school student
- €11,570 per general and technological high school student
- €13,220 per vocational high school student

TEACHER'S PRESENTATION

The plenary session will begin with a presentation of a few schools with successive zoom in:

- Elements (photos, short video clips, data, information) about the teacher's town, school and classroom.
- Place of geometry in the cursus.
- Examples of geometric activities proposed in the classroom, in the form of resource documents, materials used, video extracts, typical errors, and specific characteristics of the audience.
- The special-needs audiences they cater to.
- The technological tools they use in their classrooms.
- Their equipment.
- Problems and difficulties associated with teaching geometry.

The presentations will be followed by a discussion with the researchers of the ICMI study.

Participant teachers:

- Kindergarten: classes of colleagues presented by Christine Trouillet
- Primary school: Bernadette Da Motta (CE2), Aurélie Marche (CP-CE1) and Mélanie Binet: teachers in Reims
- Secondary school: Isabelle Audra (Paul Langevin's Middle school in Romilly-sur-Seine) accompanied by Marie-Paul Foy
- University: to be determined

DISCUSION TOPICS

TOPIC A

Theoretical perspectives

ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

UNDERGRADUATE MATHEMATICS STUDENTS' REASONING AND ARGUMENTATION IN PROBLEM-SOLVING GEOMETRIC ACTIVITIES

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The aim of this study is to analyze the extent to which a problem-solving approach in geometry activities engages undergraduate students in mathematical discussions and tasks to formulate conjectures, seek different ways to support and validate those conjectures and communicate their findings to peers and the instructor. To this end, the participants were encouraged to use a Dynamic Geometry System (GeoGebra) to model the tasks dynamically and identify and explore the relationships between objects in terms of empirical and analytic arguments. As a result, the participants initially interpreted concepts involved in tasks statements in terms of their geometric meaning to represent them. Subsequently, they proceeded to explore the model by moving select elements within it. After conducting this exploration, the participants formulated conjectures regarding objects' behaviors or relationships that were explained and supported through empirical and formal arguments. In other words, students extended their ways of reasoning transitioning from empirical to formal arguments to support and validate their conjectures.

INTRODUCTION

Over the last three decades, research in mathematics education has been interested in identifying, explaining, and understanding the difficulties that first-year students face during their integration into university-level education. For instance, Leviatan (2008) argues that these difficulties are related to the contrast between how mathematics is taught at a high school level versus at a university level. While the former tends to focus on developing algorithmic skills aimed at solving concrete and routine exercises, the latter demands abstraction skills and inquiry questioning and emphasizes non-routine problem-solving and mathematical rigor. The data obtained from diagnostic admission exams for bachelor's degrees in mathematics in Mexico indicate that the majority of students lack a basic understanding of fundamental mathematical concepts and are primarily focused on reproducing problem-solving algorithms. This means that most students begin their college education with limited knowledge of mathematical concepts and problem-solving strategies.

This highlights how crucial it is to shift the focus in mathematics education towards perspectives that prioritize the development of problem-solving and sense-making skills about mathematical concepts. That is, activities and instructions ought to be designed in such a way that they foster the development of students' mathematical thinking. Such instructional frameworks should empower students to articulate and address diverse problem types across the spectrum of their academic and social education. Santos-Trigo (2019) argues that, in a problem-solving based instruction, students can engage in mathematical knowledge by conceptualizing the discipline as a set of dilemmas that need to be represented, explored, analyzed, explained and justified, i.e., learners need to develop and value an inquiring approach to understand concepts and to solve problems. Additionally, the use of digital technologies as a problem-solving tool allows students to explore mathematical tasks in different ways. In this sense, we argue that exposing students to a problem-solving approach where students can rely on digital technologies can foster crucial skills to elaborate conjectures, justify and communicate results, which promote their integration to university education.

Therefore, in this research, we aimed to characterize, through geometry activities based on a problem-solving approach (Polya, 1965; Schoenfeld, 1985) and mediated by the coordinated use of GeoGebra, the tools, skills and difficulties exhibited by undergraduate students as they work on geometric tasks that encouraged them to formulate conjectures, explore different ways to support and validate those conjectures and to communicate their results. The research seeks to address the question: To what extent do these geometric activities, mediated by the use of GeoGebra, promote the development of mathematical processes, such as formulating, supporting and validating conjectures, and elaborating mathematical arguments among undergraduate students?

CONCEPTUAL FRAMEWORK

When students begin their university education, they face difficulties related to their lack of abstraction skills, inquiry reasoning, general sense of mathematical rigor, and non-routine problem-solving skills (Leviatan, 2008). This indicates that moving from a secondary-level of education to a university-level entails a process of transition and integration. Research on the secondary-tertiary transition in mathematics, has considered undergoing perspectives to explain and address the problems and stages that students experience during this period. In this matter, it has been found that some aspects influencing students during the secondary-tertiary transition are related to their previous experiences in learning mathematics, the quality of study strategies, the way of thinking in/for mathematics, and conceptual background about mathematics ideas (Rach & Heinze, 2016; Di Martino & Gregorio, 2019). In this sense, Schoenfeld (2022) highlights the importance of providing challenging activities to students, i.e., non-routine problems as well as generating educational environments that promote “learning to think in and with the discipline, internalizing the discipline’s habits of mind and practices of doing mathematics”. Thus, students need to have close encounters with processes linked to mathematical reasoning as the formulation of conjectures, examples and counterexamples, and arguments that allows validation of mathematical states.

On the subject of argumentation and proof in university-level mathematics education, Meyer (2020) expresses that several aspects need to be considered when analyzing how students formulate their arguments, such as the structure, the mathematical content involved, and the recipient-orientation. Furthermore, De Villiers (2010) states that it is essential for teachers to explore authentic and meaningful ways of incorporating experimentation and proof into mathematics education, to provide students with a deeper, more holistic insight into the nature of mathematics. In respect of this matter, as mentioned by De Villiers (2010), we understand that experimentation involves any intuitive, inductive, or analogical reasoning employed when:

a) Mathematical conjectures and/or statements are evaluated numerically, visually, graphically, diagrammatically, physically, kinesthetically, analogically, etc.;
b) Conjectures, generalizations or conclusions are made based on intuition or experience obtained through any of the above methods. (p.205)

In this regard, Liljedahl (2016) and Schoenfeld (2022, et al., 2016) state that it is fundamental to create robust learning environments that support students in not only developing the qualified knowledge and processes underlying mathematical thinking but also fostering the development of a sense of agency and authority to make sense of mathematical objects and practices. Thus, we considered aspects related to TRU framework (Schoenfeld, 2014) to create powerful mathematics classrooms in which exist 1) Mathematical content; 2) Cognitive demand; 3) Access to mathematical content; 4) Agency, authority and identity and 5) Use of assessments. Moreover, Santos-Trigo (2019) highlights the importance of incorporating technologies such as Dynamic Geometry Systems as the foundation to represent, explore and extend mathematical tasks when designing a powerful mathematical classroom.

Thus, first, we provide a general description of the teaching practices carried out during our research and the task design implemented; then, we present a qualitative analysis of the processes set in motion by the participants. Regarding the data analysis, we identified the resources and heuristics (Schoenfeld, 1985), as well as the conceptual and procedural tools (Melhuish et al., 2022) that students activated during their problem-solving work. For this purpose, we considered the Authentic Mathematical Proof Activity (AMPA) theoretical framework proposed by Melhuish et al. (2022). Out of the ten procedural tools expressed by those authors, we sought to identify: 1) the refinement or analysis of a proof, a statement, or definition by focusing on the attainment of assumptions; 2) the process of translating informal ideas into formal or symbolic rhetorical forms; 3) the elaboration of analogies, i.e., the process of importing proofs, statements or concepts across different domains adapting them to new schemas; 4) the use of examples, such as using a specific and concrete representation of a statement, concept or proof that represents a class of objects; and 5) the elaboration of diagrams and visual representations of mathematical objects (statements, concepts or proofs) that capture structural properties.

METHODOLOGY AND TASK DESIGN

The data gathered from this research resulted from the implementation of a geometry course with 30 first-year undergraduate students. The main objective of the course was to lead students to develop deductive reasoning through problem-solving activities through of a DGS (GeoGebra). This allowed them to make sense of the problem statements and explore, represent, and identify geometric relations between the mathematical objects involved through direct manipulation, namely dragging. Due to space constraints in this paper, we will describe the analysis of one team's work (E1) regarding one of the four implemented activities (T3). However, we will provide a brief discussion of all 15 teams. We summarize the methods followed during the tasks down below:

1. All tasks were presented as problems, whether they had a situational context or not. Tasks were adapted from rutinary textbook problems into open-ended problems that required the formulation of conjectures.
2. Students were then asked to validate their conjectures by generating an argument or giving a counterexample. Both the elaboration of the conjecture and its validation or refutation were expected to be done with the support of GeoGebra.
3. The activities were conducted by a teacher, one of the researchers, by guiding the participants' work through questions and prompts. These had the dual purpose of fostering reasoning and argumentation skills and encouraging exploration and understanding of mathematical concepts by manipulating dynamic models.

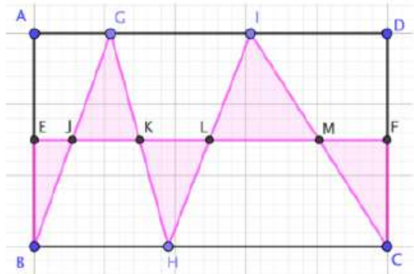
4. Students had an average of three days to tackle the task and were permitted to work in pairs or individually, depending on the task; this allowed students to socialize and share their initial ideas.
5. After exploring each task, a face-to-face session was held to discuss questions, difficulties, conjectures, and possible initial arguments expressed by students. They were encouraged to share their work with the rest of the class to gain insight into their ideas and receive feedback from their peers and the teacher. The objectives of this type of session were to lead the students 1) to identify the concepts involved in the problem as well as weak resources or cognitive obstacles; 2) to identify the conditions given by the problem's statement and its relation to the problem's representation, 3) to decide and raise an initial statement of the proposed conjecture, and 4) to exchange possible strategies to approach the argumentation of the conjecture.
6. Students were asked to register their problem-solving processes, as well as questions, ideas, or actions that arose while tackling the task and during face-to-face sessions. This constituted the main source of data for this research. For this purpose, the teacher asked students to include in their registries the following elements:
   - *Description of all explorations oriented to understanding the problem statement and generating conjectures.* For both data analysis and the statement of a conjecture, we sought to determine what concepts, strategies, processes and/or digital tools were activated by the students when exploring and understanding the problem.
   - *Description of the process followed for the proposal of a plan or strategy that leads to solving the problem.* We sought to identify what resources (concepts, mathematical content, evidence, or previous results) students recognized as helpful when tackling the problem, how they activated them when they sought a solution strategy, and what type of inquiries they made.
   - *Description of the problem-solving process,* as well as all types of algebraic or empirical procedures, diagrams, etc. We analyzed the types of arguments students used when they proved or validated their conjectures, the use of mathematical notation they exhibited, and the logical structure exhibited in their proofs.
   - *Problem extensions.* We analyzed whether students could pose new or similar questions or problems to be solved based on what they had learned, explored innovative solutions and/or generalized their strategies. Through these questions, we aimed to encourage students to engage in the activity of problem-posing.

Task T3

Now, we present the instructions and questions given to the students to guide their reasoning process in Task 3 (it is worth noting that the original language of the instructions was Spanish). Figure 1 shows the problem statement.

Instructions: The solution process for this problem must be thoroughly registered on a logbook. *Section 1. Exploring and obtaining a conjecture:* a) Create a model of the embedded figure in the problem using GeoGebra; then, explore the problem by manipulating the elements of your model using the available tools in GeoGebra; b) Register any questions you asked yourself during the construction of the model. Establish the assumptions given in the problem and those that you have considered for the construction. Write an initial guess (conjecture) about the answers to the questions stated in the problem; c) If you move the sides of the quadrilateral, does your guess still work? If you move point F along segment CD, is your conjecture still true? d) Considering the initial conditions that you think are necessary to solve the problem, determine a possible conjecture about the ratio of the shaded area to the non-shaded area. *Section 2. Elaboration of a solution plan:* a) Describe the mathematical concepts you consider are underlying in the formulation of your conjecture. Which of them do you master, and which don't? b) Use online platforms, videos, or books to do some research on those concepts. Add your query sources. c) Describe how these concepts or results could help you to solve the problem.

En la siguiente figura se muestra el cuadrilátero ABCD con BC paralelo a AD. Se traza el segmento EF donde E y F son puntos medios de AB y CD respectivamente. ¿Será posible establecer una relación entre el área sombreada y el área no sombreada? ¿Cuál sería esa relación?



Translation: The following figure shows the quadrilateral ABCD with BC parallel to AD. Segment EF is drawn, with E and F being the midpoints of AB and CD, respectively. Is it possible to establish a relationship between the shaded and the non-shaded areas? What would that relationship be?

Figure 1: Problem corresponding to Task T3

Section 3. Development of an argument and solution: a) Describe an argument that allows you to validate the conjecture you have established. If you realize that your conjecture is false, present a counterexample and then seek to establish a new conjecture and validate it. Repeat this process until you can find an argument (or proof) that shows that your conjecture is true. *Section 4. Extension of the problem:* a) Pose at least another way to solve the problem, a new similar problem, or question that you could solve based on what you found out during this problem.

Analyzing these elements, we derive several results regarding the reasoning methods and approaches students employed to solve problems and support statements.

ANALYSIS AND FINDINGS ON TASK 3

We now show the analysis of the types of reasoning and problem-solving processes followed by the team E1 when arguing the validation of their statements. We also provide some insight on the difficulties or obstacles faced by students and how they overcame them.

Team E1 initially approached the problem by employing paper drawings despite the problem indicating the use of GeoGebra. Realizing that this method complicated their exploration, the students subsequently shifted to using GeoGebra to construct the figure and represent the given problem and its conditions. Then, students manipulated (dragged) points along parallel sides, leading to the identification of various triangles with similar properties. They further posed the question “What happens when the points are evenly distributed on the parallel lines?” and explored potential changes to the figure by dragging its vertices, examining how their initial ideas were influenced. In this sense, manipulating the model was important for students to identify the existence of similar triangles. Then, students constructed rectangles by focusing on the points along the AD, BC and EF segments, viewing it as a helpful approach to reaching conclusions for the given problem (Figure 2a). Their initial state was: "The shaded area is 1/4 of the total area". Afterward, they utilized the measure area tool in GeoGebra to compare the areas of the shaded and unshaded triangles, along with the entire rectangle ABCD (Figure 2b) as they formulated the conjecture: "[In rectangle ABCD] The shaded area is 1/4 of the total area, and the unshaded area is 3/4 of the total. $\therefore$ The unshaded area is three times the shaded area." This approach allowed them to visually confirm the validity of their conjecture and state verbally that the shaded area is invariant to the position of the points on AD and BC as long as the shaded regions do not intersect each other. To support the validity of their statement, students employed pivotal concepts from the exploration and understanding phase, such as parallel lines and angle relationships, midpoint of a line segment and similarity. They considered angles formed by transversal lines (BG, GH, HI, IC) intersecting parallel lines (AD, EF, BC) (Figure 2b). This approach led to the establishment of triangle similarity. Consequently, they applied Thales's first theorem along with the area formulas for triangles and rectangles. Using these methods, they conducted a detailed analysis, comparing the areas of the shaded and unshaded triangles to the overall area of the rectangle, which allowed them to support their conjecture (Figure 2c). Although the students successfully argued their conjecture, it is evident that there is room for improvement in the use of mathematical notation, particularly regarding the establishment of similar triangles and the ratio of areas. Enhancing this notation could lead to a more concise and effective argument. Finally, team E1 did not explicitly pose a new problem or additional questions related to that posed initially.

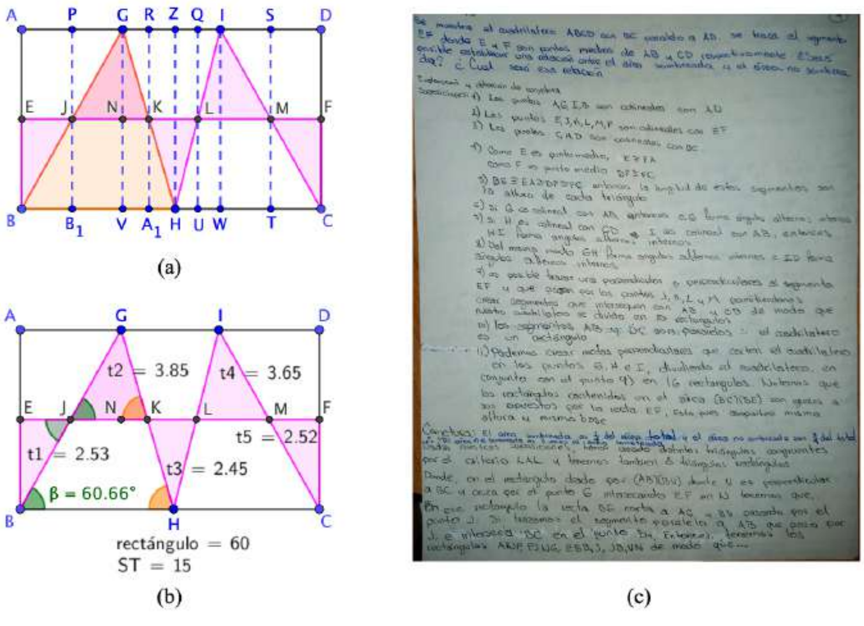


Figure 2: Illustrations of the exploration processes followed by team E1

In general terms, the students used GeoGebra to construct the figure and represent the problem and its conditions. They established initial conditions and made plausible conjectures about the type of quadrilateral involved and its area. During exploration, common questions were raised about point distribution on parallel lines and the impact on the quadrilateral when it is a parallelogram without restrictions, for example: "What happens when the points are evenly distributed on the parallel lines? What happens if there are no restrictions on the points but it [the quadrilateral] is a parallelogram?" While 80% of teams shared similar conjectures, variations on hypotheses existed; for example, Team E2 generalized to the parallelogram ABCD instead of the rectangle. Following a feedback session, students collaboratively, along with the teacher, used GeoGebra to represent and discuss the generalization or particularization of the conjecture, as well as visual principles for argumentation. The manipulation of the model played a crucial role, enabling students to articulate a more structured, albeit not entirely rigorous, reasoning.

Summarizing findings, we assert that: 1) students developed skills for analyzing statements by identifying assumptions; 2) the process of translating informal ideas into formal or symbolic rhetorical forms requires more activities and guidance from teachers to develop this particular skill; 3) students applied previous concepts and statements to this new problem; 4) they constructed and manipulated a dynamic and concrete representation to explore the problem and concepts related to stating a conjecture, allowing them to actively participate into the processes of argumentation; and 5) they elaborated diagrams and visual representations of the proposed statement and related concepts, capturing their structural properties.

CONCLUSIONS

The teaching dynamics aimed to provide students with equal access to geometric content through course notes, suggested bibliography, interactive GeoGebra applets, and the opportunity for online research. The problems and questions posed prompted students to generate conjectures, explore different ways to validate them, and consequently develop the ability to argue rather than solely rely on predetermined solutions. Group discussions fostered individual and collaborative engagement with the activities. While it is observed that students improved their reasoning and argumentation skills through the activities, it is also evident that this development is a gradual process that may evoke feelings of confusion and frustration, particularly due to their familiarity with routine problems. Furthermore, it has been noted that there is a lack of motivation for students to formulate their problems, whether derived from those suggested by the teacher or independently conceived. Therefore, the role of the teacher is considered crucial in guiding and providing feedback throughout the monitoring and control processes.

Given this, the dynamics in mathematics courses must provide activities that bring students closer to the realization of activities of a high mathematical level and empower students to recognize their mathematical resources and to make decisions on applying them in problem-solving. This entails manipulating auxiliary tools, namely technologies, to explore and construct mathematical objects and formulate conjectures. Creating spaces that cultivate oral and written arguments, using mathematical notation and symbology, and refining proofs are essential aspects of developing geometrical mathematical thinking guided by a logical sequence that progressively leads students toward the processes of abstraction and demonstration. Such practices strengthen mathematical reasoning, particularly during the transition from high school to university.

Finally, developing course content based on non-routine problem-solving or initial questions posed by the teacher, requiring students to conduct preliminary exploration mediated by the use of a GDS like GeoGebra during class, promotes: 1) the exploration and manipulation of geometrical models, as well as the understanding of the problem and mathematical content; 2) the formulation of conjectures; 3) the visual validation or generation of visual geometrical counterexamples to initial conjectures; and 4) the development of many possible solution strategies. By engaging in these processes, the structuring of arguments and proofs for the conjectures is facilitated, contributing to the development of robust reasoning within the mathematics field. It was observed that students were afforded opportunities to actively use diverse mathematical skills and explore various approaches to construct compelling arguments, substantiating the relevance and validity of their conjectures. The results indicate that students extended their ways of reasoning, enabling them to transition from empirical to formal arguments in presenting the solutions to the problems.

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SPATIAL REASONING IN AUTHENTIC CONTEXTS OF AN ENGEENIRING CHALLENGE: TAPPING INTO CHILDREN'S INTUITIVE UNDERSTANDING OF RELATIONSHIPS BETWEEN OBJECTS AND SELF

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We report on results of a study featuring kindergarten students solving an engineering challenge of constructing a shelter for a stuffed animal using carton materials and tools in a makerspace. Our analysis focuses on spatial reasoning (SR) skills that were manifested in the children's intuitive exploration of space and shapes as they were planning, realizing, testing, and adjusting the shelters they constructed. The construction context allowed the children to enact creativity and flexible thinking through spatial reasoning. We suggest that makerspaces contexts carry promising ways to surface SR. We underscore the need to make explicit the mathematics employed so that this can be recognized by teachers, students, and parents. We offer a four-part model as a working mechanism. Questions remain, however, pertaining to connections of SR in engineering tasks with aspects such as affect, materiality, agency, authorial identity, accessibility to high-ceiling mathematical concepts and the socio-ecological turn in mathematics education.

CONTEXT AND RESEARCH FOCUS

Along with other places around the world, school makerspaces are taking ground in New Brunswick (NB), Canada, K-12, emphasizing STEM education (NBED, 2016). Several non-profit organizations and charity associations in NB provide much-needed support and assistance to teachers in developing a variety of making activities to foster students' creativity, innovation, computational skills, coding capacity, and an entrepreneurial spirit across grade levels and educational curricula, both in formal and informal contexts. Among these organizations, Brilliant Labs, had initiated a research collaboration with the CompeTI.CA (Compétences en TIC en Atlantique/ICT competences in the Atlantic Canada) and had set up a partnership network team to conduct case studies of school makerspaces (Freiman, 2020).

Overall, CompeTI.CA's main objectives are three pronged: (1) to identify information and communication technological competencies continuum; (2) to research exemplary practices (case studies) in school makerspaces; and (3) to develop new practices for teachers to use. To respond to the call of the 26th ICMI Study on Advances in Geometry Education, we aim to zoom in and out on kindergarten students' intuitive SR during their work on an engineering challenge at the school's STEM Lab (makerspace). In the next section, we present key concepts that frame our work.

CONCEPTUAL FRAMEWORK

An engineering challenge and design in making activities

Learning activities that provide students with *engineering challenges and design opportunities* are considered as beneficial in supporting and further developing modeling, discourse, and natural

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 43-50). ICMI.

scaffolding processes in which students collaborate with their peers to solve real-life problems (DeJarnette, 2012). In these activities, students develop understanding through interaction and observation of their environment (Vygotsky, 1978). Design projects help students to see connections between science concepts and solutions to real-world problems (Sadler et al., 2014). In mathematics, LeBlanc et al. (2022) documented rich mathematical connections in students' work in makerspaces. Similarly, Bush et al. (2022) suggest that teaching through design is learner-driven and goal-oriented, provides authentic contexts, incorporates constraints, and allows for use of a variety of materials, resources, and tools. Teaching through design involves teamwork and also requires the use of tangible artifacts and discrete processes to suggest varied solutions (Sluis-Thiescheffer et al., 2016).

Among several types of materials and technologies available to support design activities, Deed et al. (2022) provide several arguments for the *use of cardboard* as it supports embodied experiences; creative and imaginative play; students' wellbeing—social emotional and physical safety; a process of idea generation; and reconfiguring classroom spaces to support engagement and productivity through construction processes to resolve an engineering challenge. These are qualitatively and fundamentally connected to the development of spatial reasoning (Omundsen, 2014).

Hatzigianni et al. (2021) conceptualize makerspaces as workspaces where open-ended resources are used as enablers of learning through integration of the elements of design, creation, and innovation that position young children as equally capable makers. The maker mindset is promoted in Papert's theory of constructionism, which emphasizes child-led activities, where thinking processes are more valuable and critical than end-products, and where exploration and experimentation are positioned as core to the learning process. Constructionist thinking draws on constructivism, another learning theory that considers how learners construct knowledge through interaction with experience and ideas (Hatzigianni et al., 2021). Heroman (2017) emphasizes the open-ended tinkering in making experiences. That is, as children mature, their ability to use tools, collaborate with others, experiment, observe, make discoveries, tap into prior knowledge, communicate, and persevere continue to develop and flourish. These are key developmental experiences that are made available to learners working in contexts of makerspaces. Hughes et al. (2019) offer a model for a design thinking cycle (ask, imagine, plan, create, improve; Figure 1), which we used to frame manifestations of SR (Ramey et al., 2018)—a construct that we will examine in the next sections.

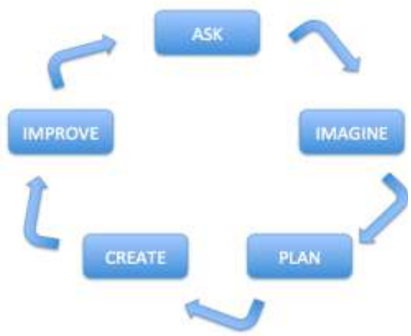

Figure 1. A design thinking cycle (Hughes et al., 2019)

Spatial reasoning: What is it? And why should it be developed from early grades?

Being considered as a critical aspect of intelligence (Smith, 1964), SR involves everything from how we perceive the world, to locating and navigating within it, to imagining, manipulating, and transforming two- and three-dimensional figures. SR is malleable and can be refined and improved upon regardless of age or gender (Stepankova et al., 2014) while tapping into "the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 147). In mathematics curricula, SR is, more often than not, assigned to the geometry strand. The issue with this is that in spite of its critical role in the development of the learner, teachers find themselves less equipped to teach geometry (Moss et al., 2015)—and by extension SR.

SR grounds "relations among two or more objects or between one's moving body and objects or landmarks in the environment" (Davis & the Spatial Reasoning Study Group, 2015, p. 5). It includes transformations in 2D and 3D shapes, angles, and direction (Mulligan et al., 2018) and takes place through interactive processes by locating, orienting, comparing, scaling, re/de/composing objects, finding/creating symmetry, rotating, classifying, balancing, diagramming, transforming, navigating, and sensing (Mulligan et al., 2018). SR can also "be practiced with limited or no use of the eyes—with the hands, with the moving body and gestures" (Whiteley et al., 2015, p. 11).

But how can SR be noticed and identified and why should we care about SR? Several studies shed light on important aspects of SR that can be noticed in a variety of out-of-school situations, and further enhanced in school situations (McCluskey et al., 2018). For instance, at a very young age (K-2), children can already develop very robust understandings of parallel lines (Sinclair & Bruce, 2014), which can be built upon in teaching/learning geometrical properties. In terms of learning outcomes, SR can be associated with mathematics performance (Hawes et al., 2022). In addition, those with developed SR are more likely to enter and succeed in STEM (Wai et al., 2009).

Pulling all these threads together, we recognize the compatibility of and interdependence between different conditions for SR to be noticed and worked on. We follow Latour's (1990) idea of objects as actants to suggest that as much as humans enact manipulations and transformations on objects, objects as well enact their affordances to direct the ways they are used. In this regard, we see discourse patterns, instructional approaches, and teaching strategies as non-reified actants. To us, the entry point to noticing manifestations of SR is located at the context-specific, situated arrangement of objects both reified and non-reified. The idea of the situatedness of SR is then a manifestation of the allowable, reproducible, imitable, transferable, yet contextually bounded spatial-related actions and reactions. Understanding SR as context dependent, relational, goal oriented, and interactive (Figure 2) helps us identify and potentially expand context-dependent affordances (Das & Winter, 2016).

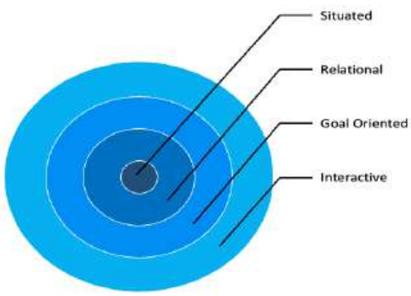

Figure 2: Elements of spatial reasoning

SR is an important part of the NB K-12 mathematics curriculum and is included in the *Geometry Strand* (MEDPENB, 2016). Over the past decades, the STEM movement pushed center stage new opportunities for noticing and fostering spatial reasoning through more informal contexts of school makerspaces (Freiman, 2020). The question we investigate in this context is What aspects of SR emerge when kindergarten students work on an engineering challenge design task?

RESEARCH SETTING AND METHODS

As part of a bigger case study of school makerspaces, we observed two groups of 20 students and their classroom teachers from a K-5 elementary school working in a STEM Lab (makerspace) in an urban area in NB, Canada. After obtaining ethics approval and parents' consent, the first author entered the Lab to video record students' work and conduct post-project interviews with some students. Our analysis focuses on one group working on *building a shelter for a stuffed animal*. The shelter needed *to be waterproof and protect the animal from high winds.* Twelve video-segments were first analyzed on the presence of spatial reasoning as per the actions listed above. Then initial codes were assigned to each video-segment. We then identified aspects across all segments.

Observing the children as they were working on constructing their shelter, we realized that we needed to adjust Hughes et al.'s (2019) design thinking cycle to reflect the design phases the children manifested in their collaborative work. We created a three-phase process: Planning → Realization → Testing and Adjusting (Figure 3).

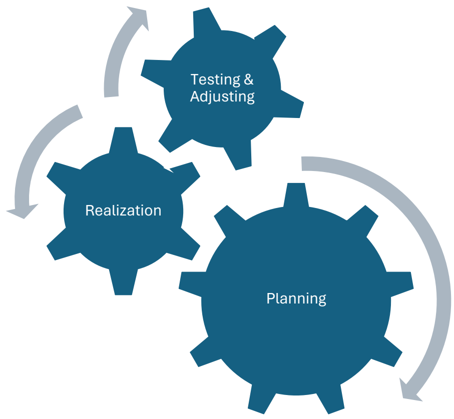

Figure 3: Interconnected cogs in makerspaces (Fellus, Freiman, & Lurette, forthcoming)

To better understand how the children engaged with SR, the following overarching questions guided our analysis that align with the three stages of planning, realization, and testing and adjusting: 1) What aspects of SR do children manifest in planning their work? 2) What elements of SR are observable during prototyping? 3) How do they modify their geometric structures in testing and adjusting their engineered product? In particular, we looked into the ways children articulated their ideas of how to approach the task (planning stage). We also looked into how children organized and managed their space to realize their construction (realization stage). Finally, we learned how children test, readjust, and improve the prototype through SR.

FINDINGS AND DISCUSSION

The process of solving the challenge is highly complex. Students were mostly working in small groups on their own with little or no intervention from the teacher. As the children were planning, realizing, and testing and adjusting, we noticed multiple instances of SR, which were built on the situatedness of the context (i.e., minimal teacher intervention, available tools and materials, the space) that allowed children to work with relation to others and to their environment towards the goal of constructing the shelter through actions of locating, orienting, comparing, scaling, composing, decomposing and recomposing objects (Mulligan et al., 2018).

In the *planning stage*, which was the initial phase of the project, students were discussing the problem and exchanging ideas. They used gestures to *estimate* the size of their shelter and organized their space as they *were getting ready to start their* construction of their imagined shapes (Figure 4).

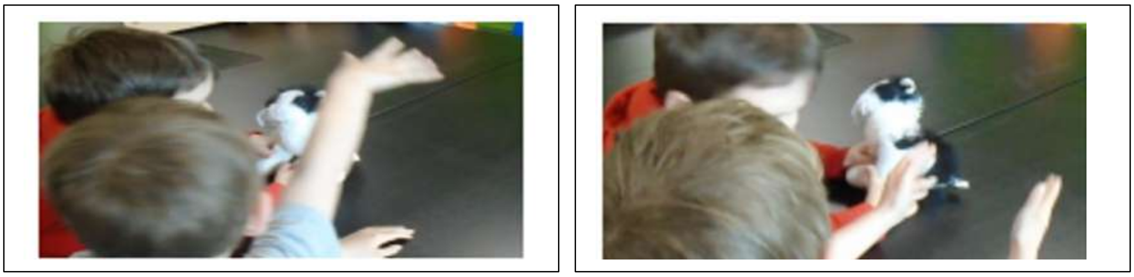

Figure 4: Children estimate and compare lengths and sizes of stuffed animals and projected designs

The second phase consisted of *building a prototype*. Students tried to construct walls using some spatial relationships. At first, students opted for a rectangular shape by synchronizing their movements to produce the desired shape. They also had to decide how to fasten the joints of the faces of the structure to make their construction more solid. They adjusted for size (pushing a toy through a "door") and trying to make the structure more sturdy by adding screws. This was accompanied by a vivant discussion and coordinated actions (Figure 5; Clements, 2004).

Figure 5: Children discussing their construction and coordinating their actions

Figure 6: Children orienting, rotating, balancing, transforming, and synchronizing their movements

Some of the students decided to follow the line of the folds in the cardboard (intuitively grasping a concept of a straight line). We also see two children doing their cuts simultaneously holding their knives perpendicular to the cardboard and trying to synchronize their movements to create parallel sides of the rectangle (Figure 6; Clements & Battista, 1990). Then they pull all pieces together to set up their shelters in a shape of rectangular prisms (Figure 7).

Figure 7: Children complete their 3D prototype

The final phase was testing and making their construction more sturdy: "waterproof" and withstanding against the "wind." Many had to readjust their construction by switching to a triangular shape, for example. Indeed, when testing their constructions, students realized that their construction needs to be more solid and well sealed to protect their animal from winds and rain. They undertook a process of ***transformation*** of their initial constructions (Figure 8; Pruden et al., 2011).

Figure 8: Children testing, rebuilding, and transforming their constructions

Children transitioned shapes of structures as they were navigating toward their goal of constructing the shelter for their stuffed animal. Figure 9 showcases an example of decomposing a rectangular prism and recomposing it as a triangular prism by bending the two side walls to create symmetrically centered, balanced walls for the shelter (Mulligan et al., 2018).

Figure 9: Children composing, de/re/composing and transforming structures

CONCLUSIONS

Overall, we found that engineering challenges enrich young children's spatial explorations by providing them with a sense of spatial insight into complex mathematical structures and relationships between objects and self (Ramey et al., 2018), at least at the intuitive level. The concept of SR is enmeshed with the expression and experience of movement (embodiment) allowing students to design, build, test, and adjust sophisticated geometric constructions within situated contexts that are relational, goal oriented, and interactive. While our findings seem to corroborate with existing literature on SR showing the importance of early STEM learning opportunities, the context of makerspaces lets us direct the discussion towards theoretical concepts that emerge when we reflect on students' intuitive understanding of geometry. Hence, the construct of agency seems to be a promising entry point into the realm of the socio-ecological turn in mathematics education (Bush et al., 2022). This is a necessary step that allows us to reflect on the multidirectional relationship between one's embodied self and their environment. This reorientation of mathematics education through the lens of agency is how we navigate public discourse in the volatile, uncertain, complex, and ambiguous world. These ideas require further exploration and discussion.

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ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
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IDENTIFYING A SEQUENCE OF CORE SKILLS FOR DEDUCTIVE PROVING IN SECONDARY SCHOOL GEOMETRY

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Developing an effective teaching sequence for deductive proving in geometry still faces many challenges. In this paper, we aim to contribute to this issue by identifying 'core skills' for deductive geometry. By analysing the survey data from 238 G8, and 208 G9 students, we identified 7 items related to the generality of proofs, the structure of proofs, the economical nature of hierarchical relationships, and spatial reasoning. Our findings have implications for the teaching of deductive proofs in geometry as our results suggest guidance in considering a sequence of geometry teaching for deductive proving.

INTRODUCTION

Mathematics education research documents how the teaching of geometrical reasoning still faces challenges, and how students continue to have difficulties to write and read geometrical proofs, to solve problems in both 2D/3D contexts, are heavily influenced by visual aspects of geometrical figures, and so on (e.g., Sinclair et al., 2016).

Given that proof is one of the forms of geometrical reasoning that is important to be taught and learnt in school geometry, and while there can be many different forms of proof (e.g., visual proofs, formal proofs, etc.), in this paper we focus on deductive geometrical proofs in lower secondary schools because a) deductive proofs are often one of the main topics in geometry curricular in secondary schools and b) it is still recognised one of the most challenging areas for both teachers and students. In particular, our goal is to explore what types of teaching sequences might be helpful in promoting the learning of geometric proofs in lower secondary schools, a theme identified by the Discussion Document for ICMI Study 26 on geometry education. In order to achieve our goal, we analyse data from 238 G(rade) 8 and 208 G9 students to address the following research questions (RQ): What skills are particularly important for students to solve deductive proofs and reasoning problems in geometry, what question items might be used to characterise these skills, and how can this information inform a teaching sequence for the teaching of deductive proofs in geometry.

We define 'core skills' as 'those that may be particularly related to students' successful problem solving' (Fujita et al., 2022), and can be used to suggest a teaching sequence for particular topics. For example, Fujita et al. (2022) identified specific core skills in 3D geometry by analysing a total of 2303 G4-9 Japanese students, and use the result to inform a teaching sequence for developing spatial reasoning. In what follows, we first present our theoretical framework in relation to our research questions. This informs what types of skills are necessary for deductive proving in geometry. Second, we explain our research methodology - survey items, participants, and analytical approaches. We then present the results of our analysis, identifying 'core skills' and critically discuss how our findings can be used to consider a teaching sequence to support the learning of geometric proofs in lower secondary schools.

THEORETICAL FRAMEWORK OF THIS STUDY

In this study we take Miyazaki et al.'s definition for a deductive proof – "a deductive proof to consist of the following components: singular propositions (premises, conclusions, and intermediate propositions between them), universal propositions (theorems, definitions, etc.), and the appropriate connectives between singular propositions and universal propositions" (p. 255). Understanding deductive proofs require certain skills and understanding of the various aspects. One of them is the understanding of the structure of the proofs (Koseki, 1987; Moore, 1994). For example, Miyazaki et al. (2017) propose the three levels of the understanding of the structure of proofs from seeing a proofs as a cluster of unrelated elements such as assumptions, conclusions, theorems used etc. to a logical network of these elements as well as logical inferences such as syllogism. Also, in addition to these structural/logical aspects of proofs, it is essential to understand the generality of deductive proofs (Koseki, 1987; Kunimune et al., 2010). For example, once a statement is proved deductively, then it is not necessary to use a few examples or measurement to verify the statement. However, this is known to be quite difficult for students to develop the skills to understand this generality of deductive proofs (e.g., Hoyles & Healy, 2007; Marco et al., 2020).

Another important set of skills relate to the nature of geometrical figures. Geometry is a domain with a dual nature – the practical and visual element of a geometrical figure can, at the same time, be studied theoretically and conceptually (Fujita & Jones, 2003). Geometrical figures are figural concepts (Fischbein, 1993), having both visual and conceptual aspects. This notion is useful when considering, for example, students' reasoning of hierarchical relationships between geometrical figures. As studies have shown (e.g., Fujita, 2012), if students rely on visual aspects of figures, then their reasoning might be restricted by such visual information, and they may not accept why squares or rectangles are also parallelograms. The skills in understanding hierarchical relationships help develop the economical nature of geometrical reasoning and proofs (de Villiers, 1994), which is also related to the generality of proofs described above.

Finally, in geometry various skills related to visualising and manipulating geometrical figures are essential. In recent years the importance of spatial reasoning skills is increasingly recognised (e.g. Lowrie et al., 2021), such as mental rotations of geometrical figures (e.g. Bruce & Hawes, 2015), using domain specific knowledge to reason about 3D shapes (e.g. Fujita et al., 2020; Fujita et al., 2022) and so on.

In summary, we consider that students' understanding of deductive proving in geometry can be studied through identifying skills relating to the structure of proofs, the generality of proofs, the nature of geometrical figures, and spatial reasoning.

METHODOLOGY

Context and survey items

In this study, we used a survey with a total of 33 items. The survey items, which were also based on our theoretical framework, were taken from our previous studies (e.g., Koseki, 1987; Kunimune, et al., 2010; Fujita, 2012; Fujita et al., 2020, Fujita et al., 2022) and have the following three main components, namely 1) Deductive proof, 2) Geometric figures, and 3) 3D geometry (see Appendix). In the first section 1) Deductive proof, students were asked to judge the generality of a deductive proof (Q1), write a simple proof (Q2), identify the definition, assumptions, conclusions and theorems used, and point out a circularity (see also Kunimune, et al., 2010). The second section 2) Geometric figures tests students' knowledge and understanding of geometric figures in the context of hierarchical relationships between quadrilaterals. The questions are adopted from Koseki (1987) and Fujita (2012) In this section, students were asked to select parallelograms (Q5), write a definition of a parallelogram (Q6) and solve problems using their knowledge and understanding of the relationships (Q7-10). The third section tests students' geometrical thinking in 3D geometry contexts. The items are almost identical to those used in Fujita et al. (2022).

Participants

It is expected that G8 students who have studied the required Japanese curriculum content would be able to answer all the questions. From February to March 2022, a survey was conducted with 238 G8, and 208 G9 students aged from 14 to 15 years from three state, nonselective lower secondary schools in two different cities in Japan. The recruitment process was exactly the same as described in Fujita et al. (2022). The schools were recruited through the authors' contacts and the study aims and survey procedures were explained to teachers, and then they agreed to participate in the survey. The period from February to March was chosen as students in each grade have experienced the prescribed mathematical curriculum (the Japanese school academic year is from April to March). The survey question items were directly distributed to the students by their class teachers with paper forms, and in general, students completed answering them within 30 min.

Analysis procedure

In this study, we attempt to identify which survey items might be particularly related to the core skills in deductive proofs in geometry. In order to identify a few items from 33 questions, we used the method developed and explored in Fujita et al. (2022), which utilises 2PLM analysis with the following procedure, in which "we divide the question items (set C) into two groups, set A from the extracted items and set B from the others (i.e., C=A+B), and calculate the Pearson correlation coefficient r between the sum of the scores of sets A and B" (p. 446). Also, this step employs the following principles (*ibids.*, p. 446):

1. Consistency with theory: The extracted question items are closely related to our theoretical framework.
2. Consistency with past research evidence: The extracted question items are also identified as important in past/other related surveys.
3. Less is more (or Occam's razor): Consider the whole set of question items as C. Suppose from C we extract five items (set A) but four items are also suggested (set A'). The rest of the items are B (and B'). It is better if the cardinal number of set A is less than set B(= C−A), that is, |A|<|B| (and |A'|<|B'|). Then we calculate correlations between (set A, set B) and (set A', set B'), and if the correlation value r' between sets A' and B' (r'A'B') is higher than r between sets A and B (rAB) (i.e., |A|>|A'| but r'A'B' > rAB) then set A' will be more likely to be

considered as a set of core skill items as we use fewer question items but have more explanatory power.

For the analysis, R 4.3.1 with "ltm" package was used.

RESULT

Overall performance from the survey

Overall, the average mean scores are 18.27 (out of 33) for G8 and 18.33 for G9, there was no significant difference between G8 and G9, t(428.79)=- 088542, p=.9295, despite G9 (M=18.33, SD=7.15) attaining higher scores than G8 (M=18.27, SD=6.77). The following Tables 1-4 summarise the detailed results of correct answers (%s, N=238(G8) and 208 (G9)).

Grade	Q1a	Q1b	Q1c	Q2	Q3a	Q3b	Q3c	Q4a	Q4b
G8	53.8	47.9	73.9	62.6	84.0	81.9	55.5	65.1	38.7
G9	41.3	33.2	69.2	63.9	82.7	80.8	52.9	70.2	46.2

Table 1: Survey results for Section 1) Deductive proof

Grade	Q5	Q6	Q7a	Q7b	Q8a	Q8b	Q9	Q10
G8	57.1	81.1	60.1	9.2	73.5	19.7	37.8	60.9
G9	42.3	73.6	53.4	9.1	70.7	15.4	24.0	46.6

Table 2: Survey results for Section 2) Geometrical figures

Grade	Q1K	Q2K	Q3aK	Q3bK	Q3cK	Q4K	Q5K
G8	92.4	80.3	89.1	78.2	96.6	31.1	32.8
G9	93.8	86.1	88.0	81.3	91.8	45.2	26.0

Table 3: Survey results for Section 3) 3D Geometry (Q1K-Q5K)

Grade	Q6aK	Q6bK	Q6cK	Q7aK	Q7bK	Q8K	Q9K	Q10K	Q11K
G8	55.0	56.3	52.1	42.0	13.9	63.9	46.6	27.3	6.7
G9	65.4	58.2	52.9	51.9	22.1	79.3	64.4	34.1	17.3

Table 4: Survey results for Section 3) 3D Geometry (Q6aK-Q11K)

The key findings from the survey are summarised as follows. First, both G8 and G9 students had a relatively good understanding of proof writing, but struggled with the generality of proofs and proofs with logical circularity: In the first section 1. Deductive Proof, while over 60% of G8/9 students were able to write a simple proof (Q2), less than 50% of them were unsure whether they could accept or reject the use of some examples to prove the sum of the interior angles of a triangle (Q1). Similarly, only 38.7% of the G8 students and 46.2% of the G9 students were able to give their reasons for not accepting a proof with a logical circularity (Q4b). This result is similar to that reported by Kunimune et al. (2010).

Second, both G8 and G9 students had a rather limited understanding of geometric figures in the context of hierarchical relationships between quadrilaterals: By the time the students participated in the survey, they had studied the hierarchical relationships between quadrilaterals, but less than 60% of the students could correctly identify images of parallelograms (Q5). Similarly, less than a quarter of G8 and G9 students were able to state their reasons for not having to prove the lengths of the diagonals of a square again once we have proved the rectangles, because the square is a special type of rectangle (Q8b, 19.7% of G8 and 15.4% of G9 gave correct answers). Only 10% or fewer students were able to give a correct answer and reason for a parallelogram whose four vertices are on the circumference of a circle, e.g. a rectangle (Q7b). Again, this result is quite similar to that reported by Koseki (1987) or Fujita (2012).

Third, while both G8 and G9 students demonstrated their good understanding of simple 3D geometry problems, they struggled to solve problems using both spatial reasoning skills and domain-specific knowledge: The results are very similar to our previous study (Fujita et al., 2022). Both G8 and G9 students struggled to identify the size of angles formed in a cube, implying they have limited spatial reasoning skills.

Identifying items which represent "core" skills for deductive proving in geometry

Whereas all the 33 items are important to grasp the current Japanese students' understanding of deductive proving in geometry, our interest is which items can be extracted from the 33, and what core skills might be related to these extracted items (NB: Core skills are those that may be particularly related to students' successful problem solving).

Our analysis with 2PLM analysis suggests each item's relative difficulty and discrimination values. We took these values with the three principles described in the methodology, and the following 7 items were suggested, summarised in the table 5 and figure 1 below.

<table>
<tr><th>Suggested item</th><th>Reasons based on the principles</th></tr>
<tr><td>From section 1
Q1a, Q3b and Q4b
From section 2
Q8b
From section 3
Q2K, Q8K and Q11K
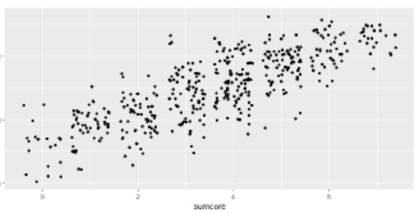
Figure 1: Jittered scatter graph for the sum of 7 items (x-axis) and 26 items (y-axis)</td><td>• These 7 items have relatively higher discrimination values from 2PLM analysis.
• rAB = 0.81 (p < .001), that is, the sum of the seven items (set A) correlates very high with the sum of the other twenty six items (set B), see figure 1.
• Section 1. Deductive proof: Q1a is related to the generality of proofs (e.g., Kunimune et al., 2010; Marco, et al., 2022), and Q3b and Q4b are related to the elemental and holistic aspects of the structure of deductive proofs (e.g., Moore, 1994; Miyazaki et al., 2017).
• Section 2. Geometrical figure: Q8b is related to the economical nature of hierarchical classification of geometrical shapes (e.g., de Villiers, 1994; Fujita, 2012).
• Section 3. 3D geometry: The three items were also suggested as the items related to core skills in Fujita et al. (2022).</td></tr>
</table>

Table 5: Suggested items which represent the "core" reasoning skills

DISCUSSION AND CONCLUDING REMARK

The results presented in the previous section indicate a few remarks. From Section 1) Deductive proof, we extracted 3 items. For Q1a, if a student accepts an explanation for this question but can write a proof, then s/he might not fully understand the generality of proof, which has been reported as one of the challenges in the learning of proofs (e.g., Koseki, 1987; Hoyles & Healy 2007; Kunimune et al., 2010; Marco, et al., 2022). Q3b asks why AB=CD (a property of parallelogram) can be said in a proof. Also, Q4b asks to state a reason why a proof with logical circularity cannot be accepted. These questions are related to the structure of deductive proofs (Moore, 1994) and identify the assumptions of a proof and the logical circularity are both indicated as essential for understanding of the structure of proofs (Miyazaki, et al., 2017). From Section 2). From Section 2) Geometrical figures, Q8b is indicated to be extracted. This question asks if 'the lengths of the two diagonals of a square are equal' can be true without any proofs or measurement if we have proved for rectangle. If a student can answer this question correctly, then this student is likely to understand the economical nature of hierarchical classification of geometrical shapes (e.g., de Villiers, 1994; Fujita, 2012). Also, the students who answered this question correctly are likely to understand what diagrams drawn for a proof might be representing in problem contexts, and have a good understanding of geometrical figures in a sense of Fishbein (1993). From Section 3) 3D geometry, Q2S asks students to draw a cylinder from a front view, and a drawing skill is suggested as important by Sinclair et al. (2018). Q8K and Q11K are related how the students use their spatial skills such as spatial visualisation and mental rotations as well as property-based reasoning, which are again important skills in 3D geometry (e.g., Bruce & Hawes, 2015; Lowrie et al., 2021). Also, the suggested items are the same as the items which were also suggested in our previous study with the different data set (Fujita et al., 2022).

Our findings suggest that our students still have challenges in their understanding of deductive reasoning in geometry, and a more effective teaching sequence might be needed to improve our students' understanding of proof in geometry. We can identify an effective teaching sequence by taking the core skills and activities derived from the 7 items. First, the high correlation coefficient value (0.81, $p < .001$) indicates that if students could answer these 7 items, then they are likely to answer the other 26 questions correctly in the survey. This implies that the skills to be used for answering these chosen 7 items might be particularly important for success in not only constructing and reading simple deductive proofs in geometry, but also answering problem solving which require deductive reasoning and proving in wider geometry problem solving contexts.

We can use these 7 items for considering the teaching sequence. When designing a teaching sequence, one way is to locate activities and questions related to these 7 questions explicitly as such activities might be related to the core skills in deductive reasoning in geometry, i.e., the generality of proofs, the structure of proofs, the economical nature of hierarchical relationships, and spatial reasoning. For example, when introducing what a geometry proof is, not only check the fundamental structure of deductive proofs such as definition, assumptions and conclusions (Q3a-c in our survey items), but also provide an opportunity for students to critically reflect why such an argument might be necessary or valid (Q1a or Q4b). Also, the roles of geometrical figures should be checked carefully at some point, and the economical nature of the hierarchical relationships between geometrical shapes should be explicitly taught (Q8b). Even where proof tasks might be heavily related to 2D geometry, checking if students can visualise geometrical figures, mentally manipulate them by using 3D geometry tasks (e.g., Q2K, Q11K etc.) can be done before students learn how to write and read deductive proofs in geometry. As our tentative conclusion, a teaching sequence might be considered as follows: 1) first, learning opportunities for the generality of proofs and structure of proofs are included (e.g., Q1 and Q3, Deductive proof) with some spatial reasoning skills training (e.g., Q2K, Q8K, Q11K, 3D geometry); 2) students then learn how to write and read proofs, but logical circularity (e.g., Q4, Deductive proof) and the economical nature of hierarchical classification of shapes should be explicitly addressed (e.g., Q8, Geometrical figures); 3) their understanding of deductive proofs and reasoning can be strengthened through more proof tasks (e.g., Q2), and problem solving in 2/3D geometry contexts (e.g., Q7, Q7K etc.). Also, these 7 items can be used to quickly check the extent to which students have understood the basic elements of deductive proofs. For example, in the case of Japanese students, we could use these 7 items with G9 students to quickly check their understanding, who have already completed their study of geometric proofs in G8, but will begin to learn more proofs in similar triangles.

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APPENDIX

Section 1 Deductive proof

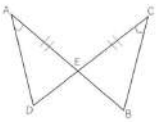
Q1 Which explanation do you agree with as a proof (you can accept as many as you like)?

A: 'I have measured each of the angles and they are 50, 53 and 77. 50+53+77=180. So the sum is 180 degrees.'

B: 'I drew a triangle and cut off each angle and put them together. They formed a straight line. So the sum is 180 degrees.

C: Demonstration using the properties of a parallel line (an acceptable proof).

Q2 Prove that AD=CB when ∠A=∠C, and AE=CE.

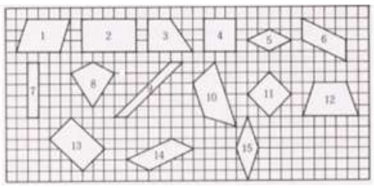


Q3 Reading a proof of the diagonals of a parallelogram intersect at their middle points', and point out why we can say a) AB//DC, b) AB=CD, and c) ΔABO≡ΔCDO?

Q4 Do you accept the following argument which demonstrates that in an isosceles triangle ABC, the base angles are equal? 'Draw an angle bisector AD from ∠A. In ΔABD and ΔACD, AB=AC, ∠BAD=∠CAD and ∠B=∠C. Therefore, ΔABD≡ΔACD and hence ∠B=∠C'. If you do not accept, then write down your reason (Q4b).

Section 2 Geometrical figures

Q5 Choose parallelograms from the figure below.



Q6 What is a parallelogram? Describe it in words.

Q7b Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle? State your reason as well.

Q8b We know that 'the lengths of the two diagonals of a rectangle are equal'. From this, is it also true that 'the lengths of the diagonals of a square are equal' without having to draw a diagram, measure the lengths, etc.? State your reason.

Q9 Name the shape described by the following statement – 'A parallelogram with one vertex as a right angle.'

Q10 Tick the ones you think are correct.

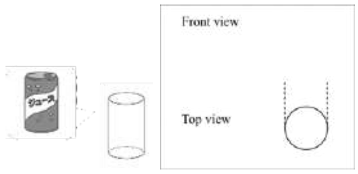
(a) A rhombus is different from a parallelogram.

(b) You can say that a rhombus is a parallelogram.

(c) It is acceptable to say that a parallelogram is a rhombus.

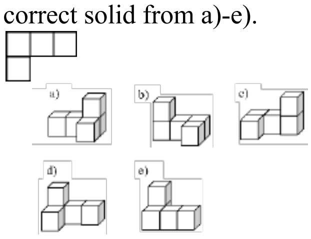
Section 3D geometry

Q2K Draw and complete the front view of the cylinder in the given box.



…

Q3-2K Next, we stack 5 cubes and make a different solid. When we look at this new solid from the top view, then it is like fig. 4. Chose the correct solid from a)-e).

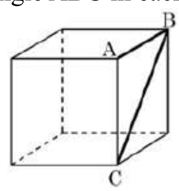


…

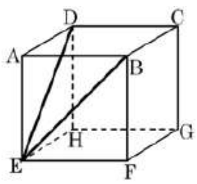
Q5K Sketch possible two solids whose plan and elevation.

…

Q8K In a cube, find the size of the angle ABC in each cube.



Q11K What is the size of the angle DEB in a cube? Write your reasons why.



ICMI Study 26
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THE ROLE OF VISUAL MEDIATORS IN GEOMETRIC LEARNING PROCESSES IN UNIVERSITY EDUCATION

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Our research aims to investigate how the presence of visual mediators can affect the resolution process of university students. Specifically, our study is focused on analyzing students' problem-solving processes in reference to tasks that require them to determine the straight lines intersecting a given parabola. This study allowed us to identify two different categories of strategies implemented by the students, which are framed by Sfard's theory of Commognition. This categorization allowed us to investigate the impact of the presence of visual mediators in the text of the tasks, and their role in the resolution processes enacted by students.

RATIONALE

Geometrical spatial thinking is the ability to visualize, represent and manipulate objects in our minds; as shown by literature (e.g. Presmeg, 2006), spatial visualization skills are those cognitive abilities that an individual can acquire to process images in his or her mind and, thus, also provide solutions to various problems. These are fundamental skills for understanding and solving geometry problems and due to their crucial contribution in the process of advancing mathematical knowledge, they are currently further studied in mathematics education (Presmeg, 2020).

As envisaged in the *ICMI Study 26 Discussion Document,* this contribution aims to explore the following key question: “In what ways are visualization and geometry linked, and how does visualization support geometric thinking?”. This delicate and crucial issue plays a fundamental role in the teaching and learning processes of geometry and is linked to the management of semiotic transformations.

The management of different representations of a mathematical object, and the difficulties in managing different semiotic representations that may hinder student learning have been the subject of several studies in the literature (e.g., Duval, 2006). As per Duval (2006), in order to address the challenges in comprehension that numerous students face during mathematical tasks, it is necessary to take a cognitive perspective and delve into the fundamental traits that underlie the various mathematical processes. The access to mathematical objects themselves, as well as their manipulation, are strictly linked to the possibility of representing them; the impossibility of direct access to mathematical objects also gave rise to Duval's (1993) well-known paradox.

Sfard's "Theory of Commognition" (Sfard, 2008) highlights the significance of visualization in the context of acquiring these mathematical concepts. This theory is focused on the mathematical discourse; the combination of the terms “cognition” and “communication” highlights the fact that cognition and communication are closely related to each other and represent different manifestations, one intrapersonal and the other interpersonal, of the same phenomenon. In this theory, four main properties characterize mathematical discourse: the visual mediators, the routines, word use and narratives. In this research study, we focus attention on the role of routines and visual mediators in resolution processes enacted by university students.

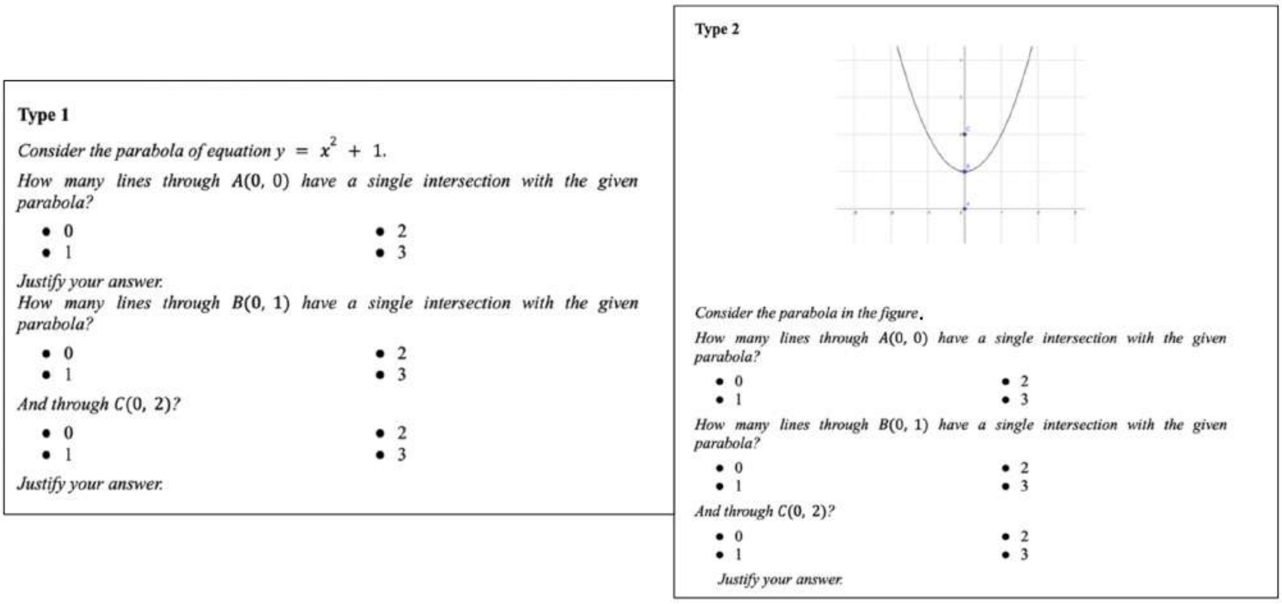
The routines – which in a very broad sense can be seen as the repetition of a previously performed action – enact the mathematical concept, since the invariant that identifies the concept provides the schema used to repeat the action done before. Lavie, Steiner and Sfard (2019) propose a refined and complete definition of routines, which we will adopt throughout our study, and conceptualize the learning of mathematics as a process of routinization of learners' actions. Sfard (2008) identified three kinds of routines: deeds, explorations, and rituals. The first type occurs when there is a physical change in the mathematical object; the second type occurs in the production of the narratives; the last type regards the discursive sequence aimed at maintaining a relationship with people. As we will see later, rituals will play a key role in our categorization.

The visual mediators are means by which objects of discourse are identified within mathematical discourses, and their communication is coordinated. While colloquial discourses are usually mediated by images of independently existing material things, mathematical discourses often involve symbolic artifacts created specifically for this particular form of communication, including graphical representations. Visual mediators are visual elements that play a role during the communication process; they can be gestures, graphs, drawings, but also algebraic representations or material objects used during the teaching of mathematics. The visual mediators help during the communicative process, whether they are actually present or only imagined; in fact, they have been defined as "providers of the images with which discursants identify the object of their talk and coordinate their communication" (Sfard, 2008, p.147). Sfard shows several examples where visual objects (described as realizations of signifiers) are used during communication. The signifiers are the object of discourse, while the realizations of signifiers allow us to perceive the signifiers. There are different types of realizations of signifiers in mathematical discourse, with one of the most important being the visual type as this category allows us to gain a lot of information about the manipulation and representation of mathematical objects (Sfard, 2008). In the field of mathematics education research, there are many studies about the role of visual mediators, especially in their use in group communication (Ryve et al., 2013) and in the students' resolution processes (Gambini et al., 2023). In Dell'Agnello et al.'s study (2022) it is shown how the use of visual mediators could help students to recognize the presence of polynomial roots.

As pointed out by *ICMI Study 26 - Discussion Document,* visualization plays an important role in the students' learning process, and it is very important to work with students with different inputs. Our study fits in this line of thought; in fact, we presented the same task to students in two different versions with two different inputs, to analyze how this difference could affect their solvisng processes. Our research aims to investigate the role of visual mediators in university students' resolution processes for a task that requires them to determine the straight lines intersecting a given parabola. The research questions that guided our study are: in an analytical geometry task concerning the parabola and straight lines intersecting it, are there differences (both in terms of performance and the solution procedures chosen) if the same task is administered with the aid of different visual mediators? Does the use of different visual mediators in solving processes affect performance?

METHODOLOGY

In our study, our attention was focused on the role of visual mediators and routines in the sense of theory of commognition (Sfard, 2008). To investigate the role of visual mediators, a task composed of three multiple-choice questions was designed and administered in two different versions (Type 1 and Type 2, Figure 1). In the first version (Type 1), we provided the algebraic representation of a parabola, while in the second version (Type 2) we showed the graphical representation of the same parabola. In both iterations, the questions remain identical and require the identification of intersections between straight lines passing through specified points and the parabola. Each multiple-choice question offers four answer choices: 0, 1, 2, 3. At the conclusion of each question, students are prompted to provide a rationale for their selected response.



Type 1

Consider the parabola of equation $y = x^2 + 1$.

How many lines through $A(0, 0)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

Justify your answer.

How many lines through $B(0, 1)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

And through $C(0, 2)$*?*

- 0
- 1
- 2
- 3

Justify your answer.

Type 2

Consider the parabola in the figure.

How many lines through $A(0, 0)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

How many lines through $B(0, 1)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

And through $C(0, 2)$*?*

- 0
- 1
- 2
- 3

Justify your answer.

Figure 1: Tasks of Type 1 and Type 2

The subjects in this research are 48 freshers of the Faculty of Engineering; the study was conducted at the end of the Calculus course in the first semester. To carry out a comparison between resolution processes and choices in relation to the proposed stimulus, half the students were given Type 1 and the other half Type 2; the students were randomly divided. The answers to the closed questions and corresponding explanations were collected and analysed with the aim to investigate the role of the presence of different visual mediators in the task text and their use in the solving processes.

CATEGORISATION OF THE ANSWERS

To answer the research questions, we consider students' protocols by performing a qualitative content analysis (Mayring, 2015). Specifically, we sorted the resolution processes according to a concept-driven categorization in the sense of Kuckartz (2019), identifying two main concept-driven categories: Guided by Graph (GG) and Guided by Routines (GR).

The GG category consists of those protocols in which a graphical element, in terms of a visual mediator, is explicitly used and wherein use of the graph is relevant to the resolution process. This category includes both Type 1 solving protocols, in which one or more graphic elements were used even though they were not present in the text, and Type 2 procedures, in which the graphic element is also explicitly present in the text. An example of a protocol that belongs in this category is shown in Figure 2.

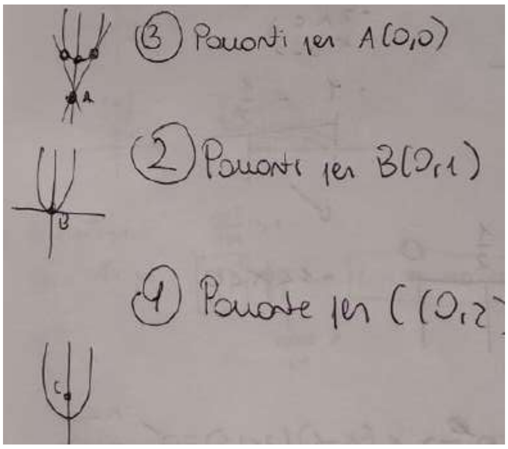


Figure 2: Example of a GG protocol of Type 2

In the protocol (Fig. 2), the student used graphical representation not only to solve the task but also to explain his answers. In fact, for each case, the student showed the intersection of the line with the parabola and the passage to the assigned point. Specifically, in the figure, the student wrote: "3 lines passing through A(0; 0)"; "2 lines passing through B(0; 1)" and "1 line passing through C(0; 2)".

In Figure 3 we can see an example from a student who solved Type 1 task and whose resolution strategy is included in the GG category.

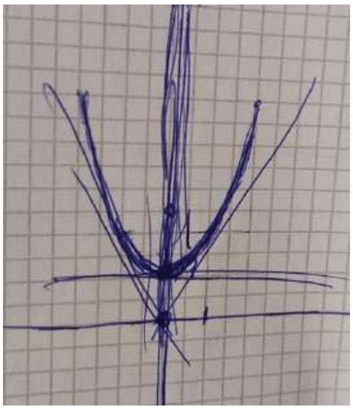


Figure 3: Example of GG protocol of Type 1

This example is in reference to a Type 1 task, where there was no figure in the text, but only the algebraic representation of the parabola. The student drew the parabola and represented on this the given points and straight lines passing through them. He supported his answers with this drawing and an explanation regarding the position of the point. In particular, he stated: "As we can see in the figure, each point stays on the y-axis".

The protocols that show an attempt to find the straight lines with the algebraic method and without the aid of a graphic element belong to the GR category. The techniques predominantly used in these protocols allow them to find only tangent lines and, more relevantly, only lines that are functions. These protocols reveal use of the routine for applying the standard solving formula, which consists of setting up the equation of the parabola and the bundle of straight lines passing through a point, and then finding the angular coefficient, m, by setting the discriminant equal to zero of the second-degree equation dependent on m. This procedure is commonly used and is widespread in Italian high schools; we can consider it both as a ritual technique and, in a broader sense, a Sfard routine. We have named the protocols in which this solving procedure is used 'Guided by Rituals – GR'. Also in this case, both the protocols referring to Type 1 and those referring to Type 2 can belong to this category. In Figure 4 we can see an example of a protocol belonging to the GR category.

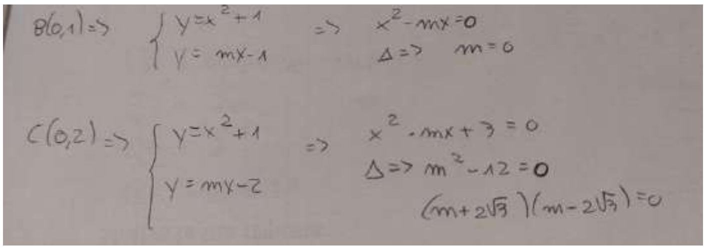


Figure 4: Example of a GR's protocol of Type 1

In the example (Fig. 4), the student tried to find the angular coefficient m of the tangent lines of the curve; in this way, he failed to find the vertical line. In addition, he used the technique to find the tangent lines of the parabola even when the point is internal (point C). As a result of adopting this procedure, the student concluded that there are two straight lines that intercept the parabola through C. This is an example of using routines to solve the task and also to justify one's answers.

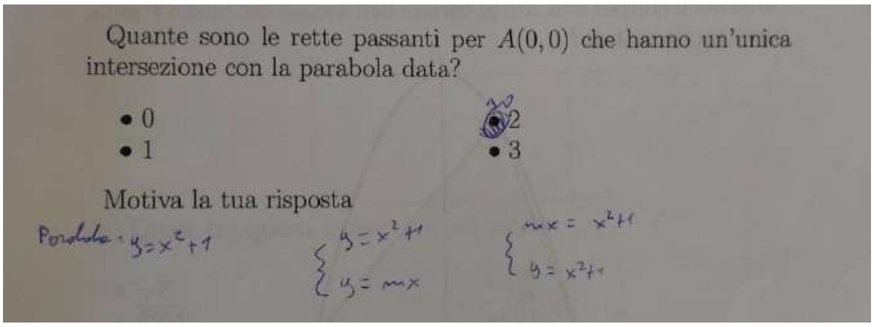


Figure 6: Example of GR protocol of Type 2

Figure 6 shows an example of a Type 2 protocol, in which a strategy belonging to the GR category was implemented. In this protocol it is possible to observe that the student tried to find the equations of the straight lines through the ritual procedure, but this technique was not completed, and the student was not confident of the given answer. Despite the presence of the graph in the text, the student made no reference to, or use of, this and he enacted a ritual procedure (probably because he or she was used to applying it in similar situations in high school).

ANALYSIS

Through the analysis it was possible to investigate, on one hand, the impact of the presence of different visual mediators in the text of the task and, on the other hand, the influence of the use of graphical representation in the resolution processes.

The presence of the graph in the text of the task affected the percentage of correct answers. We consider "correct" answers as those given correctly to all three questions of the task. The overall correct answer percentages are very low: in total, 35 % of students answered correctly but there is a relevant difference between the correct answers in reference to the two Types. In fact, only 25% of students responding to the Type 1 task answered correctly, while almost half (46%) of the students who tackled the Type 2 task (in which the graphic element is present in the text) provided the correct answer. Moreover, the use of graphical representation also played a decisive role in the resolution processes. Comparing the GG and GR categories allows us to highlight how graphical representation can facilitate the resolution process and the production of a justification.

Students who used a GG strategy were those who explicitly used graphical representation to solve the task or to justify their answers. Among all students who answered correctly, 88% used graphical representation in their solving processes. Specifically, all students who performed the Type 1 task (without a parabola graph) and used graphical representation all answered correctly (Tab. 1). In GR's category protocols, students used routines to solve the task; in particular, they used the technique to find the angular coefficient of the tangent lines, and this procedure prevents finding the vertical line because it is not a function. Conversely, none of those students in the GR category answered correctly either Type 1 or Type 2.

Table 1: Percentage of correct answers in relation to the GG category

	% Correct Answers	% GG	% CA of GG
Type 1	25%	54%	100%
Type 2	45.8%	58%	78.5%

This is also evident from the different approach to solving the task and interpreting the questions. In fact, in this task it was not required to find the equations of the lines, but to indicate how many lines intersect the parabola at a single point passing through a given point. Those who used the graphical representation were able to determine the number of the straight lines more easily and were able to draw the straight lines qualitatively. Students who implemented a GR category strategy, on the other hand, had difficulty visualizing the situation, and the algebraic approach did not allow them to find all straight lines. The technique implemented in the protocols of the GR category is a widespread procedure in Italian teaching practices and was most likely used in a ritual manner at high school by the students involved when they encountered tasks with intersections between a parabola and straight line.

CONCLUDING REMARKS

Spatial visualization skills are fundamental for understanding and solving problems in geometry; in this study we investigated these abilities framing within two elements of the commognitive theory of Sfard: visual mediators and routines. We investigated and analyzed solving analysis tasks interpreting them with these theoretical elements and investigating their influence in the solving procedures implemented by university students. The inclusion of graphical representation in the task had a discernible impact on students' performance, resulting in a notably higher percentage of correct responses compared to the same task conducted without visual mediators. Specifically, it was evident that students who engaged in the Type 2 task achieved a higher proportion of accurate answers compared to those who undertook the Type 1 task. These results are in line with what is stated in the

ICMI Study 26 document where it is emphasized how much "it is important to consider how students can be offered a much wider variety of experiences (and inputs)" (p. 4).

Regarding the second research question, it became evident that the choice of visual mediators during students' performance significantly influenced their ability to provide correct answers. This phenomenon likely arises from the fact that the use of graphical representation facilitated students in visually discerning the number of straight lines intersecting with the parabola. Conversely, when employing the algebraic approach, students attempted to determine the angular coefficient, a method that did not enable them to identify all the requisite straight lines. This study underscores the efficacy of incorporating and utilizing graphical representation in successfully tackling the task at hand. According to Duval's (2006) research, this study showed the importance of management of different representations, particularly the graphical representation of a mathematical object. The correct use of graphical representation to solve the task showed students' ability to handle the representations of a parabola. Moreover, those who used the GG strategy correctly in the Type 1 task revealed an ability to switch between algebraic and graphical representations. This grasp and management of different representations failed when routines took over.

When tackling a new mathematical topic, the student is involved in mathematical activity pushed by social needs such as acceptance, expectations and consensus of the teacher or schoolmates. The mathematical routine has no link with the objective of the task situation and instead of being outcome-oriented (as it should be in nature), it becomes process-oriented. Students who adopted a process that belongs to the GR category (probably because they preferred to use a familiar practice that meets the expectations of the teacher) instead focus attention on the request made by the task.

The use of the GG or GR categories strategies affected not only the percentage of correct answers, but also the approach to the problem and the aim of this approach; in one case, it is geared toward solving the task, while in the case of routines it is focused on the procedure employed. Further studies, combined with interviews, will provide further insight into the causes and features of the choice and implementation of respective resolution strategies.

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ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

FROM GEOMETRIC THINKING TO GEOMETRIC PRACTICE: THE POTENTIAL OF REPRESENTATIONS OF PRACTICE FOR TEACHING AND LEARNING GEOMETRY IN SECONDARY SCHOOLS

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This contribution to the theoretical strand and the visualization substrand provides a commentary on possibilities for research and development in secondary geometry. We argue that visualization tools need to include tools for representing the human practices of doing geometry, particularly in the service of problematizing and supporting the evolution of concrete experiences with shape and space into mathematizing those experiences in social interaction and reflecting on those experiences.

TWO DIFFERENT POSITIONS IN RESEARCH IN GEOMETRY EDUCATION

Poincaré classically described geometry as "the art of reasoning well from badly-drawn figures." In their graduate textbook on the subject, Herbst et al. (2017) build a definition of secondary geometry on that quote, assuming that, after elementary school, youth have had formative experiences with shape and space that knowledgeable adults might recognize as geometrical. They propose that

> the development of geometric knowledge at the secondary level consists of the progressive sophistication of students' intellectual means to model, predict, and control geometric representations — that is, the progressive sophistication in students' ways of organizing those artifacts (words, diagrams, and others) so that they can be reliably used in making and transacting meanings (p. 3).

In proposing that definition, the authors complicate our field's relationship to two positions that have currency in the education research community. The first of them can be called the *study of geometry* and takes for granted that geometry is a scholarly body of knowledge produced by mathematicians over centuries, converted into content of studies for students to acquire. The second can be called *experiencing geometry* and takes for granted that human experiences with shape and space across the lifespan confront them with challenges that they undertake by acting in ways that mathematically educated observers might describe as differentially knowledgeable of geometric concepts and properties. In the latter, it is worth noting that *space* alludes to any of the three spaces described by Berthelot and Salin (1995): The microspace of the page or screen (with objects much smaller than the human handling them), the macrospace of sea and landscapes (which much smaller human beings traverse), and the mesospace of objects of size commensurate with the human body.

Each of those positions, the study of geometry and experiencing geometry, has warranted distinct research agendas and educational proposals. In the area we call the study of geometry, the research questions concern understanding how students learn and teachers teach the academic content derived from that store of scholarly knowledge (Du & Zhang, 2019; Lawson & Chinappan, 2000; Levav-Waynberg & Leikin, 2012). This research has also included studies of the cognition involved in reading and doing geometric proofs (e.g., Braithwaite, 2022; Cirillo & Hummer, 2021; Koedinger & Anderson, 1990; Heinze et al., 2008; Lin & Yang, 2007). This research has tended to focus on modeling the cognitive operations students make use of in tasks designed to study geometric knowledge and investigate the effectiveness of teaching and learning approaches to promote this study. This research on the study of geometry has led to the development of interventions (e.g., cognitive tutors) whose effectiveness might be measured in terms of students' successful completion of school tasks. In contrast, the current we call experiencing geometry has warranted research on how children and youth's activity in the physical (or in virtual) world(s) calls for them to act and develop more sophisticated ways of acting geometrically (Hall et al., 2014, Herbst & Boileau, 2018; Nathan & Walkington, 2017; Dimmel, et al., 2021; Soto, 2022). This research done by educational psychologists and mathematics educators has concentrated on interpreting human activity using geometry concepts and casting human actions as attesting to geometric meanings even if those actions are not valued in school. In this line of research, the extent to which students can consciously relate their activity to geometric knowledge has been less of a focus. This line of research has also led to the design of experiences, spaces, and technologies in which students can experience geometry and act in increasingly sophisticated ways (e.g., Ludwig & Jesberg, 2015).

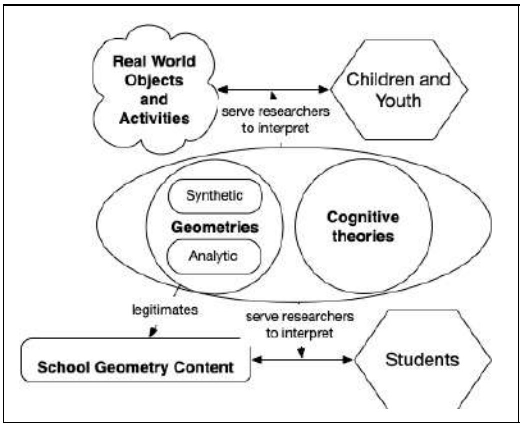

Figure 1. Experiencing geometry and the study of geometry

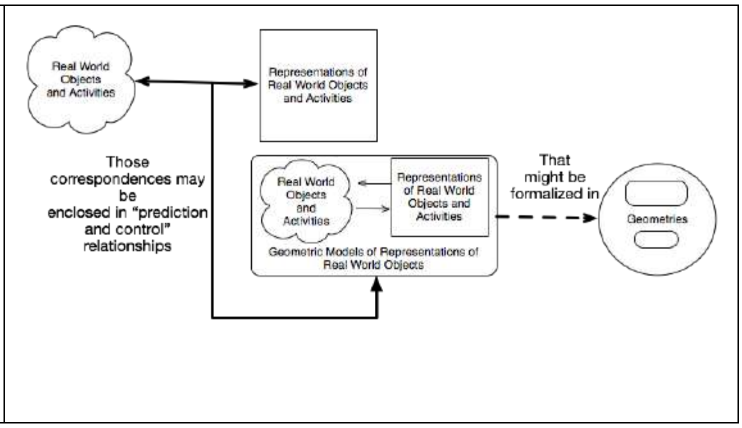

Figure 2. A modeling perspective (from Herbst et al., 2017, p. 4)

The diagram in Figure 1 can help represent these two research perspectives in their differences and their similarities. While research on the study of geometry is concerned with understanding the interactions between school students and school geometry content (bottom of Figure 1), research on experiencing geometry is concerned with interpreting the interactions between children and youth and the real world (top of figure 1). In both cases, geometry is a resource for the observer: In the first, the discipline of geometry legitimizes the content of studies as well as informs researchers' interpretation of what students do in those tasks; in the second, the discipline of geometry informs the researchers' selection of the real world objects and activities in which they observe youth and the researchers' interpretation of what the youth do therein. Geometry is wielded by the observer to sanction youth's meanings either by certifying matches and mismatches (in the study of geometry) or by providing the key for projecting correspondences (in the case of experiencing geometry). In both cases, one still has to ask on what basis may one assert that the subjective experience of the youth can be described as geometric.

Both approaches seem to ignore that the geometries that serve such sanctioning are the products of mathematicians practice and that mathematical practice is what endows the eventual mathematical knowledge with this mathematical status. Poincaré's notion of "thinking correctly about poorly drawn figures" is an epigram of this mathematical practice and alludes to how mathematicians' work with representations (e.g., diagrams, inscriptions) seeks to overcome the limitations of representations through disciplined thinking. The communications about that thinking that have served to build geometries as bodies of knowledge (using publications and letters) over more than 5000 years (Scriba & Schreiber, 2015) in which concrete problems (e.g., architectural design) interacted with theoretical problems (e.g., the possibility of a construction) have been key for constituting geometry as a body of knowledge. While the approach we call the study of geometry seems to extract the product of such evolution to use it as an academic text of studies, the approach we call experiencing geometry takes for granted how mathematical practice has historically connected human experiences with shape and space and imposes on these experiences a geometric status that is in fact contingent on mathematical practice. The modeling perspective Herbst et al. (2017) propose seeks to create in classrooms the practices that would allow students to turn their experiences with shape and space into discourses that might be comparable to the geometries produced through mathematical practice.

A MODELING PERSPECTIVE ON SECONDARY GEOMETRY EDUCATION

Figure 2 represents that mathematical activity as a process of modeling: The construction of representations of real world objects and activities (e.g., drawing perspective diagrams of buildings) and the visualization of real world phenomena (e.g., buildings) in terms of such diagrams is at once a first concrete modeling activity (what Kuzniak, 2018, would call the first geometric working space) and the source of a second modeling activity where the key features of those representations are proposed (e.g., in the form of definitions and assumptions; such as Piero della Francesca’s theory of perspective; Peterson, 1997; or what Kuzniak, 2018, would call the second geometric working space). This second, geometric modeling, as it accumulates and gets organized by logical principles (e.g., axiomatics), may give rise to the geometries that eventually legitimate the school geometry content (what Kuzniak, 2018, calls the third geometric working space). A key feature of the mathematical modeling described is an epistemological shift in the relationship with representations (e.g., diagrams): While at one time representations portray the real and hence they can be read as disclosing (e.g., through visualization) other features of the real, at another time, these representations are constrained by stipulations (e.g., definitions, norms for visualization) that allow some visualizations as reasonable but discourage others, and furthermore necessitate some visualizations as reasonable even if they are not perceived. This is what Herbst (2004) called a generative interaction with diagrams; other representations (e.g., configurations of objects taken as representations of real-world objects and activities, or media representing geometric objects over time) are open to the same considerations.

The conceptualization of secondary school geometry offered by Herbst et al. (2017) seeks to organize the teaching and learning of geometry so that students can engage in a similar mathematical modeling of their relationships with diagrams and other representations as Poincaré and other geometers have done. They represent this engagement with the diagram in Figure 2. In this conception of secondary geometry, the geometric experience in the world (e.g., the human creation of artifacts that others might endow of geometric meaning) is the beginning of a modeling process that aims at producing those meanings as ways of controlling inferences about the represented world. If the research and development agendas that motivated geometers to create geometric knowledge in the first place are not available to teachers and students at present, the general question one needs to ask is in what experiential context can specific geometric concepts become sensible ways of giving students control over the meanings they make of those experiences.

The general question raised above is not new and indeed it represents a whole field of research in geometry education–the didactique of geometry (Houdement & Kuzniak, 2006). An important question within that approach concerns students' development of the need to represent (including the need to characterize discursively) the properties of concrete representations of their geometry experiences. Poincaré's notion of "reasoning well" does not stress enough the importance that actual statements (i.e., semiotic productions about those poorly drawn diagrams) have in the production of such reasoning (Duval, 1995). If the study of geometry may take those statements as canonical elements of the content of studies, experiencing geometry may take those statements as unnecessary to constitute the geometric meaning of experience. In contrast, the didactique of geometry asks in what conditions might the statement of properties about geometric representations be instrumental for students to control how those representations are made and interpreted.

Didacticians of mathematics are mathematics education researchers concerned with the meaningful construction of mathematical knowledge in schools. Didacticians of geometry can see value in both types of research described above (research on the study of geometry and on experiencing geometry) while still noting that there is a gap that practitioners are bound to recognize: Students at times might produce the right answers to school tasks without providing any evidence that their actions in the real world are informed by geometric understanding; students might also encounter school tasks for which their actions in the world could provide valuable prior knowledge to use but fail to recognize those connections. That is, it is possible, even common, to see that knowledge fails to inform action and that skilled action fails to inform knowing. A compelling illustration of this gap in practice can be read in Schoenfeld's (1989) observation of students who had proved theorems about lines tangents to circles but failed to use them to control the ways in which they constructed a circle tangent to intersecting lines. Equally interesting are observations of adults who know geometry but fail to use this knowledge to understand practical problems in housekeeping such as moving furniture or fixing exercising equipment. For sure, we are not saying that those practical activities require an explicit use of geometry, but that a way to make geometric knowledge and its meanings valuable to people is to create the conditions in which they recognize this knowledge as enabling them to act differently.

The separation between problem types that Schoenfeld (1989) alluded to suggests not only an undesirable outcome in geometry education, but also a gap in education design and research. The definition of secondary geometry proposed by Herbst et al (2017) seeks to bridge the gap illustrated by those examples by proposing a modeling approach to geometry. By this they mean

> the development of the capacity to intellectually organize, predict, and control the world of representations of physical objects and experiences. We describe this as a *modeling perspective* insofar as it sees as central the process of creating models of diagrams and other concrete representations (p. 3)

Fleshing out this conception of secondary geometry requires the development of a kind of instruction that connects meaningful experiences in the spaces in which students (could) live with the mathematical ontologies, epistemologies, and axiologies produced by the discipline of mathematics. The reasons one might want to do that have been articulated over more than a century of geometry instruction (González & Herbst, 2006). They boil down to the belief that geometric knowledge can make lives better in a variety of ways (where lives include those of the mathematics enthusiast, the manual worker, the city dweller, the computer gamer, etc.). Inasmuch as secondary schools can be places where students explore various possible lives, the design of such instruction does not need to be limited to any given conception of the real (material or social) world but can make different worlds (including virtual worlds in immersive digital environments) available to students' exploration.

The added complexity of secondary geometry instruction is how to make experience in real worlds an object of mathematical modeling in which geometric ontologies, epistemologies, and axiologies, can emerge. By geometric ontologies we refer to the notion that geometric objects and concepts as construed in the discipline of mathematics are different kinds of beings than the beings students encounter in their daily, real or virtual, experiences with space and shape; and that one role of secondary geometry is for students to create models of the beings they encounter in their experiential world that are comparable with the concepts and objects of geometry. By geometric epistemologies we refer to the notion that coming to know in the discipline of mathematics (including calculating, constructing, exploring, proving) involves a different relationship to knowledge than coming to know in concrete experiences with shape and space: While the latter may seek timely effectiveness or efficiency in context, the former seeks enduring truth and reproducibility across contexts. By geometric axiologies we mean that the production of geometric knowledge in the discipline is sustained by ethical and aesthetical values that are not necessarily present in the experiences with shape and space in which youth are involved daily: While the latter may privilege the self's well-being (succeeding, feeling good) and their preferred relationship with their temporal and physical surroundings (seeking usefulness or comfort), the ethical and aesthetical values of the discipline seek to add an outer orientation (to develop knowledge that enlightens the world) and depersonalizable ways of appreciating the world (seeking intersubjective truth and beauty, for example). Clearly, that secondary geometry may pursue the expansion of students ontological, epistemological, and axiological relations with the world, does not denigrate earlier experiences with shape and space or suggest that the mathematics discipline provides the only way of expanding those experiences–the visual and performative arts clearly are other such disciplines that expand similar earlier experiences toward alternative relationships with the world. Just as the recognition of a drawing as art requires an artistic processing of the drawing, the recognition of experiences with shape and space as geometry require a geometric modeling of those experiences. If outside observers may bestow the labels of art or geometry to such drawing or experiences, the possibility that the producer of such drawing or the protagonist of such experience might be able to construct meaning for such labels, use them productively in later practice, and be able to allocate them appropriately are still different, compelling possibilities. And we are saying that secondary geometry is the place where learning to do such geometric modeling needs to happen–a geometric practice may unfold there: A reflective discourse of action with shapes and spaces in social circumstances that require communication progressively commensurate with that of mathematicians. In what follows we are much more concrete, focusing on construction as part of geometric practice and for the sake of illustrating the intricacies of a pedagogy (or, should we say, a didactique) that seeks to induct students into a geometric practice.

USING REPRESENTATIONS OF EXPERIENCES WITH SHAPE AND SPACE TO TEACH

Students' induction into a geometric practice can be supported by a different kind of curriculum development, one that engages students with representations of practice. We propose that for experiences with shape and space to become objects of geometric practice in which students may produce knowledge compatible with what is typically aimed for in the study of geometry, students needs to engage in work that problematizes those experiences, making them subject of inquiry. Ontological, epistemological, and axiological transitions can be promoted through multimodal argumentation via a a semiotic infrastructure of practice.

Consider as an example the theorem that characterizes an angle bisector as the locus of the points equidistant from the sides of an angle. Earlier in their studies, students have likely constructed angle bisectors using a protractor. How can that experience be turned into an object of reflection and argument in which the theorem listed above can be conjectured and become a useful control? Arguably, this requires appropriating the goal to describe the drawing as a set of points with some properties. Figure 3 shows a storyboard that can support discussions promoting students' conjecturing of the theorem and their use to engage in arguments in which points are singled out and described in terms of being or not being on the angle bisector as well as in terms of their distance to the sides.

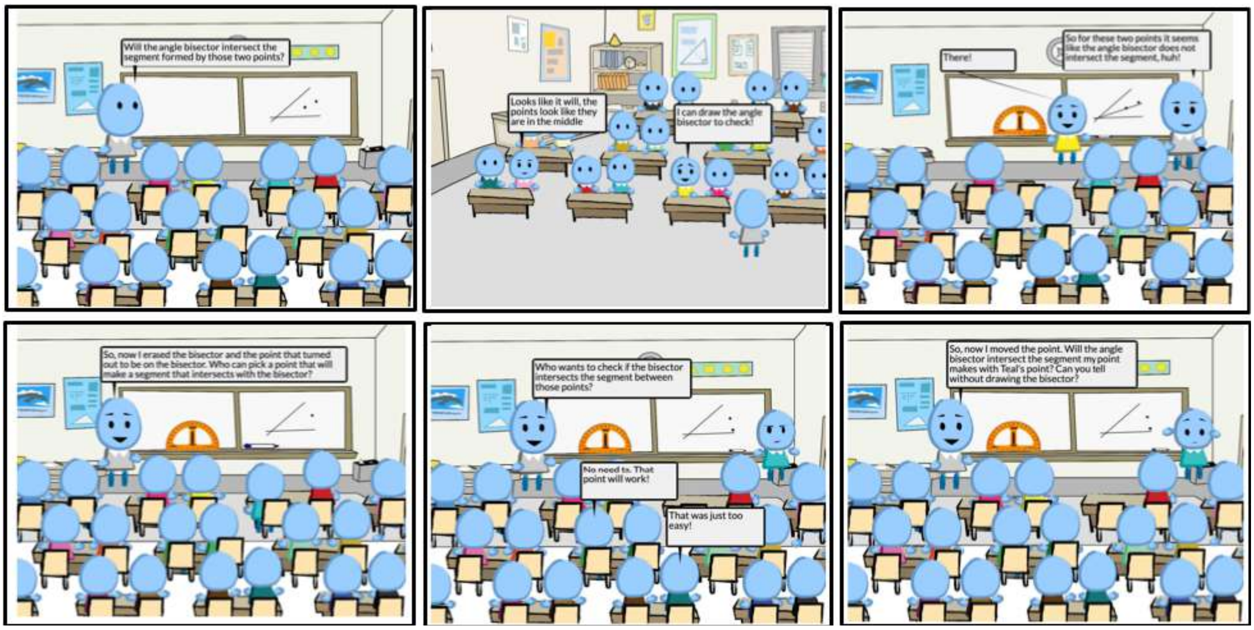

Figure 3. Toward seeing the angle bisector as made of points with particular properties.

Consider high school students being asked to annotate this storyboard, including answering questions such as "What is the point of posing these questions at the board as opposed to giving them to each student on a worksheet?", "How is the teacher choosing which points to erase and to keep in Frames 4 and 6?", "Would there be a way to know whether a segment determined by two points will cross the angle bisector without drawing the angle bisector?", "What could you do to find a point that would turn out to be on the angle bisector?," or, "If a segment is such that you think it would intersect the angle bisector, could you find out where they would intersect before drawing the angle bisector?" These questions would likely get students to make a conjecture along the lines that for a segment to intersect the bisector, one of its extremes would need to be closer to one side than the other and the other extreme closer to the other side and then that for a point to be on the angle bisector, it should be at the same distance from both sides of the angle. The task itself could be used to promote the conjecture as a theorem-in-action; it might be sensible to start the lesson by having students try themselves the question in Frame 1.

Offering the storyboard and posing questions that involve the interpretation of the storyboard may help initiate the above-named transitions. The storyboard instantiates a semiotic infrastructure (icons, indices, and symbols organized in a multimodal grammar; see Herbst et al., 2023) that supports talking about points as both actual and potential depending on events in the storyline (e.g., between Frames 4 and 5) and of constructions as anticipated and performed (e.g., between Frames 1 and 3). In the context of a discussion of that semiotic environment (e.g., the narration or critique of the story), there is a possible transition in the ontological status of points: Points are no longer only specific objects in the diagram, but also can be described (i.e., modeled) as possible contingencies among properties (i.e., points can be characterized as closer or farther from one or the other side). Further, the storyboard supports an epistemological transition away from drawing to see whether something is the case to anticipating what could or would happen if one were to draw. The storyboard medium provides also access to the storyline not only as a vicarious experience but also as a replayable experience that can support an applied rationalist (as opposed to empirical) construction of reality along the lines of Bachelard's "Le réel n'est jamais ce qu'on pourrait croire, mais il est toujours ce qu'on aurait dû penser."[4] The surveyability of experiences with shape and space that a storyboard provides allows for someone who would initially only know from observing what happened to one who might come to realize they should have anticipated the outcome. Finally, and perhaps this is the most important aspect of the storyboard representation of geometric experience, this semiotic mediation supports decentering knowledge from the function of serving a selfish goal (e.g., solving one's problems) to solving problems that anyone could have, without eliding the human need for knowing. Thus, while the reader of the storyboard is not herself on the spot in Frame 6, they might feel empathy with the student at the board that could compel them to offer a characterization of the points as a solution to a human need. The ethical option of helping the student at the board may further support the aesthetical value of obtaining a good definition of the distance between a point and a line.

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ADVANCES IN GEOMETRY EDUCATION
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PROOFS IN GEOMETRY TEACHING IN THE BRAZILIAN CONTEXT: A REPRESENTATION OF YESTERDAY AND TODAY

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The topic regarding advances in geometry teaching leads us to reflect on the past of this teaching, mainly about changes in curricular proposals and how they were reflected in textbooks, aiming to identify progress or setbacks. This study focuses on investigating, from a cultural historical perspective, how the teaching of geometric proofs has been proposed and discussed in Brazilian regulations, as well as in textbooks. We analyzed three educational reforms from the first half of the 20th century, the 1998 National Curricular Parameters and the 2018 National Common Curricular Base, and we examined four textbooks representing different moments of changing approaches to teaching geometry, from 1930 to the present day. One result throughout the various regulations and pedagogical approaches found in textbooks is the persistent tension between intuitive and deductive geometry. The importance of the transition between pragmatic proofs and conceptual proofs in geometry teaching is a topic that remains central and relevant in the Brazilian reality.

INTRODUCTION

The call for ICMI 26 (Advances in Geometry Education) encourages us to think about the past of geometry teaching, changes in pedagogical proposals and how they were translated into didactic-pedagogical tasks in textbooks, to identify progress and setbacks. It is in this context that the present work is inserted, by choosing the theme of proofs in geometry teaching and analyzing, from a cultural historical perspective (Chartier, 2009), how this theme has been proposed and discussed in official curricular regulations in Brazil, as well as in textbooks. Thus, we took a period of almost one hundred years, from 1930 to the present day, beginning our analysis at the time of the unification of the fields of Algebra, Arithmetic and Geometry into a single school subject called Mathematics for secondary school programs (aimed at students aged 11 to 14, after primary school). The research sources were the regulations of three educational reforms from the first half of the 20th century (1931, 1942 and 1951), the National Curricular Parameters (PCN) at the end of the same century, and the last regulation, approved in 2018, the Base National Common Curricular (BNCC). We also selected four textbooks as representative of important moments in changing perspectives, the books by Osvaldo Sangiorgi from the 1950s and 1960s, to examine the approach of the Modern Mathematics Movement and the books by Edwaldo Bianchini from the 2010s and 2020s, to investigate possible changes between the last two regulations, the PCN (Brasil, 1998) and the BNCC (Brasil, 2018).

It is necessary to highlight that the study considers and examines national regulations, however Brazil is a country of continental dimensions, with 27 states distributed in 5 regions, with important social, economic, cultural, and educational differences, each state having the autonomy to organize its school programs under national regulations. Thus, the choice of textbooks, which will be explained below, represents a sample that we consider significant, but without any intention of representing the entire

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 75-82). ICMI.

national reality. The focus of the investigation is not on an in-depth study of specific moments, but rather on analyzing long-term processes with the necessary methodological care, to support the historical narrative of the issue at hand. The objective is to analyze how the proofs were and are proposed in this selection to discuss the different conceptions of proofs for teaching geometry to students aged 11 to 14, in the period.

Educational Reforms in Brazil in the first half of the 20th century

The reflections of the international movement to reform mathematics teaching, discussed at the IV International Mathematics Congress, held in Rome in 1908, under the leadership of Félix Klein (1849-1925), were appropriated in Brazilian regulations only in the 1930s. Euclides Roxo (1890-1950), engineer and teacher at Colégio Pedro II (a reference in Brazilian secondary education) was the leader of the renovation in the Francisco Campos Reform in 1931, which instituted, for the first time, the subject Mathematics for the secondary course.

According to Roxo, this reform brought together the trends of the international movement, both from a methodological point of view, as in the contents, by merging the different fields of mathematics: arithmetic, algebra, and geometry into a single discipline. The instructions for teaching mathematics recommended that:

> Starting from the living and concrete intuition, the logical feature will grow, little by little, until it gradually reaches formal exposition; [...] knowledge will be acquired, at first through experimentation and sensory perception and then, slowly, through analytical reasoning. Thus, regarding geometry, the formal demonstrative study must be preceded by a propaedeutic course, aimed at intuitive teaching, of an experimental and constructive nature (Bicudo, 1942, p. 157).

The detailed analysis of the regulations emphasizes the importance of intuitive and experimental teaching preceding geometric proofs. The 1931 Reform lasted a short time and was reorganized in 1942, by the Capanema Reform. In 1951, the Simões Filho Reform kept the 1942 structure. For geometry teaching, the three reforms designated deductive geometry, that is, the introduction to the study of geometry proofs for the final grades (students aged 13 to 14) as we can see in Table 1.

Reform	Geometry organization	Instructions
Campos (1931)	*Geometric Initiation* *Geometry*	- Geometry propaedeutic course - Starting from intuition to **gradually reach** formal exposure: from experimentation and sensory perception to analytical reasoning
Capanema (1942)	*Intuitive Geometry* *Deductive Geometry*	- Intuitive geometry as a **smooth transaction** between experiences with shapes and deductive conception of Geometry
Simões Filho (1951)	*Geometry* (Only to students 13 and 14 years old)	- Do not ignore appeal to intuition - **Gradually awaken** the feeling of the need for justification and proof

Table 1: Geometry Teaching in prescribed programs (1931-1951). Source: The authors based on the regulations.

The three reforms emphasized the need for an intuitive geometry, with experimentation, and sensory perception, as preparation for understanding a deductive geometry, but the 1951 reform excluded the space allocated to introductory and intuitive part of previous reforms, maintaining only the deductive approach. Chervel (1990) warns us about the difficulties in distinguishing objective finalities and real finalities and, in this sense, the study of regulations indicates the objective finalities of teaching, however, pedagogical practices are those that reveal the real finalities, and they do not always converge. For instance, in the 1950s, despite the regulations signaling attention for intuitive study and gradually awakening the need for justifications, the reports on practices in classrooms that took place at the II National Congress of Mathematics Teaching, in 1957, constitute traces of what the approach to deductive geometry was like, associated with memorizing the proofs of theorems:

> The lack of initial logical concatenation of the theorems and the intuitive character of most of them produces the harmful impression, in the student's mind, that the proofs constitute juggling on the part of the teacher. […] It also does not form a clear idea of what a theory is or does not have a theory at all. [...] The ill-fated theorems have been the lifeline of mediocre students who, in exams, achieve the minimum grade thanks to a memorized formal proofs (Congresso …, 1959 *apud* Búrigo, 2015, p. 8).

Despite the attempt to insert the principles of the international movement of the beginning of the 20th century into Brazilian regulations, the innovations suffered a lot of resistance from a traditional school culture (Julia, 2001), based on the rigor and abstraction that was expected from a study of deductive geometry. We reached the middle of the 20th century with the teaching of deductive geometry characterized by the exposition of a set of postulates, followed by numerous formal proofs of a significant set of theorems, to be memorized and taken in exams, as we will see below.

Textbooks by Osvaldo Sangiorgi in the 1950s and 1960s

Osvaldo Sangiorgi (1921-2017), mathematician and teacher, author of best-selling textbooks in several editions since the 1950s, was a relevant name in the process of appropriation and dissemination of the ideas of the Modern Mathematics Movement (MMM) in Brazil. He did an internship at the University of Kansas, in the USA, in the mid-60s and when he returned to Brazil, even without changing the regulations, he created a new collection of textbooks with modern proposals and expressive editions.

The study of Jahn and Leme da Silva (2023) compared the first textbooks by Osvaldo Sangiorgi (OS) that introduced students to deductive geometry (13-year-old students) in the 1950s, considered pre-modern and, in 1960, modern. An initial quantitative analysis indicated that practically the number of pages dedicated to the study of geometry remained stable, however the approach between the two textbooks proved to be completely different. When comparing the topics related to the teaching of geometry - in common in the two textbooks - it was identified that the same concepts started to occupy 28% more space in the modern collection. On the other hand, the number of theorems proved in the two textbooks went from 54 to 28, that is, a reduction of around 50% in the number of theorems. For the study of proofs, the same have indicated changes in the selection of theorems and in the approach to presenting them. The qualitative analysis made it possible to understand the numerical differences: 120 preparatory pages were inserted before starting the presentation of postulates and theorems, and the author explains the change:

> At this stage, the student is already "accustomed" to seeing that some properties are consequences of other more elementary ones. It is the preparation for the beginning of a simple axiomatization, where the situations encountered in the **exploratory exercises** will be brought together in the form of postulates (axioms). A theory (recommended by Hilbert) is thus constructed from primitive concepts, postulates, and theorems, easily proved, through a logical chain of reasoning. [...] The current treatment used to begin deductive geometry is very different from what was traditionally done. **Only after much experience** can the student be initiated into the **proofs** themselves (Sangiorgi, 1966, p. 51, emphasis added).

OS clarifies that the exploratory exercises were introduced so that the student can experiment with verifying some properties as a result of others, an attempt to insert preparatory experimental geometry before the deductive study. The author also indicates the construction of a new axiomatization for proofs in which the exploratory exercises carried out by students become postulates when proving the theorems. Everything indicates that the main point is the understanding of a deductive process, which incorporates “new postulates”, to allow a more adequate understanding of logical-deductive reasoning.

An example of exploratory exercises proposed by OS in the 1960 textbook were the geometric constructions of triangles involving cases of congruence, in which students were invited to carry out the constructions of triangles with the respective data, for example, the case side, angle and side (LAL) and then compare their triangle with their classmates in order to identify congruence. In the 1950 textbook, the four cases of triangle congruences were stated as theorems and proved; while in the 1960 textbook, after the exploratory study carried out with geometric constructions, they were then taken as postulates. The list of postulates between the two textbooks was significantly altered, both in quantity and choice (Jahn & Leme da Silva, 2023). Another example that helps us understand the change in the proposal made by OS is the way the theorems are presented. The theorem on the property of the base angles of an isosceles triangle, in the 1950 textbook, was done in a classic way: the theorem is stated, with the figure on the side, the hypothesis and thesis are identified, and the proof is carried out. In the modern approach, students were first invited to carry out exploratory exercises, which correspond to constructing an isosceles triangle, measuring the angles, comparing with their colleagues whether the result obtained was also verified by the group and stating the identified property. Other exercises complement this stage, such as attention tests, which ask students to identify the corresponding vertices in different triangle congruence situations. OS presents three distinct proofs: the first as an “action plan”, which corresponds to drawing the bisector of the angle between the congruent sides; the second, as “another way” for the proof, tracing the height relative to the base of the isosceles triangle and the third, through “drawn diagrams”, colored preferably, where a series of deductions appear, through constructions (→), of equivalences (⇔) and implications (⇒). The emphasis of different resolutions and different representations to proof the same theorem, so that the student can understand the diversity of paths in the deductive process, is an innovative aspect in Brazilian textbooks from the 1960s. OS encourages students to develop their own proofs, warning: “Don't 'memorize' proof of theorem! Value yourself, using any of the methods presented. Do it your way by using these methods and you will be accomplishing in Mathematics!” (Sangiorgi, 1967, p. 258).

It is necessary to consider that the appropriation made by OS for the study of modern deductive geometry must have been influenced by the North American trend based on Birkhoff; however, other trends were present in Brazilian textbooks, such as the proposal to teach geometry through geometric transformations linked to Klein. Anyhow, OS's modern approach to deductive geometry significantly changed the representation of geometry teaching in the 1950s of a formal deductive geometry, of memorization, both within the scope of Euclidean geometry and in the methodological didactic aspects, by proposing the insertion of exploratory exercises, in addition to different records of representations as a way of make understandable the deductive geometry.

In summary, in the period examined, from 1930 to 1960, it can be said that deductive geometry was present, both in Brazilian regulations and in textbooks, from the perspective of *conceptual proofs*[5]. However, the moment described above, of the incorporation of modernist ideals by OS, seems representative to us, which, even without abandoning the conceptual proof, the exposition of the postulates and the many theorems to 13-year-old students, innovates in the textbooks, through a differentiated proposal to prepare the study of deductive geometry, by inserting *pragmatic proofs*, a call to the student to build their own proofs, to argue, to justify. Everything indicates that the insertion of this new approach was not incorporated into school culture, however, it seems to have been the moment to decrease the formal approach, that of many theorems to be memorized.

Educational reforms in Brazil at the end of the 20th century and beginning of the 21st century

From the 1970s onwards, criticism of the MMM became increasingly accentuated and, in parallel, a movement of reflection on mathematics teaching began, bringing together study groups, teachers, mathematicians, psychologists, educators, without, however, result in new curricular programs at a national level. At the end of the 20th century, and with the presence of mathematics educators participating in the development of curricular programs, the National Curricular Parameters (PCN) were implemented. Recently, in 2018, with controversy and criticism from educators, the National Common Curricular Base (BNCC) was implemented, which we began to examine comparatively, on how deductive geometry has been proposed.

The PCN (Brasil, 1998) continue to advise that the study of deductive geometry should be started for students aged 13 and emphasize that the skills of arguing and proving in Mathematics are very important for the formation of a critical spirit:

> Therefore, it is desirable to work on developing argumentation, so that students are not satisfied with just producing responses to statements but **have the attitude of always trying to justify them**. [...] Geometry activities are very suitable for the teacher to build, with his students, a path that, **based on concrete experiences, leads them to understand the importance and need for proof** to legitimize the hypotheses raised (Brasil, 1998, p. 71 and 126, emphasis added).

The document explains the relevance of tasks that favor the path from concrete experiences to the need for evidence, in this case, in the sense of pragmatic evidence. The formal proofs are not explained in the regulations. It is understood that the focus must be on arguing, justifying, and proving with usual language.

Twenty years later, the BNCC (Brasil, 2018) removes any discussion about proofs processes, the need for proof and the transition between intuitive and experimental geometry to deductive geometry. The regulations emphasize tasks involving concepts of congruence and similarity so that students "know how to apply this concept to perform simple proofs, contributing to the formation of an important type of reasoning for mathematics, hypothetical-deductive reasoning" (Brasil, 2018, p 272, emphasis added). However, we do not discuss or comment on what is meant by simple proofs. In this way, we can infer that the BNCC, compared to the PCN, presents a setback for the study of proofs,

[5] We mobilize the concepts of pragmatic proofs as proofs that include resources of action and empirical observation on mathematical objects, aiming to verify or prove the mathematical property and conceptual or intellectual proofs are based on the formulation of relevant properties and connections are established between them in accordance with Balacheff (1988).

which ceases to occupy a prominent place and becomes specific applications for the study of congruences and similarities of triangles.

Textbooks by Edwaldo Bianchini in the 2010s and 2020s

Edwaldo Bianchini (1935-2018), science graduate, mathematics teacher and textbook author since the end of the 20th century. Bianchini's collection was selected for its positive emphasis on the approach to geometry at PNLD[6] 2011, which proposes the study of geometry in a rigorous and extensive manner, with frequent formal proofs. We seek to comparatively analyze the introduction of deductive geometry in two textbooks by Edwaldo Bianchini (EB), aimed at 13-year-old students, one from the 2010s (book 1), in which the PCN were in force, and the other, from the 2020s (book 2), shortly after the implementation of the BNCC.

An initial quantitative analysis showed that the number of pages dedicated to teaching geometry decreased, from 46% of the total in book 1 to around 27% in book 2. However, the same does not occur when comparing the common themes that deal with of deductive geometry, in which the number of pages in both textbooks was practically the same, with a very similar approach. For example, regarding postulates, the same list appears in both volumes, with no mention of students' familiarity or prior experience with such propositions. This choice may represent the stability of the Euclidean Geometry model at this level of education.

Regarding the theorems proved, in both textbooks, EB begins with a topic on geometric proofs, in which he defines what a theorem is, but he no longer uses this term in the sequence, adopting the designation of “property” as a synonym. The number of properties proved in each textbook remains the same, with 16 proofs. In both textbooks, the author explains the need to prove without resorting to measurements or observations on the figure: “measurement or simple observation are not always sufficient to confirm whether a geometric property is true or false. As evident as it may seem, it can only be considered true after being proven” (Bianchini, 2015, p. 156). Complementarily, in book 2, the author drew attention to the fact that “Verification is not proof” (Bianchini, 2022, p. 181), referring to verification through measurements with instruments (ruler and protractor). We can interpret it as a way for EB to differentiate pragmatic proofs from conceptual proofs and to highlight the need for mathematical proofs, that is, validation using a hypothetical-deductive method. On the other hand, in both textbooks, pragmatic proofs are not explicitly identified in the exercises proposed to students. For example, in the two volumes, after the introduction of the concept of congruence of triangles, the cases of congruence were presented in an expository way, stated, accepted as true without proof and exemplified. In other words, such cases are not explored, discovered, or verified experimentally by students, they must be accepted as truths, and are used in identification exercises as an assumed result, similar to applying a formula. Only when he properly begins the proofs of the properties, EB states that: "In the following situations, we will consider that cases of triangle congruence are already proved truths[7]" (Bianchini, 2015, p. 158; 2022, p. 180). We also note that in the book 1 Teacher's Manual there are no suggestions for exploratory tasks or experimental proposals

[6] From the 1990s onwards, Brazilian textbooks started to be periodically evaluated by a team of experts designated by the Brazilian Ministry of Education, in the National Textbook Program (PNLD) and distributed at no charge to public schools.

[7] This choice was not justified, EB bluntly states that: “we chose not to present the proofs of congruences of triangles cases as we consider them didactically inappropriate for this moment” (Bianchini, 2105, p. 158).

that could represent pragmatic tests, as was done in the book 2 Teacher's Manual, which contains suggestions for students to represent triangles with a ruler and protractor, cut out these triangles and compare them with the productions of other colleagues, in order to verify that all the triangles produced are congruent (Bianchini, 2022, p. 166).

The theorem on congruence of the angles of the base of an isosceles triangle, in both textbooks, was made through the classic presentation: statement, identification of hypothesis and thesis, an illustrative figure and the proof steps for which the student is called to "to follow". In this sense, we can recognize the same approach as the pre-modern OS textbook. Furthermore, EB exposes only one way of proving properties, both in its deductive path and in its way of representing, which can reiterate the idea of a single solution to be memorized. This way, students, in general, are not called upon to reflect or carry out any deduction steps, much less to try to produce their own proof.

In summary, the examination of the textbooks revealed that the tone of the proposal reinforces the conclusion of Martins and Mandarino (2013): in most textbooks, formal proofs (or *conceptual proofs*) are presented ready and finished, not giving space for the student to develop their own conjectures and conclusions. The exercises, in most cases, approach the content under analysis with a procedural focus, "where calculations and formulas are the basis of the questions with no apparent intention of developing the spirit of argumentation in the student" (Martins & Mandarino, 2013, p. 112).

CONCLUSIONS

The objective of this study, in creating a long-term retrospective is to construct a broad trajectory on proofs in geometry, emphasizing significant historical moments and seeking to identify tensions, ruptures, and continuities. We reinforce it is important not to generalize the notes obtained in the analytical exercise but rather to discuss the complexity of the teaching of proofs in geometry.

The tension between intuitive and deductive geometry in geometry teaching is evident in the regulations and pedagogical proposals of textbooks. Curriculum programs state the importance of intuitive approaches for students aged 11 to 14 but also highlight the need to introduce deductive practices from the age of 13. Striking a balance between these approaches is challenging, requiring strategies for the transition between pragmatic and conceptual proofs. The reforms of 1931 and 1942, the modern ideas of the MMM appropriated by Sangiorgi, along with the PCN, constitute convergences in valuing intuitive geometry. They also indicate pertinent perspectives for the current moment. The results reinforce the relevance of spaces such as the 26th ICMI to internationally debate advances in geometry teaching and collectively discuss indicators of continuity from different realities and cultures in the search for balance. On the other hand, we identified moments of ruptures and setbacks, such as the 1951 reform, which removed the space allocated to the study of intuitive geometry, as well as the current BNCC, which in normative terms, retreats by giving little emphasis to the transition between tense approaches – intuitive and deductive. However, according to Chervel (1990), objective finalities do not always follow teaching finalities; the production of OS is an example of the insertion of new approaches, even if they are not explained in the regulations. Regarding EB production, few differences were identified, despite regulations indicating detachment.

The study invites us to reflect on how the past can teach us by examining old proposals that evoke current issues, such as the expressions "intuitive geometry as a smooth transition between experiences and deductive conceptions" from 1942, or the OS's option in 1966 to postulate cases of triangle congruence after experimental conjectures. All these debates precede the carefully developed study by Balacheff, who, since the 1980s, highlighted the complexity of teaching mathematical proof. In summary, it is crucial to reinforce the importance of the transition between pragmatic proofs and conceptual proofs in geometry teaching, at least in Brazilian reality.

The study also allows for identifying connections, interactions, and appropriations of international movements (particularly Klein and Birkoff) by Brazilian educators. However, such appropriations were not fully integrated into the school culture, and what becomes relevant is the complexity of changing pedagogical practices, especially, breaking with the tradition of conceptual proofs, balancing them appropriately with pragmatic proofs enables a mathematical practice supported by perspectives from the field of Mathematics Education. We can conjecture that factors such as teacher training, cultural differences, and educational policies may not contribute as significantly to advances in geometry teaching as desired.

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A DYNAMIC AND SPATIAL APPROACH TO ENRICH THE TEACHING AND LEARNING OF GEOMETRY IN PRIMARY SCHOOL

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We propose a more dynamic and spatial way of teaching and learning geometry in primary school. Based on the NCTM's "Essential Understanding Series", which sets out four main ideas and ten essential understandings for the teaching of geometry and the report on recent research in this field (see Sinclair et al., 2012), we propose strategies to enrich classical geometry tasks. This contribution is based on research showing the central role of spatial reasoning in learning geometry (Battista, 2020) as well as the need to further value the dynamic aspects of geometric objects (Sinclair et al., 2016). Our reflection on new ways of implementing a geometry lesson is anchored in the theory of grounded cognition (Barsalou, 2008) and more specifically on the "Activity Generating Structure" (Marchand, 2020) which takes into account, among other things, the model of Uttal et al. (2013) which highlights the static/dynamic duality at play in the development of spatial reasoning. Our study exemplifies how this framework can lead to more dynamic and spatial experiences of geometry.

INTRODUCTION

The teaching of geometry has always been a part of the mathematic elementary curriculum, but its importance may vary according to the different reforms in recent decades. In Canada, geometry has received little attention in the curriculum and in classroom teaching (e.g., Bruce et al., 2012). However, there has been a recent push for spatialized the curriculum (Hawes et al., 2023), which is part of the more general trend in research focusing on "advances in the understandings of visuospatial reasoning", as identified in Sinclair et al. (2016). This push is based on growing research that shows that spatial reasoning[8] (SR) is correlated with mathematical success and that it is malleable (see Davis et al., 2015; Lowrie et al., 2017; Mulligan et al., 2020). Our work fits within this spatializing trend but focusses specifically on geometry. Although the role of spatial reasoning should be an obvious part of geometry, if teachers focus only on naming shapes and measuring them, little spatial reasoning is actually needed. We are particularly interested in dynamic forms of spatial reasoning, given their importance in mathematics, and the availability of digital technologies to support dynamic imagery and interactions. In our work, we have been focussed on how to transform static and quantitative geometric activities and tasks into dynamic and spatial ones.

The role of the SR is central to develop geometric reasoning (GR), but it is also distinct yet inextricable from it. Behind the understanding of a geometric object is also a spatial understanding of the object in question. Students must create a mental image of the geometry object, which they may do by simulating the experience of seeing and/or manipulating an object (Barsalou, 2020), connect this image with their geometric understanding and transform them according to the situation

[8] Globally, SR referred to "the reasoning [...] that helps each individual better anticipate the effects of their actions on space, to control these effects, and to communicate spatial information" (Berthelot & Salin, 1992, p. 9, freely translated).

to conceptualize them. SR can represent not only a gateway to the study of geometry, but it constitutes also a powerful cognitive tool for the study of formal geometry (Moss et al., 2016; Clements & Battista, 1992). For example, it is used to anticipate the figure-image of a rotation or establish the relevant properties to find a missing measure of a figure from geometric statements. Research conducted in Italy by Giofrè et al. (2013) cited in Sinclair et al. (2016) shows that SR is not only central, but it is essential for geometric learning: "According to these researchers, the importance of visuospatial working memory is critical in learning geometry" (p. 697). Our research is rooted in this work and has as its starting point the conclusion that Sinclair et al. (2016) had following their review of the research conducted so far and recommendations for future research to be carried out:

> Since geometry is a school mathematics topic that explicitly engages visuo-spatial reasoning, several authors have argued that an increased focus on geometry in the curriculum would be well advised (Sinclair & Bruce 2015). In particular, given the importance of dynamic forms of spatial reasoning, highlighted in the work of Newcombe and others, and the relevance to motion and time across ages and cultures (humans live in a world that is constantly moving, changing), there is reason to conjecture that developing dynamic and haptic forms of visuospatial reasoning would have a positive impact on children's learning. There are interesting and long-standing mathematical challenges in moving away from paper-based static representations, taking inspiration perhaps from non-Western geometric ways of making sense of the world, but as shown in the Sect. 5, digital technologies can provide alternatives to such representations that offer learners more opportunities to create and reason with dynamic imagery. (pp. 698-699)

The research question asked in this proposal is: How can we conduct a classic geometry activity in class to enhance a more spatial and dynamic approach? To respond to this question, we present the frame of reference on which we based our analysis and examples of experiments we have carried out.

FRAMEWORK

In recent years, frameworks for studying SR have been developed in different fields of research (psychology, cognition, didactics, etc.). The framework of this project fits globally into embodied approaches that ground cognition in the body and the environment (see Barsalou, 2020) and more specifically on the exploitation of a dynamic approach focused on the development of SR to enrich the teaching and learning of geometry. Grounded cognition guides several choices in the design and analysis of the activities proposed in classrooms for both the mathematical teaching components. This framework is not based on the study of information processing, but rather on the study of the sensory experiences of learners, including their actions and perceptions. In grounded cognition, all learning is based on a perceptual and physical interaction of the student within their affective, material and social environment. Actions and perception shape and are shaped by sociocultural forces such as language, tools and values (de Freitas & Sinclair, 2013).

And more specifically, the activity generating structure (AGS) guides us in our choices to enhance the development of SR in the proposed activities (Marchand, 2020). This framework provides different benchmarks for generating and analyzing SR development in the classroom. Thus, the benchmarks chosen for this project will be detailed in the methodology since they constitute the analysis variables of our *Design-based research*. The Uttal et al. (2013) model offers a typology of SR according to four categories, as in Figure 1 (the hammer could be any object, such as a cube):

	Intrinsic	**Extrinsic**
Static	Relationships between parts of an object; static geometric characteristics. 	Coding the location of objects relative to other objects or a reference frame. 
Dynamic	Folding, bending, rotating, scaling, cross-sectioning, comparing 2-d and 3-d views. 	How one's perception of the relations among objects changes as one moves through the environment. 

Figure 1. A 2x2 classification of spatial skills (adapted from Uttal et al., 2013, p. 354)

The development of SR is based as much on activities involving objects that remain **static** as on activities involving **dynamic** objects by moving or transforming them or by observing them from different perspectives. As mentioned above, the geometric activities usually offered to students are static. This absence of dynamic consideration in activities is also rooted in a number of education programs, including in the mathematic education programs of Québec (Ministère de l'Éducation, du Loisir et du Sport, 2003). Dynamic and spatial approaches to teaching and learning geometry at school are rare, despite the fact that psychology research has for many years shown a correlation between, for example, the development of SR related to mental object rotations, which falls under the dynamic category, and student performance in elementary and high school (Hawes et al., 2015).

The other angle used to characterize SR is the type of spatial relationships at play. These relationships can be between a whole and one or more of its parts (intrinsic) or between different wholes (extrinsic). The emphasis for us here is to consider the management of these relationships as a central element in the design and implementation of geometric activities in the classroom.

By articulating this framework with our overall approach based on grounded cognition theory, it is possible to highlight another element of Marchand (2020). For each of these four categories, the three type of action will necessarily interact with each another. The development of SR and GR is an iterative process between action (physical and mental) on the one hand, and reflection and abstraction on the other (Battista, 2007). In the activities that we propose, students will be in action or be called upon to manipulate physical or virtual objects; they will therefore be in what we have called an archaeological level of action/reflection (Marchand, 2020): they will have to move, observe, manipulate and deduce spatial and geometric regularities, relationships or properties. This action must be accompanied by spatial and geometric mental work. To ensure that students can have access to this internalization, the activities we propose must create this bridge. It is therefore necessary to value activities that require students to create mental images, photographic level of action/reflection (Marchand, 2020) of the manipulated objects. A good example of an activity that enhances this bridge is the Tangram activity of Yackel and Wheatley (1990) in which students are asked to observe an image composed of Tangram pieces for three seconds, and then reproduce the image in his absence. Students can observe the image two or three times; emphasis is placed on the spatial and geometric strategies they use to reproduce it (What do you see in your head? How do you take into account the position of the figures and their articulation? What were your actions between each observation?). This activity is located in the static/intrinsic category and it allows students to connect, several times through a back and forth between action and reflection (observation, construction and verbalization), through archaeological and photographic tasks. For both dynamic categories, it will be necessary to extend the action/reflection to tasks that stage the geometric objects studied in motion (by moving, decomposing, transforming, folding them, etc.) physically and mentally. This requires a dynamic and articulated vision of the objects in play (the scenographic level of action/reflection). This study falls into both dynamic categories and therefore we will focus on these aspects henceforth.

METHODOLOGY

Broadly speaking, we draw on the *Design-based Research* methodology for the lesson design (Anderson & Shattuck, 2012). However, in this paper, we present already-designed lessons and analyze them based on the TWO DYNAMIC CATEGORIES OF ACTIVITIES PROPOSED BY UTTAL ET AL. (2013) AND THE THREE TYPES OF ACTION/REFLECTION PROPOSED BY MARCHAND (2020). WE CONSIDERED THREE VARIABLES THAT APPEAR TO BE CRUCIAL, GIVEN THAT THEY ARE RECURRING IN OUR WORK AND THAT OF OTHER RESEARCHERS. WE CHOSE THE TWO LESSONS ANALYZED IN THIS PAPER BOTH BECAUSE THEIR STATIC VERSIONS WILL BE FAMILIAR TO MOST MATHEMATICS EDUCATIONS RESEARCHERS AND BECAUSE THEY HAVE BEEN USED NUMEROUS TIMES IN OUR RESEARCH.

The **first** variable is the conception of space (micro, meso and macro) (Berthelot & Salin, 1992; Brousseau, 1983). Briefly, micro-space is the space of small objects that can be manipulated, or the space of a sheet of paper. This is the space usually exploited in mathematics classrooms. Meso-space is the space encompassing the movements of a student or object and is fully accessible via the student's field of view. The Tangram activity mentioned before was in the micro-space, but it can be carried out in the meso-space using figures constructed from sheets.

Figure 2. Example of a Tangram constructed from sheets

Macro-space is too large for a student "to be able to take in at once" (Brousseau, 2001, p. 6, our translation). Only a local, partial view of the space is possible. The connection between micro- and meso-space seems to be a promising avenue for conceptualizing both SR (Braconne-Michoux & Marchand, 2021; Brousseau, 1983) and GR and this variable is consistent with grounded cognition theory. The **second** variable is the nature and the chronology of tasks that can be given to students. Davis et al. (2015) HAVE IDENTIFIED 29 POSSIBLE TASKS (EX.: OBSERVE OR TOUCH, COORDINATE PERSPECTIVES, MOVE, BREAK DOWN, FOLD OR REORGANIZE). One of the main findings of our previous research is that the chronology of tasks is a decisive factor in the level of difficulty of the activity. Specifically, the following two chronologies seem promising to enrich the teaching/learning of geometry from a dynamic and spatial point of view: 1) anticipate the sufficient and necessary materials required before proceeding to any construction, build, and then describe the solid obtained (Marchand, 2006, 2009) and 2) observes to anticipate geometric properties and spatial relations to be retained for construction, construct without reference to the object to be made and describe the Tangram (Yackel & Wheatley, 1990). In these two cases, the anticipation task is carried out differently, but always takes place before the manipulation or resolution (Michoux-Braconne & Marchand, 2021). Finally, the **third** variable is the nature and dimension of the objects in play, which can be 0D (points), 1D (lines), 2D (figures) or 3D (solids) and can be prototypical or non-prototypical and positioned in a typical or non-typical orientation. This variable plays a key role in geometric conceptualization. The fact that we are in a dynamic approach necessarily implies that students will be confronted with geometric objects in motion and therefore in non-typical positions and spatial relationships.

A FIRST STEP TOWARDS A DYNAMIC AND SPATIAL APPROACH TO ENRICH THE TEACHING AND LEARNING OF GEOMETRY IN PRIMARY SCHOOL

We explore this avenue through the analysis of two classical activities: the reproduction of an image formed by geometric figures (as in the Tangram activity) and the representation of the possible developments of a solid, such as those of the cube.

The Tangram activity discussed earlier is a classic in the SR literature and involves students identifying and manipulating 2D shapes[9] to create certain designs. Changing the orientation and chirality of the shapes allows students to experience transformation directly, which might help them mentally rotate these shapes. Geometrically, the main pedagogical benefits might be (1) to identify the names of shapes, such as triangles, squares and parallelograms) and (2) to decompose and recompose shapes in such a way that preserves area. Given the research showing that students tend to develop prototypical images of shapes (squares that are sitting on their sides, triangles that are equilateral, etc.), and the role that dynamic geometry can play in which students experience a wider variety of these target shapes, as well as a sense of their properties (Sinclair & Yurita, 2008), this sequence of DGE tasks[10] was designed to support both SR and GR.

In this activity, students are asked to use a set of arbitrary triangles (triangles that can be dragged into any shape) and fit them into a given shape (such as a square, a pentagon, a start, etc.) (see Figure 2a). Instead of manipulating a fixed figure, they are producing triangles of various shapes and sizes. While we often see the flat figures statically at school, here they are dynamic as much in relation to the intrinsic spatial relationships, between the triangles and the reference figure (square, pentagon, star), as extrinsic between each of the triangles that make up this figure since they can take any shape and they must fill, without leaving a hole, the reference figure (see Figure 2b). Students use archaeological, photographic and scenographic actions/reflections here. They transform the triangles, they position them in the reference figure (archaeologic), anticipate how to position it in relation to the reference figure (photographic) and in relation to the next triangle to be placed (scenographic). The main tasks required are deformation, anticipation, decomposition and recomposition. The activity takes place in micro-space and students drag the vertices (0D), segments (1D) and figures (2D) to fill in the reference figure (2D). Finally, since students will necessarily deform the triangles to fill the figure, their positions and appearance will not be typical. A conceptualization of the figures involved (triangle, square, pentagon) and the notion of area will thus be enriched.

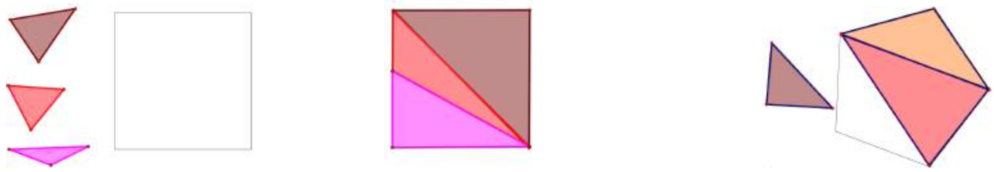

Figure 3a: Triangles to fill the square; b: one solution; c: fitting the last triangle into a pentagon

[9] They are meant to be treated as figures, as they are called squares and triangles, and so on, but are in fact solids, which introduces a complexity that is rarely discussed in the mathematics education research.
[10] http://www.sfu.ca/content/dam/sfu/geometry4yl/sketchpadfiles/Triangle%20Designs/index.html

The concept of folding a net into a solid is in the curriculum at the primary and secondary levels. And, in this sense, several textbooks present the classic static task: Observe these nets and say which can be folded into a cube. We propose a more dynamic and spatial approach, as follows. Provide students with eleven nets of the cube, each on a different page, all having semicircles on the centre of each side of the six squares of the net (see below) and inform students they must remain on their desks and cannot be cut or folded. The eleven nets are also available at the front of the classroom and can be folded, but they cannot be written on (the semicircles are also not coloured). In teams, the students must colour each net, each semicircle of twelve distinct colours so that they form a circle of the same colour when the cube is formed (one side with a blue semicircle joins another side with a blue semicircle to form an edge of the cube). The students can manipulate the net, are challenged to go do to at the front of the classroom as few times as possible.

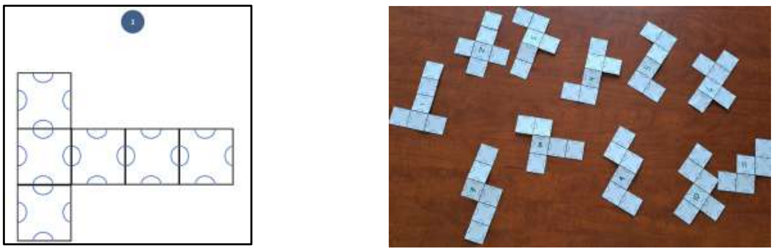

Figure 4a: One page from the student leaflet; b: the nets available in front of the classroom

The didactic choices of not being able to cut the net to colour the semicircles and not being able to manipulate the net so easily function to force students to resort to other strategies that are not only rooted in archaeological action/reflection, but also in the photographic and scenographic. They have to imagine the cube formed, to take into account each of the rotations made by each of the six squares that constitute the cube and retain the relationships that each of the segments of the squares maintain once folded to form the twelve edges of the cube. To do this, students can perform these movements mentally, but also with the help of gestures. These gestures express their starting point (the square on which the cube is placed), the transformations they anticipate, one rotation at a time, the relationship between the different segments to identify the colour to be inscribed and the end point of the folding envisaged. These gestures are very telling about how each of the students folds and designs the cube, as are their verbalization to the other members of the team.

In this activity, the actions/reflections also represent a back and forth between archaeological (the net in front of the class), photographic (the image of the cube to obtain) and especially scenographic (anticipation of how to fold the cube). Students are in the micro-space; they mainly deal with the 2D-to-3D passage, but to identify the corresponding semicircles, they must also resort to the study of segments (1D). In addition, activity lies within the dynamic/intrinsic quadrant since students analyze the relationships that each of the segments has with the others forming the whole, the cube. Usually, of the eleven developments in the cube, students see the one shown in Figure 3a, but as they complete this activity, they will explore the eleven nets and understand how each one forms the cube. This more dynamic and spatial way of approaching this mathematical concept seems to be conducive to a richer understanding of nets.

CONCLUSION

Given the important role that dynamic thinking plays in SR, which in turn correlates strongly with mathematical success, we are interested in not only spatializing the curriculum, but also dynamizing it. With a specific focus on geometry, we have provided two examples in this paper of ways to render well-known primary school tasks more dynamic. One involved inviting students to work with dynamic objects, with the help of digital tools, while the other involved drawing on pedagogical strategies to increase students' dynamic thinking. In both examples, we highlighted the presence of archeological, photographic and scenographic forms of SR. We also highlighted opportunities for working across dimensions. Both activities include direct feedback—in the first case from the dynamic environment itself and in the second case from self-checking at the front of the classroom—which we see as a significant aspect of task design, as per Laborde (2001). In our presentation, we will provide other examples of dynamising geometric tasks for primary school learners, including ones not confined to the micro-space, and share results of our classroom experimentations.

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ADVANCES IN GEOMETRY EDUCATION
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GEOMETRIC PREDICTION AS A BRIDGING PROCESS BETWEEN TRANSFORMING AND UNDERSTANDING

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Building on the theory of figural concepts, this paper describes geometric prediction as a process that lies at the interplay between the pure manipulation of figural elements and the pure selection and use of theoretical ones. In this paper I use a geometric prediction model to analyze the cognitive processes of an expert solver, while she is solving a geometric open problem, showing how it is instrumental in addressing the issue of zooming into the tension between spatial reasoning and geometric understanding.

SPATIAL REASONING BETWEEN TRANSFORMING AND UNDERSTANDING

Spatial reasoning is strongly intertwined with geometrical activity, as highlighted by the amount of research that lies at the intersection of these two topics. As for research in mathematics education, almost invariably, the definitions of *spatial reasoning* have in common the activity of imagining static or dynamic objects and interacting with them through mental transformations (e.g., rotation, stretch, reflection, to name a few). However, during the resolution of a geometric task, mental or physical transformations alone may not be sufficient, and they need to be accompanied by "analytical thought processes" (Presmeg 1986, p. 45). This is particularly evident when the solver must manage spatial and theoretical dimensions, for instance in the case of *open problems*, where the solvers are free to explore the problem and draw their own conclusions. In such problems the solution process often ends with the formulation of a conjecture after a (physical or mental) exploration of the situation (Mariotti et al., 1997), but the complexity of the processes elicited by the task cannot be fully explained by the intervention of mental transformations alone (Mariotti & Baccaglini-Frank, 2018).

Also for this reason, recently researchers have taken a transdisciplinary perspective and defined *spatial reasoning* as an overarching term used to refer to several dynamic processes that benefit from the constant challenge between two major dimensions: *transforming* and *understanding* (Davis & Spatial Reasoning Study Group, 2015, p. 140). Although this is a fresh take on a traditional theme of the research in mathematics education, the intertwining of spatial reasoning and geometric understanding has not yet been fully unraveled. In this paper I tackle such an issue within the domain of 2D Euclidean geometry.

The main assumption is that specific *bridging processes* lie at the interplay between spatial transformations and geometric understanding. *Geometric prediction* (GP) is one of such bridging processes. It builds on the *Theory of figural concepts* (Fischbein, 1993) and some of its effects can be captured as solvers deal with geometric open problems. This paper focuses on unraveling aspects of the intertwining of transforming and understanding thanks to a GP model (originally introduced in Miragliotta, 2020) developed specifically to analyze the prediction processes of an expert solver.

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THEORETICAL FRAMEWORK AND RESEARCH QUESTION

In the case of Euclidean geometry, the tension between transforming and understating is particularly intrigued, since it mirrors the well-known tension between spatial and theoretical aspects (Mariotti, 1995). The *Theory of figural concepts* (Fischbein, 1993) capitalizes on this tension by considering the cognitive counterpart of a geometrical object as comprising a *theoretical* and a *figural* component. Indeed, as part of a mathematical theory it is ideal, abstract, perfect, and general (conceptual or theoretical component), but it also reflects spatial properties like shape, position, and magnitude (figural component). Ideally, figural and conceptual components should be fused together in the *figural concept*, that is the actual object we are reasoning upon. During the resolution of a geometric task, the solver might alternatively operate on the figural components, through manipulations, or they might consider the theoretical nature of the objects in focus; however, it can be difficult to identify the contribution of each component when they act in harmony (Mariotti, 1995). In light of this theory, in the following we operationally consider:

- *transforming* as manipulating figural components through physical or mental transformations, that can be more or less isomorphic to geometrical transformations,
- (*geometric*) *understanding* as the act of recalling and using suitable theoretical components which belong to the mathematical reference theory at stake,
- in order to manage the complex mutual relationships between figural and theoretical components that comprise the figural concepts in focus.

In geometry, one of the processes at the interplay between figural manipulation and theoretical elaboration, triggered by *open problems*, takes place when the solver's exploration is carried out by mentally transforming a figure and imagining how it will change given certain constraints (Mariotti & Baccaglini-Frank, 2018). When this happens within the specific domain of Euclidean geometry, we have identified the process of *Geometric Prediction* (GP) as the generation of a new figural concept through the manipulation of its figural elements that maintain invariant certain theoretical elements belonging to the solver's mathematical reference theory (Miragliotta, 2022, p. 16). The process has been modeled, as shown in the visual diagram in Figure 1a, in order to highlights its main components and all their possible connections (see Miragliotta, 2022, for further details).

In a nutshell, the model is composed of two main sides, connected in the center through a sort of funnel, which metaphorically shows the dynamic interaction of the elements that flow into a GP process, producing a new object that is more than the sum of the original elements, but which retains some traces of both. On the left side of the visual diagram, there are *theoretical elements* recalled by the solver; they include all the properties that solvers attribute to the figure (or part of it), theorems and mathematical results. On the right side there are *figural elements*, on which the solver can focus, and which can be manipulated; these are elements of the solver's productions that belong to the figural domain at a specific moment as the solver looks at the figure represented.

Manipulations can be accomplished in two ways: *continuously*, that is solvers can imagine, perform, or mimic a continuous movement of one or more parts of the configuration (i.e., points, segments); *discretely*, that is solvers can locate these parts at a specific position on the plane and reconstruct the corresponding configuration. The solver's interaction with the drawing (if any) can be direct mainly by *bottom-up* (perceptually driven) or *top-down* (theory driven) processes (Gal & Linchevski, 2010).

Speech and gestures can occur at any time, shaping the process and providing the researcher windows onto the elements in focus.

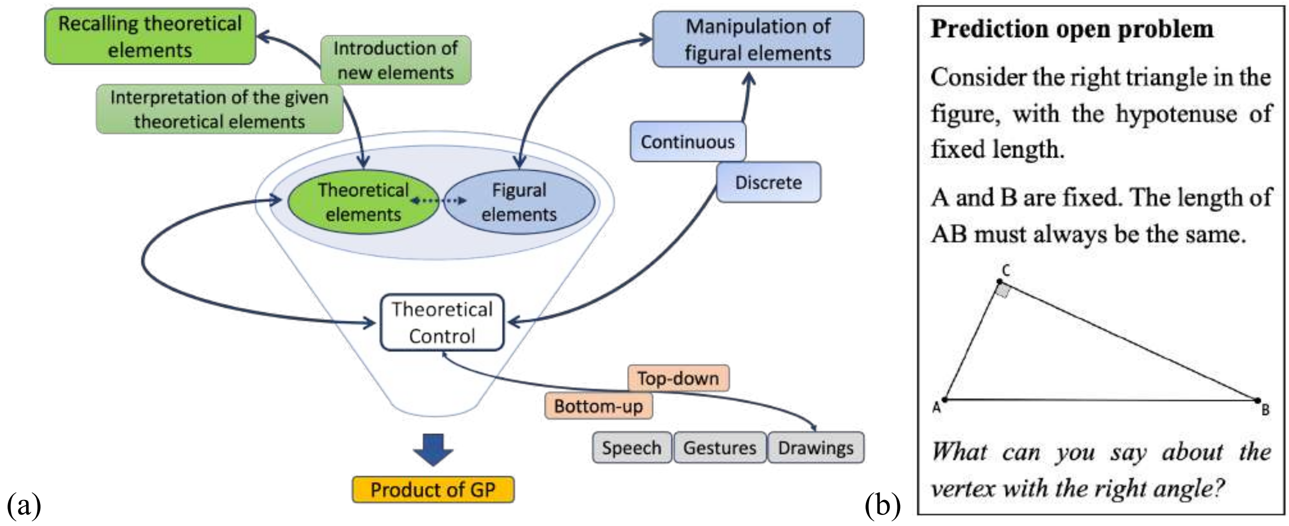

Figure 1: (a) Visual diagram of the GP process (b) An example of *prediction open problem*

The solver's act of mentally imposing onto the figure theoretical elements that are coherent in the mathematical reference theory at stake, i.e. *theoretical control* (Mariotti & Baccaglini-Frank, 2018), is central and guides each phase. In order to reach a coherent product of GP, a solver has to cover the whole path: starting by recalling theoretical elements referred to particular figural elements, and by continuing to manipulate the corresponding figural elements, under the strict supervision of the theoretical control. The product of GP is a new figural concept (for the given task) enriched by the figural and theoretical components that emerged during the process.

Although prediction processes may be different because their components may have different interactions – resulting in multiple paths through the model – they are all characterized by the interplay between pure manipulation of figural elements and pure recollection and use of theoretical elements.

Research question

Given the theoretical framework outlined above, this paper addresses the following question: What insight does the model of geometric prediction provide into the intertwining of spatial reasoning and geometric understanding, in the case of dynamic processes involved in the resolution of geometric open problems?

METHODS

The data analyzed was part of a larger data collection aimed at observing and describing prediction processes of volunteer solvers with a mathematical background, i.e. *expert solvers*. Individual task-based interviews were carried out and focused on *prediction tasks*, that are particular kind of open problems in which the solver is asked to describe possible alternative arrangements of a geometric configuration (imagined, given by a drawing and/or by a step-by-step construction) maintaining given properties. Each solver spent 60 min with the interviewer and worked through as many prediction tasks as they could. Data was collected in the form of video recordings behind the solver, in order to capture all of their productions (verbal, gestural, diagrammatic). Here we focus on a solver, fictionally named Giulia, who at the time of the interview had just graduated with a master's degree in mathematics.

This paper presents excerpts from Giulia's interview as an *instrumental case study* (Stake, 2003), since it provides a telling example of GP processes in a quite short time span. The selected excerpts allow us to navigate the multifaceted aspects of the GP model in depth. More specifically, excerpts report on Giulia's resolution of the prediction task shown in Figure 1b. Interpreting the resolution process using only figural manipulations could be quite challenging. Indeed, the solvers need to consider what has to remain invariant (the right angle at C, the length of AB, the hypotenuse is at AB) while other elements can be modified (the position of C, the legs length); this involves many theoretical aspects. For example, four positions of C can be figurally reached by imagining symmetries, but the right-angled and isosceles triangle can only be considered as a possible configuration by means of a conceptual elaboration.

During the resolution of this task, several GP processes can be accomplished, leading the solver to describe possible alternative configurations. We have called *configuration* an instance of a geometrical figure expressed by a drawing and/or a gesture (see C_i in Figure 2A). Excerpts are analyzed using the GP model (Figure 1a), in order to describe in greater depth how figural manipulation and theoretical elements come into play. The unit of analyses is the interview while the solver is solving each task. Each transcript is divided into segments, delimited by two configurations given or elaborated by the solver. We focus on the segments in which one or more products of GP are communicated by the solver. A *product of GP* is labeled using a progressive number (GP_i) when the solver refers to elements (a geometrical object or part of it) which are not present in the drawing or to a new arrangement of the configuration without drawing anything. Gesture and speech are considered jointly as a window onto the solver's processes. A wider description of data analyses along with the analytical scheme is available in (Miragliotta, 2022).

ANALYSIS

Two selected excerpts from Giulia's interview are presented here and analyzed using the GP model.

First excepts: Giulia communicates GP_1 and GP_2

The excerpt starts right after the first question (see Figure 1b and watch the video here).

Giulia: [She points at C] That I can move C so that it makes…[using her finder, she traces an arc from C to A and then a semicircle, starting from A, passing through C and ending at B, see Figure 2A]…a half circle [she repeats the same gesture as in Figure 2A].

Giulia: Where does this semicircle [she moves her finger upon AB from A to B] has its center...[she points at a position on AB between A and B, see Figure 2b]…at the midpoint of AB [she points at A and B one after the other].

Giulia: There, the hypotenuse AB always has a fixed length, because I am only moving C.

Giulia: Since C is...the vertex of a triangle that lies on a semicircle circumscribed to the triangle, it is always right, so the triangle always stays... right. In any place I move C [using her finger, she outlines the beginning of a circular path].

The first product of prediction (see GP_1 in Figure 2) is reached quite soon in the first line. At the beginning Giulia looks at the drawing silently, while she points at *C*. The initial stage of interpretation of the givens is so fast and silent that we cannot gather information on how the configuration is interpreted; however, looking at the whole excerpt we can say that an interpretation took place and the solver refers to the figural components of the figural concepts in focus using specific theoretical elements (i.e., hypothenuse, right triangle, right angle). The most observable part of the process is the continuous manipulation of figural elements (point *C*). Giulia engages in a dialogue with the given drawing: after she mimics the semicircle for the first time, she looks at the drawing silently; then she concludes that C "makes a half circle" and simultaneously she repeats the circular gesture. We can infer that the theoretical control intervenes for checking the coherence of the figural manipulation outcome (i.e., the circular path) and then recalling the most suitable theoretical counterpart (i.e., the semicircle). The dynamic and silent gestures coupled with the following utterance suggest that the outcome of the process is not a mere recollection from long-term memory of a theoretical element in a crystallized form (as she will do later), but an actual construction of a new figural concept, i.e. a prediction. She repeats the same gesture twice (Figure 2A), but with slight differences: the first gesture is performed silently and slowly, it is fragmented and quite tentative; the second gesture is fast, confident, and coupled with speech. These differences might correspond to different roles. The first gesture seems to be performed in order to look at a possible manipulation; so far, the outcome is quite fuzzy, and the gesture provides figural support to solver's processes of prediction. The second gesture seems to be made only to emphasize the solver's verbal production and to confirm her now less vague prediction. Finally, the gesture serves to crystallize the product of Giulia's prediction: the dyad composed of the drawing and the circular gesture is the initial configuration (C_2 in Figure 2) for the next GP processes. Indeed, she keeps talking about the semicircle as if it is actually part of the drawing (see the deixis "this" in the second utterance of the excerpt).

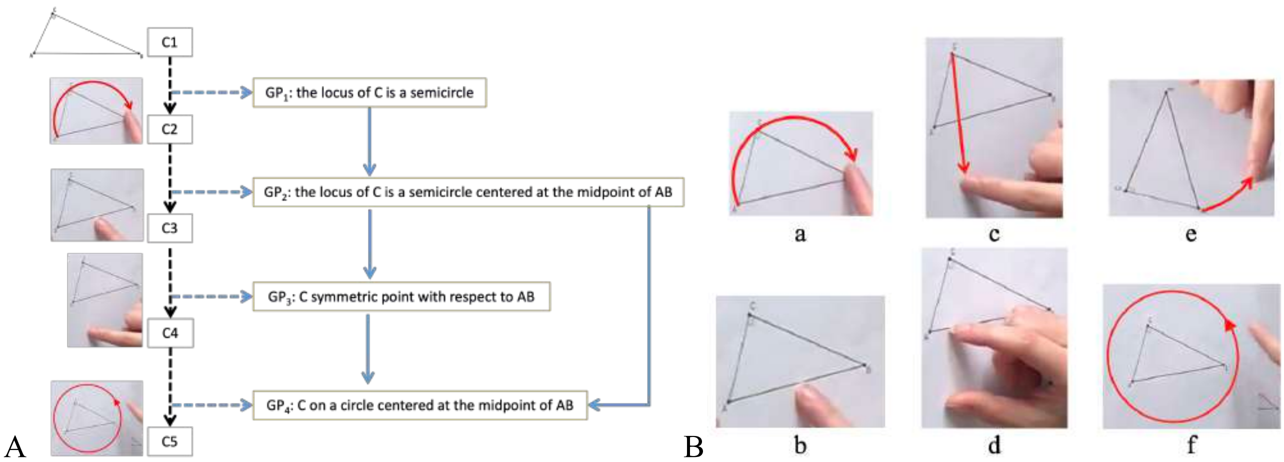

Figure 2: (A) Overview of Giulia's GP processes and their connections, each GP_i is a reformulation of the solver's productions; (B) instances of Giulia's gestural production

The second process shows an intense interplay between figural and theoretical elements. A new figural element is discretely introduced (a point on *AB*) and it is seen as part of different figural concepts, as demonstrated by the introduction of two theoretical elements (the circle center, the midpoint of *AB*). The theoretical control allows the solver to manage this complexity. The utterance shows the intertwinement of theoretical and figural components of the figural concepts in focus, which leads the solver to refine the first prediction by communicating additional geometrical properties of the semicircle: its center and radius. GP_1 is refined in GP_2 (see Figure 2A) through the top-down imposition of additional properties. This GP process is condensed into a single static utterance ("this semicircle has its center at the midpoint of AB"), but the claims that follow suggest a more dynamic dimension that is only verbally recalled ("I am only moving C") and not gesturally performed. Indeed, Giulia says that when *C* is moving "there" on the semicircle she has imagined, the hypothenuse *AB* "has a fixed length", stressing that the given constraint on *AB* is maintained.

The tone of the voice and the long pauses suggest that Giulia is quite hesitant, until she recalls a theorem that connects the properties of right angles, circles, and triangles. It is not formally stated, rather Giulia simply recalls the theoretical elements that support her prediction. The tone of her voice suggests that she is recalling a piece of previous knowledge, suggesting the intervention of theoretical control. We notice that Giulia uses a statically stated theorem to justify a dynamically reached prediction. As already stressed, although the positions of *C* were dynamically explored, when the semicircle emerges as a product of prediction it becomes part of the configuration, to the extent that the point now "lies on" the semicircle that was created by that point instead. In the same way, Giulia seems to start from a generic triangle that is inscribed on a semicircle: the right-angled triangle, which was a given, becomes a result that is drawn from the rightness of the angle at *C*. Therefore, the product of GP becomes a hypothesis, while the right angle at *C* is the thesis.

Excerpt 2: Giulia undertakes two additional prediction processes that lead to GP_3 and GP_4

A new process of prediction starts when the interviewer asks for positions for point C (video here).

Giulia: So that it maintains this configuration? If I draw...[using her finder, she points at C and traces a line downwards to a specific point, see Figure 2c]...point transformed from C [she points at A and B] with respect to AB, so I send the perpendicular from C to AB [she points at C and traces a line towards AB], that is I draw [she uses two fingers, see Figure 2d] a segment that is always perpendicular to AB of the same length [she uses two fingers between C and AB and then as in Figure 2d] of the one that I drew before.

Giulia: So, C can...can also stay on the opposite side [she traces a small arc from A to the imagined symmetric point, Figure 2e]. And so, in the end C can be on the...[she traces a circle, see Figure 2f]…can be, yes, lies on a circle with center at the midpoint of AB.

GP_3 is reached quite quickly and it incorporates all the theoretical and figural elements that were part of GP_1 and GP_2. Giulia starts reconsidering all the given constraints (see the first interrogative statement), she focuses on the figural element point C, and mimics the motion of C towards and below AB (Figure 2c). The gesture follows continuously a straight trajectory (as if the solver were holding a fictional pencil), but it is quite fast and confident as she were anticipating the final destination, so in this sense the manipulation is also discrete. The final pointing gesture communicates GP_3, which is silently contemplated for a while. There is a silent top-down and bottom-up interaction with the drawing, and finally the verbal production informs us that the new point is "transformed with respect to AB", that is according to a line symmetry. We can infer that the theoretical control supports the solver in evaluating the coherence of this manipulation, before continuing the exploration. The theoretical control also supports the introduction of new theoretical elements which add detail to the prediction. Indeed, Giulia claims that the new point is obtained by a geometric transformation of C and then she starts explaining how the symmetric point can be geometrically constructed. In doing this, she introduces a number of new theoretical elements (perpendicular line, length transport, opposite position), coupled with new figural elements, which blend together in a dense interplay. Here again some figural elements are interpreted as part of different figural concepts: now the hypothenuse and diameter AB is conceived as an axis for the line symmetry.

The process moves in a constant interplay between the left and right side of the model, driven by theoretical control. The output of this process, which blends the recollection of theoretical elements and the manipulation of figural elements, culminates in GP_3.

The products of all these prediction processes flow into a final one (see Figure 2A). Right after the symmetric point is described, Giulia summarizes her findings ("So, C can […] and so, in the end"). She starts from the last GP, stressing the new position of C, and outlines a small arc with her finger (see Figure 2f). This gesture blends the initial locus (GP_1, GP_2) and the symmetric position of *C* (GP_3) in a new locus. GP_4 constitutes a refined and extended version of the previously communicated predictions (see Figure 2A). Finally, we observe that the last claim on *C* is purified of any dynamic dimension and crystallized into a static statement. Indeed, at the very beginning of the interview, the solver has stated that she "can move C so that it makes" a certain locus, while now, at the end of the exploration, *C* "lies on" a certain fixed locus.

DISCUSSION AND CONCLUSION

We set out to describe *GP* as a bridging process between spatial reasoning and geometric understanding in order to gain a deep insight into their interplay in the case of Euclidean geometry.

Following the GP model while an expert solver is solving an open problem, we can see how the process contains a constant interplay between the theoretical and the figural elements as they are embedded into the two sides of the model. In particular, we can identify two catalyzers of the GP process: the prominent role played by the theoretical aspects and the dynamic approach. Indeed, from the beginning Giulia's figural manipulations seem to be prompted by the aim of looking at possible positions of point C; this dynamic view guides her GP processes. At the same time, she needs fixed products of the manipulation to be reinvested into a new process, giving rise to chains of GP processes (Figure 2A). We can confirm that GP processes involve many components of spatial reasoning, but only a strong geometric understanding can support and suggest spatial transformations that are effective and controlled. This is particularly evident when Giulia recalls theoretical elements that are part of a theorem she knows and when she describes the isometric transformation of point C. On the other hand, figural manipulations could guide the introduction of new theoretical elements, suggesting that spatial transformations might affect geometric understanding. Indeed, before introducing the theoretical elements of the theorem in a crystallized form, Giulia seems to need the figural manipulation of point C. Moreover, very often the figural component, that is mainly communicated gesturally, precedes the verbal description of the theoretical counterpart of the figural concepts at stake.

More generally, expert solvers demonstrate a productive combination of theoretical and figural aspects, which come together into play in an immediate and condensed processes of GP. An essential part of the process consists in giving (multiple) geometrical interpretations to spatial elements. Indeed, looking diachronically at the elements in focus during the four GP processes, gestural and verbal productions reveal that the figural elements (e.g., AB) change their theoretical status (e.g., hypothenuse or side of a right triangle, diameter of a circle, axis of a line symmetry). The graphical element is always the same (e.g., the straight line between A and B), but each time it is addressed as a different geometrical object. Giving different geometric meanings to the same object could involve the spatial transformations (e.g., decomposing, rearranging), but it deals also with the geometrical interpretation of the configuration and, in particular, with the (meta)cognitive act of seeing the same figural element as part of different figural concepts. In the case of GP processes, theoretical control collects many metacognitive acts: monitoring and checking the coherence of a figural manipulations, managing the temporality (from dynamism to crystallization and back), supporting the selection of theoretical elements to be recalled and recognizing invariants, (re)interpreting given figural elements, considering or rejecting the outcome of a manipulations as suitable for a new dynamic process.

Zooming out from the particular case in focus and considering the GP model as a prototypical example of bridging processes, we can say that the interplay between spatial transformations and geometric understanding is driven by specific metacognitive acts that deserve our explicit attention in order to be further explored and identified; the GP model allows us to highlight that theoretical control is one of them. Therefore, the study shows the explanatory potential of analyzing processes which lie at the intersection between different cognitive activities, but further research is needed in order to identify other bridging processes within and beyond Euclidean geometry.

From the educational point of view, this study provides further evidence of the importance of supporting the blending of theoretical and figural components of figural concepts. This must become an educational goal. Since open problems seem to spontaneously trigger processes that elicit a dialectic between the two components, we can advance the hypothesis that their use in geometrical activity could help educators in pursuing this goal. Investigations on open problem is not new, however the new perspectives on spatial reasoning can stimulate a new fresh insight into their use both for researcher and educators.

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BUILDING MATHEMATICAL MATURITY THROUGH ALGEBRAIC TOPOLOGY CHALLENGING PATHS

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A central educational goal of higher-level mathematics undergraduate courses is to foster the development of competences that enable students to use mathematical abstraction and formalism and, at the same time, to develop mathematical creativity. Does it mean mathematical maturity? Research in university mathematics education suggests that to grasp the characteristics of maturity, it might be beneficial to introduce students to theoretical objects by engaging them in the autonomous construction of examples, conjecturing, and proving, especially in geometry. This approach is generally fruitful, but students sometimes have difficulty providing counterexamples to false conjectures, formulating conjectures, or proving statements. Starting from the point that there is no way to acquire mathematical maturity except by doing math, we introduce a path that stimulates students to construct knowledge by themselves in the realm of topology. We face the issue of promoting students' maturity through designing paths that stimulate geometric reasoning. We present the first findings on some students' development and self-perceptions of their maturity.

INTRODUCTION

Mathematical maturity is one of those expectations in higher education that is often mentioned but not yet clearly defined (Steen, 1983; Lew, 2019). There is no empirically based description of mathematical maturity currently. Lew explores existing descriptions of mathematical maturity as well as descriptions of the related concepts of mathematical intuition and mathematical beliefs, investigating how mathematicians describe mathematical maturity. Interviewed mathematicians report using this term in various ways: ways of thinking about mathematics, mathematical intuition, and comfort with and competence in mathematics. Even more obscure is how exactly to reach mathematical maturity, both for educators and students. Research in mathematics education agrees that encouraging students to work on their own to generate examples of mathematical concepts, to conjecture, to prove or refute a statement, or to prove a theorem is an effective way to reach maturity. We firmly believe that two components, both stimulating conceptual understanding, can decisively contribute to its achievement at every school level: the first one concerns the topic, and a powerful candidate is geometry, which with its visual push can be an excellent springboard, and the second one deals with the teaching practices to be implemented, designing suitable paths that start from the construction of examples and pass through conjecture, proving or disproving, and giving alternative proofs of known theorems. In geometry, properties of figures derive from definitions within an axiomatic system: a figure is "controlled by its definition" (Fischbein, 1993). The construction of a visual representation of a key definition or concept in the statement of a theorem can lead to the recognition of the key idea of the proof (Gallagher & Infante, 2022). A primary goal of advanced mathematics courses at the university level is to have students become proficient at writing proofs. However, research has reported that this goal is rarely met. The difficulty seems to be in passing from

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the stage in which the operational aspects of a mathematical concept and, more generally, the procedural knowledge of mathematics is gained to the stage at which the structural aspects and conceptual knowledge of mathematics are acquired (Sfard, 1991). Tall and Vinner (1981) relate the understanding of a concept to the distinct notions of concept image and concept definition. Moore integrates the construct with the notion of concept usage, “which refers to the ways one operates with the concepts in doing proofs” (Moore, 1994). An efficient way to develop conceptual understanding is to approach it through examples. The exploration of example spaces is essential to learning mathematics, and this power manifests particularly in geometry. Watson and Mason (2005) propose that “examples can be perceived or experienced as members of structured spaces” and introduce the term "example space." Example space exploration and extension activities play an important role in students’ mathematical thinking as they make formal mathematics more familiar. Dahlberg and Housman (1997) argue that creating examples can be a powerful “learning event” in which students make real progress in understanding and that it might be beneficial to introduce students to new concepts by starting with a concept definition and going on to enrich the concept image. Conceptual understanding is necessary for advanced mathematical thinking (Tall, 1992) and creativity, two essential characteristics of maturity. We could say that:

> mathematical maturity refers to cognitive, communicative, metacognitive, and affective problem-solving competences dealing with generating examples and counterexamples, conjecturing, proving, and mathematical creativity in constructing original examples, modifying some already known or generating new ones, producing original proofs of already known theorems using different techniques or new theorems, and, in the meantime, to a sense of self-efficacy and mastery, even in collective discussions.

Once given an attempt at the definition of maturity, it comes naturally to face the following questions: how can we encourage students’ mathematical maturity? What specific paths or trajectories during the process of learning could help them to be able to independently learn and apply geometry? In what ways can students be engaged in formulating conjectures and doing geometric proofs? Our attempt at an answer lies in the design of specific paths that engage students in self-constructing knowledge. We address the issue of promoting students’ maturity by designing suitable paths in the realm of geometry, which in our setting is topology. Topology is a tool encouraging undergraduate students’ maturity, as counterexample constructions (Steen & Seebach, 1970), intuition, abstraction, and generalization are highly valued. Its dual figural-conceptual power arises from the impulse to abstract essential features from complex situations and then to let our curiosity roam while striving to truly understand what is essential about fundamental ideas (Fischbein, 1993; Tall, 1992; Weber & Alcock, 2004; Selden & Selden, 2013). The designed path aims to engage students in example construction, conjecturing, and proving in topology, helping them to make deeper sense of known theorems and techniques and to discover new ones. In the paper, we give a taste of students’ answers to highlight the figural-conceptual power of topology and analyze how students show and perceive their own maturity after the implemented path.

RESEARCH FRAMEWORK AND RESEARCH QUESTIONS

The theoretical model to which we refer to design the path is that of *concept usage* (Moore, 1994), in the form proposed by Dahlberg and Housman (1997), integrated with the construct of *example space* (Watson & Mason, 2005) and with the notion of *figural concept* (Fishbein, 1993), essential in the realm of geometry. Dahlberg and Housman assign students some exploration "pages," starting from a concept definition, to develop their own related concept images: to define the concept (*definition* *page*); to generate two examples of objects, one satisfying that definition and the other not (*generation page*); to verify whether some given objects do or do not satisfy the definition (*verification page*); and to explore some statements linking that concept to others (*conjecture page*). We make Dahlberg and Housman's model more powerful by adding a page named the *bridge page* (Miranda, 2023b), requiring the example space extension through *bridging examples* suitable to overcome students' difficulty in providing counterexamples and in formulating conjectures. The lens with which we look at students' maturity and how students perceive it refers to the descriptions reported by Lew (2019). In this proposal, considering how mathematicians describe mathematical maturity (Lew, 2019), we give a taste of students' answers in a particular step of the path, the last proving/disproving conjecturing phase, highlighting the power of figural and conceptual aspects in developing an aware concept usage and investigating students' perceived maturity. We address this research question: *What is the impact of such an approach on students' outcomes and perceptions of advanced mathematics and their personal maturity in knowing or constructing it?*

METHODOLOGY

The research is part of a project concerning the transition to advanced mathematical thinking (Miranda, 2023a), supported by paths starting with the example of space extensions and going towards generalization and abstraction (Miranda, 2023b). The implementation, held in the academic years 2021–22 and 2022–23, involved 34 students attending an introductory algebraic topology course within a Bachelor of Mathematics, in which we gave definitions of algebraic objects that are topological invariants, such as the fundamental group. The path was designed to encourage the transition from informal to formal, from intuition to logic deduction, from empirical to theoretical thinking, and to stimulate those thinking processes that are activated when facing a conjecture. Our design choices were inspired by the learning model of *concept image/concept definition/concept usage* (Moore, 1994; Tall & Vinner, 1981), the *example space* construct (Watson & Mason, 2005), and the theory of figural concepts (Fischbein, 1993). The path was packaged by integrating the pages (definition, generation, verification, and conjecture) of the Dahlberg and Housman task model (1997) with another page, the *bridge page*. The path is composed of three phases. In the first phase, the example phase, it starts with a given definition and requires generating examples and non-examples satisfying the definition (generation page) and verifying if a given object belongs or not to the example space (verification page). The bridge page connects the example phase with the conjecturing phase, requiring comparing the definition with some constraints that must or must not be satisfied and the conjecturing phase. The path concludes with proving or disproving a given statement (conjecture page). In the bridge page, assuming that one of the constraints is the beginning definition, we associate with each constraint the value 1 if it must be satisfied and the value 0 if it must not:

Give an example, if possible, of a mathematical object satisfying/not satisfying Constraint 1, Constraint 2, Constraint 3.

This requirement generates a matrix containing eight rows, corresponding to eight subtasks, and three columns. Students explore the subtasks to solve them. If, in a particular case, they fail to find an example, they must understand why they cannot do so. It could be difficult to find an example or impossible to generate one. In the latter case, it is necessary to prove this impossibility. The students worked individually on each task in the first phase and collectively in the second phase to discuss, compare, or improve their solutions. Students constructed personal examples and proofs. They were then asked to explain and comment on their solving process to the class, explaining the reasons why a particular example does or does not satisfy the required property, whether an example exists, and why no example exists. Personal examples were collectively analyzed initially to compare all the outputs and identify common aspects and differences, and then a second time to identify interesting features, such as accessibility, and to choose a representative of a larger class. The first phase concludes with the collective choice of the less complex example, if it exists, for every row. If they found a difficulty, all the components acted as prompts to modify an example or recall a useful result, with the aim of constructing the solution together. The second phase deals with conjecturing by looking at the matrix columns. The collective discourse is a very strong pedagogical tool because it encourages questions and discussions (Sfard, 1996) to explore the links between the properties expressed by the constraints, to discover whether these properties are independent, to open a conjecture-oriented exploration, and to foster generalizations. As an example: Why is this an example? Why is there no example? Does a mathematical object that satisfies constraint *i* satisfy constraint *j*? And vice versa? Some examples then become counterexamples, showing that certain implications are false. Others generate conjectures. Finding that a set of constraints seems mutually incompatible is an excellent way to generate a conjecture. Looking at the columns and comparing them, some conjectures are produced. The last phase deals with proving or disproving given conjectures. The model represents a teaching-learning opportunity to explore students' understanding and usage of definitions. It aims to encourage fluency, flexibility, and originality through different solution paths and making use of different representations (Duval, 2006), therefore shaping maturity.

About a specific path

The definition running through the selected path (Table 1) is simply connectedness (*c2*), a concept in which two complementary operational-structural views and figural aspects coexist and can be combined to ensure its reification (Sfard, 1991; Fishbein, 1993).

<table>
<tr>
<td>Definition page
<u>Instructions.</u> You will have a few minutes to study the following definition.
Definition A subspace of a topological space is said to be simply connected if it path connected and has a trivial fundamental group.</td>
<td rowspan="3">Bridge page
<u>Instructions.</u> Consider conditions c1, c2, c3, associated with the value of 1 if you require that it be satisfied by the example you are looking for and the value 0 if it shouldn't be. Construct examples, if they exist, according to the requests of the following dichotomous matrix. Choose, comparing collectively, a generic example. What can you conjecture by comparing the columns?
<table>
<tr><td>c_1 : X contraibile</td><td>c_2 : X semplicemente connesso</td><td>c_3 : X stellato</td><td>Example, if it exists</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td></td></tr>
<tr><td>1</td><td>1</td><td>0</td><td></td></tr>
<tr><td>0</td><td>1</td><td>1</td><td></td></tr>
<tr><td>1</td><td>0</td><td>1</td><td></td></tr>
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<tr><td>0</td><td>0</td><td>0</td><td></td></tr>
</table></td>
</tr>
<tr>
<td>Generation page
<u>Instructions</u>. Answer the following questions.
1. Give an example of a space that is simply connected and explain why it is.
2. Give an example of a space that is not simply connected and explain why it is not.
3. In your own words and/or images, simply explain what a simply connected space is.</td>
</tr>
<tr>
<td>Verification page
<u>Instructions.</u> Determine which of the following spaces is a space simply connected. Explain why.
1. R   4. $\{(x, \sin(1/x)): x \in R\}$
2. $\{(x, \cos x): x \in R\}$ 5. Z
3. S^1 6. S^2</td>
</tr>
</table>

Table 1: A specific path from the example phase to the conjecturing phase

Explored the first three pages, the bridge task requires comparing the property *c2: simply connected* with *c1: contractible* and *c3: star-shaped:* give an example, if possible, of a topological space satisfying/not satisfying constraints *c1, c2, c3.* In some cases, it is possible to generate examples, but this is not possible for subtasks such as (0,1,1) because "a star-shaped space is contractible." Different kinds of tasks entail different processes and, consequently, different behaviors. Looking at the columns, students are required to generate conjectures about the relationships between the constraints. Exploring the links, they could ask themselves: “Is a contractible space simply connected?” The bridge page opens the door to the conjecture that “a contractible space is simply connected”. In the last conjecture page, students are involved in a prove-disprove activity dealing with the concepts of contractibility and simple connectedness (Table 2), intuitive concepts sometimes wrongly identified.

Data collection and data analyses

All data related to the training activity were digitally stored via the Moodle platform: written notes, recorded discussion, and a feedback questionnaire. To shed light on the impact that the practice had on the students’ development and self-perception of mathematical maturity, we focused our analyses on some selected answers to conjecture page and questions 1 and 2 of the feedback questionnaire.

> **Conjecture page** *(Prove/disproving)* Instructions. Say which of the following statements are true. Justify the answer.
> *If a topological space is contractible then it is simply connected.*
> *If a topological space is simply connected, then it is contractible.*
> *The quotient of a simply connected space is simply connected.*
> *Identity id:* $S^1 \rightarrow S^1$ *can be continuously extended on disk* D^2.

We searched for sentences providing evidence about the characteristics of conceptual understanding (concept usage) and the meaningful processes towards maturity students experienced and perceived.

Question 1. The activities included the construction of examples, counterexamples, and proofs. Have they been useful for you to interpret a definition, to acquire a concept, or to acquire autonomy in proving? Did you happen to produce a proof by yourself? What has changed compared to previous experiences?
Question 2 Beyond the exam, what do you have left of these experiences? Would you recommend a friend to have similar experiences? Why?

FIRST FINDINGS

Due to space restrictions, we confine our analysis to some selected answers to the mathematical problem in the last conjecture page (see Table 2) and to two questions of the feedback questionnaire. Students’ first attention is focused on the topological object and properties that must be visualized or empirically produced to sketch the first attempt at a proof; they, through the graphical representations, explored and found a preliminary proof. This was the prelude to its formal proof. We search where concepts have been used in the sense of Moore (1993), with what improvement towards conceptual understanding, and which processes have determined maturity.

In Fig 1. St5 starts from the intuitive idea that a contractible space is a space collapsing into a point and then recalls the notion of deformation and gives a logical proof. She uses her prior knowledge (definitions) to create a new one, a logic proof (concept usage). 2. The graphical register through which the simple connectedness of the sphere can be intuitively deduced is convincing and preludes to its proving. She uses graphical and then verbal register (concept usage) to prove that the implication is false. In 3. StT3, mimics the passage to the quotient while explaining and says: 'It is as if you consider a string and join it to the extremes’ (empirically reasoning), and makes previsions (geometric predictions). Then she justifies formally and generalizes (concept usage). Some difficulties can be seen in other protocols: StT10 seems to confuse the properties of a quotient of topologic space with those of the quotient group of its fundamental group and deduce that the quotient of a simply connected space has a trivial fundamental group by quotienting the group. Finally, to prove that *the* *identity id*: $S^1 \rightarrow S^1$ *is not continuously extendable to the disk*, StT13 translates the negation of the statement in terms of retraction to explicit semantics (Weber & Alcock, 2004) and produces a contradiction. This reveals a crucial choice, giving impetus to a syntactically valid proof (concept usage) different from other students' attempts. At the beginning of the proof, which is a delicate step (Moore, 1994), some students need to review the definitions. We observe that students who reinterpreted the meaning reached proof success (StT13), while StT6 invoked the *Brower Fixed Point Theorem* without arguing correctly. Some findings delineating signs of maturity emerged from the last step of the path: students' use of different semiotic representations (Duval, 2006), among which the figural one (Fishbein, 1993) is strategic to apply geometric reasoning to prove the conjecture; the transition from an intuitive to a more structural approach to understanding (Sfard, 1991).

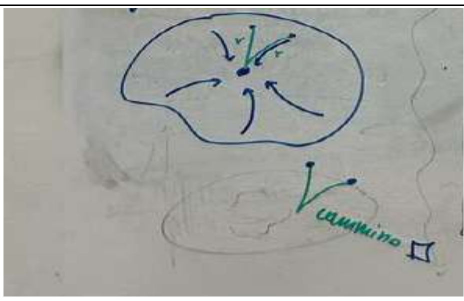
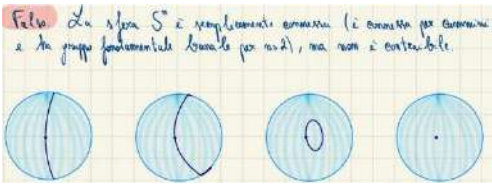
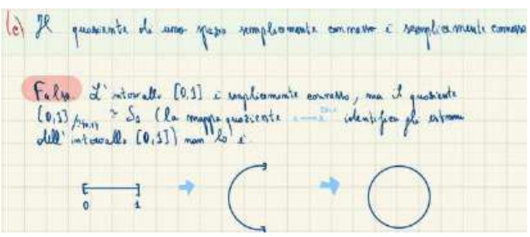
1. 	St.5.: *True*. If X is a contractible topological space, then it has as a deformation retract a point, so it is homotopy equivalent to a point (since the retraction is a homotopic equivalence); therefore, X has the same fundamental group as the point which is the trivial one. Furthermore is path connected, as you see in the figure. St. 4: *True:* A topological space, which is homotopy equivalent to a point, has its fundamental group. Furthermore, the path connection is a homotopy invariant, so since the point is connected by paths, X is too. It follows that X is simply connected.
2. 	StT3: *False.* The n-sphere is simply connected (it is path connected and has trivial fundamental group for n≥2), but it is not contractible. St. 6: *False*. The surface of any polyhedron is simply connected but it is not contractible.
3. 	StT3: *False.* The quotient space of a simply connected space is simply connected False. Interval [0,1] is simply connected, but its quotient $[0,1]\big/_{\{0,1\}} \sim S^1$ (the quotient map identifies the extremes) is not. St. 5: *False*. The square is simply connected but the torus, the cylinder, the projective plane are not. St.10 *True.* The quotient of a trivial group is trivial, therefore it is simply connected
4. StT6: *False.* $\nexists$ a way extending the identity map id_{S^1} to a map $D^2 \rightarrow S^1$, if it was possible to extend it, **it would have no fixed points** $\Rightarrow$ I would violate Brouwer's fixed-point theorem	StT13: *False*. If *id*: $S^1 \rightarrow S^1$ was continuously extendable over the disk D^2, then S^1 would be a retract of D^2, which is a simply connected space, and therefore S^1 should be simply connected, too, but it is a contradiction.

Table 2: Some excerpts selected from the students' answers to the proving/disproving task

Identifying some characteristics of maturity from answers to question 1 of the questionnaire

The protocols show that the activities were very useful to internalize a definition, to acquire awareness (StT7), autonomy, mastery, self-confidence, and self-efficacy, and to develop a natural propensity towards abstraction and formalization (StT1).

StT1: [...] were of fundamental importance for the focus of the notions. Without them, I probably would not have grasped the more sophisticated and interesting aspects of some concepts. [...] In many cases, *intuition* was a guide, and *formalization* was quite natural.

StT7: Absolutely, there is no better way to *internalize a definition* in mathematics than by *producing examples and counterexamples*. [...] The activities were very important to acquire greater awareness of the topics covered in the course. [...] Thanks to them, I was able to have more *mastery of many concepts* and *proof techniques*.

The vision of the proof changes. The term "methodical" in the following excerpt would seem to recall an idea of a reproductive, procedural approach to proof, with respect to which the student applies according to a procedure, while the term "reasoned" is linked to a productive idea with respect to which the student autonomously reason by himself and therefore has an active thinking approach:

StT1: [...] Compared to previous experiences, I had a less *methodical* and more *reasoned* approach to proof. [...]

Another fundamental ingredient to deepen maturity is inspired by the discourse, communication, negotiation, and participation in a mathematical community that are dominant (Sfard, 1996):

StT7: The experiences that will surely remain more impressive are the *independent reflections* and the *moments of collective discussion*. The former because they fully immerse the student in the subject, while the latter allow for *many different ideas* and *points of view*, even just in *understanding where two ideas of different proofs* for the same result come from. The *discourse* is fundamental, I believe, a comparison I found in many moments of the path, and these are formative experiences for the our career.

Identifying some characteristics of maturity from answers to question 2 of the questionnaire

From the first analysis, it seems that the students perceived the experience as very demanding but received many benefits. They declare that they have lived a rewarding experience, with serenity and as a protagonist (StT1), leaving the *students' comfort zone*:

StT1: This experience has been challenging for me, but really *rewarding*. I am satisfied with the contents studied and the method used. I lived the journey with serenity, never feeling the weight of the study load or the pressure of deadlines. [...] I would recommend similar experiences to a friend because they allow you to live a course fully, keeping in mind the objective of passing the exam, but making you the real *protagonist of* the *path* made for the mere pleasure of studying and *doing mathem*atics.

StT3: Personally, the more we got to the heart of the course, the more stimulating it was to be engaged in the activities [...].

StT2: Surely being *out of the comfort zone* of the student and having put his hand to a topic not mediated by a professor was beneficial. [...] to exhibit their work was very interesting, and certainly I would recommend someone with similar experiences.

StT3: I would recommend it to colleagues; indeed, *I would also* include it in the other courses.

Students recognize that it was a formative experience although difficult:

StT5: Pages paths are a great idea; the difficulty of some tasks should be moderated, but I think that stops are very formative for a math student and quite pleasant.

CONCLUSIONS

Topology is a topic in which visual reasoning is a prelude to maturity because it allows the key idea to be identified to go towards formal reasoning and proving. Generating examples reveals itself as a good way to achieve an understanding of definitions; exploring the links between constraints is an excellent way to generate conjectures; and, finally, transporting personal example choices to a collective discussion forces students to become aware of a more general class of examples and aids proof productions. Our results confirm that example-space activities are crucial in developing students' theoretical thinking and supporting their undergraduate-graduate transition to maturity. We find that the collective manipulation of the personal examples contributes to a deeper understanding and shapes the premises for abstracting and generalizing. The value of geometry for enhancing competences in conjecturing and proving came out. However, we believe that further comparative studies are needed at every level to improve the design of activities and the power of geometry to develop a structural approach to knowledge towards maturity. The question of how to define mathematical maturity and encourage students to build it remains open.

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ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
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ON THE ROLE OF SHIFTS OF ATTENTION AND FIGURAL APPREHENSION IN THE EVOLUTION OF GEOMETRIC PERCEPTION

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Geometry learning pivots on perceiving and reasoning upon geometrical objects and relationships. This paper develops theoretical and didactical ideas on ways to evolve students' perceptive and analytical discernment ability for learning spatial geometry. I locally integrate Duval's framework on geometric apprehension with Mason's theory on shifts of attention for analyzing 10th-grade student R's experience with spatial geometry problem-solving facilitated by a 3D pen. This analysis suggests that active engagement with a 3D medium and tools, such as the construction and manipulation of models, play a vital role in the evolution of students' geometric perception and reasoning. Shifts in attention lead to the need for different forms of figural apprehension. This study underscores the need for enactive, dynamic learning experiences in geometry education.

ENCULTURATING PERCEPTION AS A POSSIBLE KEY TO GEOMETRY LEARNING

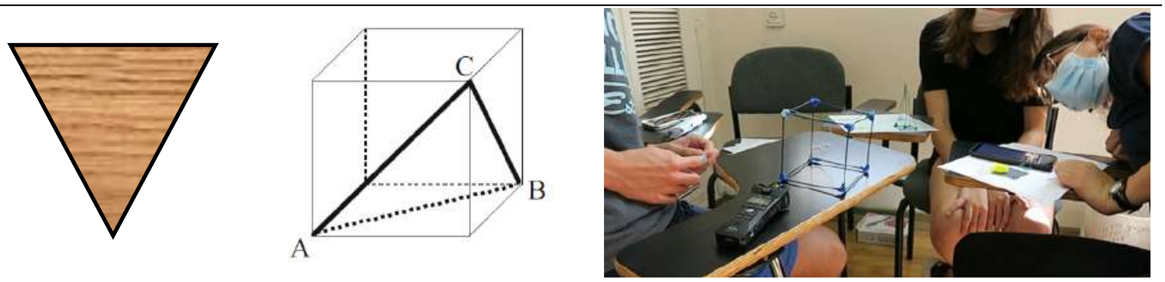
Perception is critical in learning, especially in geometry. Johnston-Wilder and Mason (2005) articulate that the crux of geometric learning is rooted in the ability to "see," i.e., to identify and comprehend geometrical objects and their relationships. However, what enables to educate this ability of perceptive and analytical discernment, this learning to "see," remains less understood.

Bryson (1988) posits that learning involves enculturation of the senses. He argues that one's perception is not directly interacting with the environment but is mediated by a system of socially constructed symbols and discourses. This notion resonates with Gibson's (2014) ecological approach to perception, in particular the concept of affordances as potentialities for action, and with scholars of cognitive and learning sciences addressing the construct of culturally conditioned vision—'professional vision' (Goodwin, 2018), 'educated perception' (Goldstone et al., 2010). Goldstone et al. (2010) claimed, "A credible and worthy hope for education is to teach students to take the natural affordances of our long-tuned perceptual systems, which are at their core spatial and dynamic, and retask them for new purposes" (p.280). This contribution aims to develop theoretical and didactical ideas on possible ways to retask students' perception for spatial geometry learning.

The geometric characteristics of the environment, such as right angles and parallel lines, are not inherently obvious. Instead, they become apparent through interacting with and manipulating cultural artifacts (Bartolini Bussi & Mariotti, 2008). Consequently, adopting these cultural mediators alters one's perception (Goodwin, 2018). Johnston-Wilder and Mason (2005) recount the case of a young girl struggling to recognize a specifically oriented wooden shape as a triangle (Figure 1a). This narrative exemplifies one of the ways in which geometric perception may be educated: "I'm not sure, but it is a triangle for her', pointing to her friend on the other side of the desk." (p. 3). By acknowledging a different perspective, the student comes closer to grasping the invariant essence of geometric forms, such as triangles. Our preliminary research (Palatnik & Abrahamson, 2022)

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 107-114). ICMI.

exemplified how students solved a spatial geometry problem (Figure 1b) by shifting their perception through moving, sketching, manipulating the model, and being attentive to others' actions and discourse (Figure 1c), thereby enculturating each other perception and reconciling their spatial and analytic reasoning (Fujita et al., 2020).



a b c

Figure 1: Learning to perceive a(n equilateral) triangle

Role of figural apprehension in learning to perceive

Raymond Duval's seminal work provides a useful lens on the mechanisms of seeing and interacting with figures mathematically (Duval, 1995, 2005, 2017). Duval (1995) distinguishes four distinct types of apprehending geometric figures: perceptual, sequential, discursive, and operative. According to Duval, *perceptual apprehension* is the most basic level and pertains to recognizing geometric figures and their constituent parts at first glance. *Sequential apprehension* focuses on the procedural aspects of how a geometric shape is formed and the order in which the figure's elements are assembled, relates to the figure's construction, and depends on technical constraints that change with construction tools. *Discursive apprehension* involves recognizing and processing attributes of geometric figures mediated through language, definitions, and mathematical properties. It requires the learner to make discursive statements that define and determine the figure's properties. According to Duval, *operative apprehension* is the most advanced level of apprehension. It requires learners to actively engage with a figure by modifying it mentally or physically, often to gain insights into problem solving or proving. Duval asserts that learners must distinguish and coordinate between these different forms of apprehension to understand geometric figures comprehensively. However, various forms of apprehension may collide. For example, emphasizing discursive apprehension may impede the insights obtained through operative apprehension. These classifications provide a useful framework to study the cognitive processes involved in learning geometry and to design environments facilitating students' coordination of different forms of apprehension.

Developing Duval's (1995, 2017) ideas, Mithalal and Balacheff (2019) explored conditions in which construction tasks in 3D DGE (dynamic geometry environment) stimulate students' transition from working with drawings to perceiving geometric properties of figures. Recently, Palatnik (in press) adopted a figural apprehension framework to characterize the affordances of the different novel medium—3D pen (a hand-held device enabling spatial sketching with hot, fast-hardening plastic) for spatial geometry learning. Preliminary findings suggest that 3D sketching as a form of embodied and enactive math activity (Ng & Ferrara, 2020; Palatnik et al., 2023) may evoke coordination between various types of apprehension, thus facilitating learning and teaching spatial geometry. The importance of operative apprehension aligns with other key ideas from Duval's (2017) framework.

Specifically, Duval proposes two figural operations that facilitate students' progression from "discriminated recognition of forms to the identification of the objects to see" (Duval, 2005, p. 13, translated by Herbst et al., 2017, p.86): the mereological division of shape (transformation of the geometric figure into figural units of the same dimension) and the dimensional deconstruction of shape (decomposition into elements of dimension lower than in given figure). However, Duval remarks that introducing a new element into a figure does not necessarily mean the student becomes aware of a new property. This insight mirrors John Mason's (2008) ideas about attention and awareness: "someone may be attending to something in a particular way but unaware explicitly of the what or the how" (p. 12). The What and the How pertain to the focus and structures of attention.

Role of shifts of attention in learning to perceive

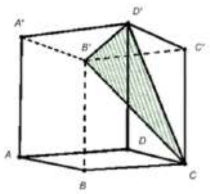
Mason (2010) describes learning as a transformation of attention that involves 'shifts in the form as well as the focus of attention' (p. 24). Mason (2008) distinguishes five different structures of attention: (1) holding wholes, (2) discerning details, (3) recognizing relationships, (4) perceiving properties, and (5) reasoning based on perceived properties. The structure of *holding wholes* is related to a macroscopic view, where learners consider the overall structure or context. *Discerning details* involves zooming in on specific elements, which might be akin to focusing on individual vertices or edges in a geometric model. *Recognizing relationships* emphasizes the interconnections between the elements, such as equal sides in a geometric shape. *Perceiving properties* pertains to an active search of the elements possessing the inherent characteristics. *Reasoning based on perceived properties* is the advanced structure of attention, where learners make logical conclusions or predictions based on selected properties. Mason's framework is frequently used to understand problem-solving processes.

Palatnik (2022) applied shifts of attention as an analytical framework for investigating spatial geometry learning in the context of collaborative geometric activity in which students constructed tangible models of a geometric object on different scales. This study documented shifts between foci of attention and the shifts between all five attention structures. These shifts in attention were linked to students' visual and tactile senses, proprioception, and physical interactions with the models. For example, by positioning the model at its vertex, students could visually segment the icosahedron into three distinct sets, leading to a breakthrough in establishing the number of edges.

Next, I will illustrate the idea of learning geometry as the evolution of geometric perception by analyzing an episode involving a 10th-grade student who tackled a spatial geometry question while utilizing a 3D pen. This episode is part of a larger data set from an extensive research project focusing on the design of embodied learning for spatial geometry (e.g., Palatnik & Abrahamson, 2022). In this analysis, two theoretical frameworks will be integrated: Mason's (2008, 2010) theory on shifts of attention and Duval's (1995) framework concerning the cognitive apprehension of geometric figures. By employing a local integration of these disparate viewpoints (Bikner-Ahsbahs & Prediger, 2014), I aim to provide a theoretical foundation for learning geometry as the evolution of students' geometric perception in the context of spatial geometry problem-solving activities facilitated by 3D sketching.

THE CASE STUDY

The episode illustrating the evolution of student geometric perception in spatial geometry problem solving presents the student R, facing the Triangle task (Figure 2). While R did not formally study spatial geometry, he had knowledge of planar geometry for the presented challenges. When facing the previous tasks, he already used 3D sketching to facilitate problem solving (see Palatnik, in press). He also knew that due to the structure of the task (succession of four questions), he may change his initial answer to the true/false questions. The whole episode took little less than 5 minutes. The first T/F question in the task was: CB'D' is a right-angled triangle. The right angle is ___?



ABCDA'B'C'D' is a cube. Answer the true/ false questions and explain your reasoning.

1. CB'D' is a right-angled triangle. The right angle is ___?
2. B'D' is the shortest side of the triangle CB'D'.
3. Triangle CB'D' has an obtuse angle.
4. In triangle CB'D', all angles are equal.

Figure 2: Triangle task (adopted from Widder et al., 2019)

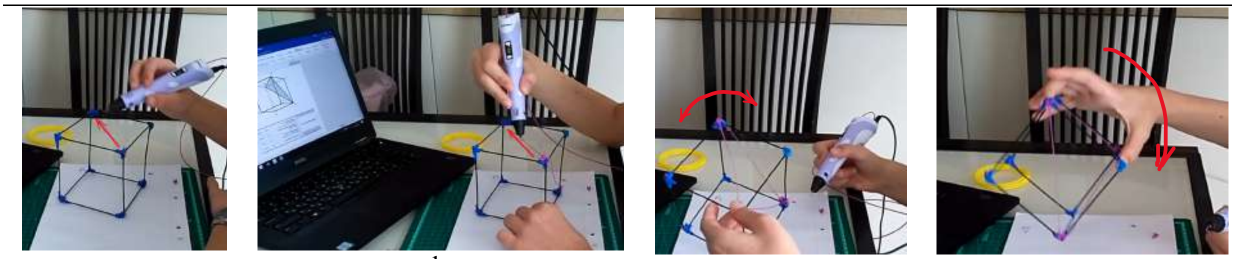
Interaction with a 3D model conflicts with the perception of a 2D diagram

R started by observing the illustration on the computer screen while tilting a head in different directions. He pointed with a 3D pen at a model's vertices C, B', and D'. Then, he drew an imaginary triangle by outlining segments CB', B'D' (Figure 3a), and D'C on a model.

R: I don't even need to (3D) sketch this to see that this is a right angle. Cause it seems… [draws imaginary angle CB'D' on a model]. Wait a minute… [tilts a head]

I: You should (sketch), I think.

R: Well, here, actually, I should. But also on the diagram, it is a right angle. [starts sketching CB']. But it doesn't say anything. [sketches] But… I assume it is true [the angle is right]



a b c d

Figure 3: Actions advancing learning to perceive: gesturing, sketching, rotating the model (red arrows are added to show the movement)

Analysis. In terms of shifts of attention, R's attention is split between two main foci: a diagram and a model. On a diagram, he *discerns a detail*—angle B' and identifies it as a right angle. R's actions—tilting his head, pointing with a 3D pen, and drawing the imaginary angle and triangle- help him establish a correspondence between the two focal objects. Through these actions, he discerns details on the model, too: vertices of a triangle, its sides, and eventually an angle CB'D' as a candidate for a right angle. In terms of figural apprehension, *perceptual* apprehension is dominant (mostly vision within a flat and static medium). In addition, R starts to apprehend the problem *sequentially* (the imaginary construction of a triangle on a cube's model is sequential).

Sketching and rotating the model shapes the perception of a geometric problem

R: [sketches CB' and then B'D' (Figure 3b)] Ok. Now it becomes a little bit different. [continues to sketch D'C]. Nice! It is good that I sketched. It is a right-angled triangle, but the right angle is not this, but instead, in fact… Wait a minute… [fixes the sketch]

…

R: [inclines a head]. Now, according to how it looks now, well… [points with a pen] D’CB’ is a right angle. Well, I don’t have to explain why.

I: Yes, it is better to explain.

R: Well, what? Acute [points at the angle D’], acute [points at the angle B’], right [points at C]. Nu, [points with a pen] C B’ D’…

I: You can rotate the trian(gle), the square, the cube if you want (to show me).

R: Ok, so I will rotate it [starts to rotate the model, looks at the model while rotating it back and forth]. Now it looks different. Now it looks as if it is not like this. [tilts a model] Wait a minute. [rotates a model again, changing axes of rotation]. Ah. No! It is not a right-angled [triangle]. It is, in fact, isosceles. [gestures two sides with two strokes of a pen].

R: [rotates a model, Figure 3c]. It is confusing. [looks at the model] No! It is not a right-angled (triangle). False.

Analysis. R is focused mainly on a model. His exclamation while sketching: “Now it becomes a little bit different,” indicates that his perception takes shape through the action of sketching. This process is gradual (R’s first candidate for a right angle now looks acute to him). Further action of rotation provides additional perspectives and reshapes perception further: “Now it looks as if it is not like this”. R *discerns* new *details*. His attention is now shifted from angles to the sides (his stroking gesture indicates this), probably by the action of sketching. R establishes *a relationship* of equality between two sides and *perceives* the triangle as isosceles. In terms of figural apprehension, sketching provides both *perceptual* and *sequential* apprehension. The 3D pen tool affords immediate materialization of an imagined geometric element (the way R sketches emulates the sequence of his previous imaginary drawings). The rotations accompanying sketching and further exploration of the model add *operational apprehension*. R states, “it is confusing”, and we can attribute this confusion to incoherence between various forms of figural apprehension.

Consolidated forms of figural apprehension support reasoning

R: [Checks the question on the screen] Where we were at? B’D’ is the shortest side (T/F?) in the triangle C, B’, D’ [points on model’s vertices with a pen]. No. These are diagonals. These are diagonals of the cube (faces). So, they all are equal. In fact, it looks now as an equilateral triangle. Yes. Well, it is [B’D’] not the shortest side because they are all equal.

R: [reads the next question] (Does) Triangle CB’D’ have an obtuse angle?

R: No. There is no obtuse angle. After all, it is an equilateral triangle, so 60, 60, 60. All the angles are equal, so 60, 60, 60. That’s it.

Analysis. R is focused solely on the model. He continues with *discerning details*—the triangle’s sides. But structures of attention evolve further, first to *recognizing relationships* “These are diagonals of the cube”, then to *perceiving properties* “…they all are equal”. R’s *reasoning is based on perceived properties*—” [B’D’] is not the shortest side”. From a perspective of figural apprehension types, the utterance “In fact, it looks now as an equilateral triangle” indicates that the perceptual, operational, and discursive apprehensions are “now” coherent.

Following a protocol, each task was ended with a question about a confidence level (out of five) in the correctness of the answer.

R: Between 3 and 4. Cause, I don’t have backing for this, except for vision.

I: Ok.

R: Well, I can bring a ruler, put out (the segments), and measure [gestures with a finger an upper diagonal]. But what (for)? Again, [takes a model in hand and rotates, Figure 3d] it is a cube, after all. And these are its diagonals. Thus, these (triangle) sides are equal.

I: Ok. So?

R: Thus, it is an equilateral triangle

I: Are you sure about this? 100%?

R: I think that, yes, I'm sure of it.

Analysis. This segment shows how R consolidates different forms of figural apprehension of triangle CB'D'. The perceptual form of apprehension is still very dominant: "I don't have backing for this, except for vision". By his relatively low initial self-confidence assessment, R acknowledges a deficiency of a visual argument. However, he is aware of the possibility of establishing the equality of segments by measuring, and he rotates the model for alternative perspectives (both actions are forms of operative apprehension). His concluding deductive chain is correct and adds a discursive apprehension. This argument structure follows R's previous structures of attention shifts: from holding the wholes (a cube with a triangle) to discerning details—cube's (faces) diagonals as sides of a triangle to recognizing relationship "equal" to confident reasoning, "it is an equilateral triangle".

DISCUSSION

R answered the questions correctly, and his argumentation was valid. In studying spatial reasoning skills about 2D representations of 3D geometrical shapes, Fujita et al. (2020) reported just a 52.4% success rate for Japanese 9th graders (N=225) on a similar task. It is easy to dismiss R's success with the task as trivial. After all, he had additional tools—a 3D model and a sketching pen. However, as Shvarts et al. (2021) note, simply providing culturally developed tools is insufficient for students to discover their affordances (Gibson, 2014), nor does the existence of motor-perceptual systems guarantee the realization of their body potentialities. Drawing from Duval's figural apprehension and Mason's shifts of attention frameworks offers insights into his success and, in general, "how students chose particular spatial reasoning skills and domain-specific knowledge" (Fujita et al., 2020, p. 254).

Contesting the initial and misleading perceptual apprehension of a problem was challenging for the student in agreement with the findings of Widder et al. (2019). While the tools afford active ways to interact with a diagram, the student was initially reluctant (see also Palatnik & Abrahamson, 2022). Thus, the interventions of the interviewer: "You should (sketch)…" and "You can rotate…" served as action-perception catalysts for shifts in foci and forms of the student's attention and triggered other types of apprehension. These 'calls for action' with an intention to change perception appeared thought-provoking. The need to coordinate a 2D diagram with a 3D model by mapping a vertex to vertex and side to side through 3D sketching led to alternate attention foci and the occurrence of unusual dual sequential apprehension. Moreover, the continuous change of a 3D display during sketching and rotating contrasted to a static 2D diagram let R notice the inconsistency between 2D and 3D representations, creating a productive doubt in perceptual apprehension. When backing based on one of the apprehension types is discredited, a need for other types of apprehension arises.

The nature of the final attention structure—*reasoning based on perceived properties* (Mason, 2008) may appear as a purely cognitive act contrasting perceptually and action-oriented holding, discerning, recognizing, and perceiving. However, an enactivist stance in which cognition is perceived as an active, ongoing engagement with the world: "All doing is knowing, and all knowing is doing" (Maturana & Varela, 1987, p. 26) resolves a dichotomy of perceptual and cognitive aspects of attention structures. Within this framework, "perception consists of perceptually guided action," and "cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided." (Varela, 1999, p.12). Indeed, R's actions led to shifts in perceptual structures of attention, which informed his further actions. A shift to discerning details (vertices, sides of the triangle) was associated with a figural operation of dimensional deconstruction. 3D sketching and rotating actions were related to sequential and operative apprehension, shifting the student's attention to the triangle sides and enabling a discursive apprehension of them as diagonals of the cube's faces. When all forms of figural apprehension appear coherent, they support geometric reasoning.

The exchange concluding the episode demonstrates that an enactive approach goes beyond the ability to recognize the model's geometric properties empirically. The student briefly considered measuring the segments but chose an alternative path. He starts the argument with a rotation of a model, helping him relieve his previous actions and perceptions and situate the application of relevant geometric knowledge. However, this material scaffolding fades out as R constructs an almost complete chain of deductions, increasing his confidence level about the correctness of his answer.

The presented episode contrasts with R's (and most of our students) usual form of learning geometry, which involves merely observing and rarely interacting with any form of diagram. It was a *learning* activity since it allowed R to use his senses to guide his actions, change his perspective on a problem, and recall adequate geometric content to improve his perception of it. To solve a problem, R overcame initially deceptive perceptions and combined different kinds of apprehension. By designing such experiences and letting students and teachers reflect on them, noticing and guiding shifts of attention, we may facilitate learning geometry by evolving our geometric perception.

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SPATIAL REASONING INTERVENTIONS AND TRANSFER TO GEOMETRY: WHAT WE KNOW ABOUT MECHANISM

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There is growing evidence that spatial reasoning learning causally supports improvements in geometry performance. However, the mechanism that explains this transfer is not yet well understood. The current study provides a systematic literature review on all existing experimental studies that examine the impact of spatial learning on mathematics. We have found that, despite an increasing interest in mechanism and uptake of experimental studies, only a few studies have explicitly assessed mechanistic pathways. The two most promising pathways are the use of specific spatial skills within specific geometric tasks and visuospatial working memory. Although changes in strategy is understudied, this may be another pathway. We identify process models, mixed-methods (including qualitative research), and mediation/moderation as important aspects of future experimental work.

SPATIAL REASONING SKILLS SUPPORT GEOMETRY LEARNING

There is growing evidence that improving spatial reasoning skills causally transfers to improvements in mathematics achievement, including within the field of geometry (Adams et al., 2022). For example, a recent meta-analysis of 28 spatial intervention studies found that spatial reasoning learning consistently improves mathematics performance with average effect sizes between .20 and .42, depending on transfer distance (Hawes et al., 2022). Despite the strong evidence supporting spatial transfer to mathematics, the mechanisms behind transfer are not yet clear (Young et al., 2018) – that is, *why* does spatial learning support mathematics achievement?

Understanding why spatial reasoning supports geometry learning has theoretical and practical implications for a range of research disciplines (e.g., mathematics education, cognition, development, neuropsychology), which are interested in characterizing the relation between the constructs. For example, there are currently qualitatively and functionally different theoretical models (described below). Characterizing an accurate theoretical model of mechanism would, in turn, inform the development of more targeted and effective mathematics learning interventions. That is, the value in understanding spatial mechanism for research on the teaching, learning, and understanding of geometry, is the optimization of instruction strategies (e.g., what spatial techniques are the most beneficial); effective integration of technology (e.g., what spatial barriers may students face); and the development of individualized learning plans, valid assessments, and curriculum.

The current study provides a systematic review of existing experimental research, to characterize what we know about why spatial reasoning learning transfers to support geometry and mathematics achievement. Correlational research was not included because, by design, it is unable to establish causal mechanisms (Rohrer, 2018). The current study then suggests innovative methodologies for capturing mechanism in future research, which includes mechanisms not yet considered.

Spatial reasoning, spatial learning, and geometry

Spatial reasoning is an umbrella term for a wide range of different, but interrelated, skills that involve thinking about two-dimensional and three-dimensional relations between and within objects. For example, being able to imagine an object rotating (mental rotation) or a perspective different from your own (perspective-taking) are two separate spatial reasoning skills (Hegarty & Waller, 2004). Spatial visualization is also a kind of spatial reasoning skill that refers to the multi-step mental manipulation and transformation of spatial information.

There is extensive empirical evidence spanning the past three decades that show spatial reasoning skills, and in particular spatial visualization and mental rotation, support geometry performance (e.g., Clements & Battista, 1992). This relation is consistent, predictive, and strengthens over time (Resnick et al., 2019). More recently, there has been experimental evidence that shows improving spatial reasoning skills causally improves geometry understanding. Some spatial learning interventions have included targeted spatial learning opportunities, such as origami (Arıcı & Aslan-Tutak, 2015; Boakes, 2009), mental rotation or spatial scaling training (Gilligan et al., 2020); whereas others have included a wider range of spatial learning opportunities, such as mental rotation, spatial orientation, spatial visualization, and the integration of these spatial skills (Adams et al., 2022; Lowrie et al., 2017, 2019).

Current theories of mechanism

We are not aware of any broad theoretical models that characterize potential pathways between spatial reasoning and geometry, and so we will focus here on general mathematics achievement. Existing theoretical models are based primarily on correlational research (Battista et al., 2018). Hawes and Ansari (2020) have identified four possible mechanisms based on this work: (a) individuals map number onto space to better understand them (spatial representation of numbers); (b) completing both spatial and numerical tasks involve the same brain areas (shared neural processing); (c) individuals use spatial visualization skills as a "mental blackboard" to organize and manipulate mathematics problems (spatial modelling); and (d) spatial reasoning is a proxy for other cognitively demanding skills (working memory). Notably, these models are not mutually exclusive and may reflect varying levels of analysis (e.g., activation in the intra-parietal sulcus during spatial and geometric tasks may be functionally equivalent to using spatial visualization skills to complete geometric tasks).

Interestingly, none of the Hawes and Ansari's (2020) models positioned mathematics as inherently spatial (e.g., numerical-spatial mappings serve as a tool for understanding; cognitive architecture originally specialized for interacting with the physical world was co-opted to understand mathematics). This may be due to their focus on numerical reasoning rather than geometry. That geometry is inherently spatial is a common description in geometry literature (e.g., Battista et al., 2018; Boakes, 2009). Although such an explanation highlights the affordances embedded within task demands, it falls short of providing a causal mechanism. That is, it does not answer questions of how, when, or which kind of spatial skills are employed to complete different geometry tasks.

To address the complexity of multiple, overlapping mechanisms, Mix (2019) characterizes broader attributes of mechanism. They argue that the relation between spatial reasoning and mathematics is general, and not limited to specific spatial and mathematics sub-skills. However, elsewhere they acknowledge that this relation is fragile and difficult to replicate. In contrast, most research to date assumes that specific spatial skills underlie specific mathematics tasks (e.g., Gilligan et al., 2018).

Mix (2019) suggested that spatial skills can perform several functions in real-time problem solving, such as decoding, mapping referents, and the use of mental models. Although there is some automatic processing between spatial reasoning and mathematics, strategic recruitment of spatial processes is more predictive of achievement. Finally, they find that the relation between spatial reasoning and mathematics is consistent but may change qualitatively as both familiarity and task demands change.

Methods for characterizing mechanism

Most research characterizing the relation between spatial reasoning and geometry (and mathematics) has been correlational. These approaches can involve completing of a range of different spatial and mathematics tasks to examine the relative correlations and emergence of factors (Young et al., 2018). For example, mental rotation has been found to predict performance in geometry and not algebra, whereas visuospatial working memory (the ability to keep spatial information active in the mind) had the opposite relation (Kyttälä & Lehto, 2008). Although it is speculated that geometry involves reasoning about transformations whereas algebra can involve mental operations (Newcombe et al., 2019), correlational analyses are unable to establish causal mechanisms (Rohrer, 2018), and factor analytic approaches have practical, statistical, and interpretative limitations (Young et al., 2018).

Experimental research is required to empirically assess hypothesized mechanisms. Unfortunately, such research is limited, especially in geometry. There are currently only nine existing experimental studies connecting spatial learning with geometry. The primary aim of these studies was to develop effective spatial learning materials to support geometry learning. Subsequently, what these studies can tell us about mechanism (*why* the spatial learning materials help) may be limited in scope. Nevertheless, these studies, and the broader context of experimental spatial learning studies, are an important starting point to examine the mechanism connecting spatial reasoning within geometry.

METHODS

A systematic literature review was conducted to characterize how experimental studies, examining the causal effects of spatial reasoning learning on mathematics achievement, have considered mechanism in their justification, experimental design, and explanation of findings.

Data collection and inclusion criteria

A recent meta-analysis by Hawes et al. (2022) identified 28 experimental studies that examine the causal effect of spatial training studies on mathematics achievement. To be included in the Hawes et al. meta-analysis, studies were required to (a) involve humans, (b) be published in English, (c) report on behavioral outcomes, (d) use a pretest-training-posttest design, (e) have at least one mathematics outcome measure, (f) include sufficient spatial training, and (g) compare a spatial training group with a control group. Although Hawes et al. included unpublished findings and doctoral theses in their analysis, we have included only those publications that have undertaken peer-review (n = 23). An additional literature search was conducted using the same criteria, to identify studies published after the Hawes et al. study (n = 7). Taken together, a total of 30 studies were included in this review.

Data coding

A thematic analysis (Guest et al., 2012) was used to identify and characterize mechanistic explanations for how spatial learning may support mathematics learning. The operational definition of mechanism used in this study was phrases that considered how or why spatial reasoning may support mathematics learning. For example, the phrase, "*children might use spatial visualization… to picture and rotate the shapes presented to count the number of sides (faces) on the shape.*," (Gilligan et al., 2020, p. 15) would be counted as a mechanistic explanation because it explains *how* a spatial reasoning skill may be used to solve mathematics problems. In contrast, the phrase, "*Spatial ability and visualization have been shown to be related with academic success in mathematics and geometry*," (Arıcı & Aslan-Tutak, 2015, p. 181) would not be counted as a mechanistic explanation because it does not provide an explanation of *why* this relation is observed.

An initial reviewer coded each instance of mechanistic explanation for emergent themes within the literature review, methods, and discussion sections (results sections were not coded because they do not include explanation or connections to the literature). New codes were created as required. Once the full set of codes were identified, the initial reviewer re-coded the full sample to ensure a subsequent code wasn't more appropriate. In a final step, two new reviewers independently coded the full sample of mechanistic explanations using the developed coding system. Although new codes could be created during this step, none were required. One code was renamed to capture that theme more accurately and another code was split into two for greater specificity. Discrepancies were resolved through discussion, resulting in 100% interrater reliability.

RESULTS

Of the 30 studies examined, 76% justified their study (Introduction section), and 66% explained the results from their study (Discussion section), by speculating a causal mechanism connecting spatial reasoning and mathematics achievement. Explored in depth below, is that these mechanistic models (a) varied, (b) were largely absent from the experimental design, and (c) changed over time. A summary of the findings where methodological design considers mechanism is also included.

Varied mechanisms that have been considered (description of emergent themes)

The most prevalent mechanistic pathways involved using improved spatial skills to complete mathematics tasks (60%). This could include using proportional reasoning skills to complete quantity processing tasks (Gilligan et al., 2020), spatial visualization to create mental models (Lowrie et al., 2019), form perception to interpret shapes and graphs (Burte et al., 2017), or mental number line to compare magnitude (Mix et al., 2021). All these specific connections reflect the idea that there are specific spatial skills that are recruited to complete specific mathematics tasks.

Visuospatial working memory (vsWM) was also identified frequently as required to complete mathematics tasks (37%). In this view, keeping in mind a spatial organization of sequences of moves and transformations when completing multi-step problems would require vsWM. Some related discussion referenced working memory and executive function skills more broadly (7%). We separated out this coding to distinguish between the two explanations.

Although 30% of studies acknowledge the possibility of priming spatial strategies, this discussion was usually framed around not being able to rule this explanation out, while favoring an improved spatial skill or vsWM model. Only two studies (7%) described strategy change as a main outcome. Another common explanation was that mathematics is inherently spatial (27%), though notably this does not actually address mechanism. Shared cognitive resources (i.e., the same brain area is responsible for completing spatial and mathematics tasks) and consideration of moderators (e.g., how mechanisms may vary based on age or task demands) were also commonly mentioned (23%).

Experimental evidence for mechanism

There were seven studies that were able to address mechanism through their experimental design. Five of these studies had separate conditions to compare the effects of training different spatial skills, finding broadly that specific spatial skills support specific mathematics tasks. Only two of these studies had geometry outcomes. Gilligan et al. (2020) found that, although mental rotation supported missing term problems and spatial scaling supported number line, improvements in both skills supported geometry performance. Gilligan et al. (2020) suggest that form perception might be useful for distinguishing between different symbols and identifying information presented in charts. Adams et al. (2022) found that spatial learning (which included a range of different visualization-based skills) transfers to geometry outcomes and not algebra outcomes. Adams et al. (2022) suggest that spatial visualization might be useful in geometry because it involves reasoning about transformations.

Visuospatial working memory (vsWM) may also provide support. Both mental rotation and vsWM training equally transferred to improved mathematics performance, with no transfer on one another (Zhang et al., 2021). Notably, Mix et al. (2020) found that age moderates the relation between spatial learning and mathematics, with spatial skills being more important early on and vsWM more important later. These contributions appear to be unique to vsWM and not improvements in general cognitive processing, with one study including cognitive skills as co-variates (Cornu et al., 2019) and another not finding transfer to general skills (Cheung et al., 2019).

There were four studies that considered mechanism in their research aims, but did not have a sufficient experimental design to adequately address it. Three of these studies, instead, addressed best teaching practices, finding benefits of embedded (Lowrie & Logan, 2023) and embodied (Gilligan et al., 2023) approaches that include strong pedagogy (Mulligan et al., 2020). There were also three studies that found socio-economic status (SES) mediated the impact of training. Although this does not provide a mechanism, it highlights the importance of including contextual factors as well as potential mediators / moderators when characterizing mechanistic pathways.

How consideration of mechanism has changed over the last three decades

Across the 1990's and early 2000's, there was no experimental research examining the impact of spatial learning on mathematics, despite a large literature showing strong correlations between the two constructs. Experimental studies begin to emerge starting in 2009. However, discussion of mechanism is limited; from 2009 to 2016, seven studies do not consider mechanism at all (with four justifying their study by characterizing mathematics as inherently spatial). Only two studies identify the potential role of vsWM and one study identifies spatial visualization. From 2017 to present, there has been a marked increased interest in mechanism, both in frequency and range of explanation. However, explanations have focused on skills and vsWM, with only one study (Lowrie & Logan, 2023) centering the possibility of changes in strategy.

DISCUSSION

As research into the relation between spatial reasoning and mathematics has progressed, researchers have shown increasing interest in the mechanisms supporting this connection. Earlier studies focused on establishing correlations, and later verifying that spatial learning causally supports improvements in mathematics. Although studies have provided speculative explanations by drawing on correlational work, only more recently have a few studies sought to assess this directly within their methods.

The most common explanations for transfer are built on the idea that specific spatial skills are utilised within specific mathematics tasks. Although this contrasts Mix (2019), who argues for a general connection, Mix's (2019) work is based on correlational research. The experimental work described here suggests there may be specific connections, including within geometry (Adams et al., 2022; Gilligan et al., 2020). This experimental work also highlights a separate role for vsWM. Hawes and Ansari (2020) consider vsWM in the broader context of improvements in general cognition; however, this is not supported by the experimental findings, which instead finds a unique role for vsWM separate from improved spatial skills (e.g., Zhang et al., 2021). Spatial skills and vsWM may work together by providing a larger "mental blackboard" to keep more spatial information in mind as well as the skills required to mentally manipulate that spatial information. Given the small number of experimental studies, and the inclusion of broad geometry measures, more experimental research is needed to characterize the kinds of spatial skills that support specific geometry tasks.

A new model for mechanism – the role of strategic choice

Independent of any gains due to improved execution of spatial skills and vsWM (as described above), it is possible that spatial learning might encourage students to apply different strategic approaches when solving geometry problems. Changes in strategy are largely absent from correlational and experimental literatures (Lowrie et al., 2021; 2023), and, when included, described as a priming effect that cannot be ruled out (e.g., Gilligan et al., 2020). However, students know multiple ways of solving problems and will balance efficiency and accuracy when selecting an appropriate strategy (Siegler, 2006). These strategic decisions are influenced by the nature of the problem, their familiarity with different strategies, and their beliefs about potential success. This is aligned with the finding that spatial skills versus vsWM are relatively more important when solving mathematics problems as familiarity and task demands change (Mix et al., 2020), and that strategic processing of spatial-mathematics relations is more predictive of mathematics performance than automatic processing (Mix, 2019). Importantly, completing geometry problems can involve a multi-step problem-solving process, and students may use different cognitive and meta-cognitive strategies at any given step.

Future directions – how to capture mechanism

We suggest that it is possible to characterize mechanism with experimental research explicitly designed to assess potential mechanisms. Such work would be benefited from process models (Young et al., 2018) to identify the specific tasks involved in geometry (previous work often uses broad measures). Mixed-methods approaches would also be helpful, with qualitative methods well suited to characterize when and how strategies may change. Also missing from the literature is analysis of changes in quality, accuracy, and relevancy of (external and internal) representations, which would inform how spatial learning supports mathematics performance. For example, future research may explore if spatial learning leads to representations becoming more detailed, abstract, or focused on relevant spatial information; or perhaps the quality of representation remains the same, but the ability to manipulate it improves. Another key approach to determining mechanism is the inclusion of key mediators and moderators to parse out how basic cognitive skills may manifest in different mathematics contexts (Resnick & Stieff, 2024).

Educational implications for incorporating spatial learning in geometry

Although more research is required to determine mechanism, there is growing evidence that spatial learning does, indeed, support geometry learning. Successful spatial learning interventions tend to be embedded within authentic mathematics contexts, and not separate decontextualized training (Lowrie et al., 2023). This likely provides better alignment between the spatial reasoning skill and relevant mathematics tasks, which enables students to see the connections. Interestingly, most of the research described in the current study came from the field of cognitive science, which tended to focus on broad mathematics achievement. However, given that geometry is the part of the curriculum most associated with spatial reasoning (e.g., Battista et al., 2018), the development of effective spatial intervention to support geometry seems particularly important.

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GUIDED PLAY THAT FOSTERS THE DEVELOPMENT OF CHILDREN'S SPATIAL ABILITY

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One type of spatial reasoning involves the skill of mental imaging, a capacity initially acquired by children through physical experiences and gradually refined through mental manipulation (Clements et al., 2004). The development of children's mental manipulation skills necessitates the cultivation of concrete manipulation skills in geometry. However, there is a significant research gap concerning the specific classroom instruction and experiences that contribute to the enhancement of children's mental imaging abilities. In this study, we initiated a lesson centered around free play with geometric toys to gauge children's geometric aptitude, and then we implemented guided play sessions enhance their geometric abilities. By incorporating discussion using appropriate materials at the beginning of guided play and allowing children to select and work on three-dimensional objects themselves, children can create three-dimensional objects at a more advanced level.

INTRODUCTION

Recent studies have highlighted the close relationship between spatial ability and mathematical performance (Casey et al., 1995; Geary et al., 2000; Mix & Cheng, 2012). Researchers have not only observed a correlation between spatial thinking and mathematics but have identified spatial thinking as a predictor of subsequent mathematics performance (Mix & Cheng, 2012). Moreover, early spatial experiences contribute significantly to the later development of spatial and geometric reasoning skills (Okamoto et al., 2014).

It is noteworthy that the spatial ability of young children demonstrates greater improvement than that of adolescents and adults, which aligns with the prediction of the sensitive period hypothesis (Baenninger & Newcombe, 1989). Research on spatial abilities in young children involves educational resources such as blocks, origami, and computer games (e.g., Cakmak et al., 2014). A variety of measures, including verbalization, pictorial imagery, and construction, have been employed to assess children's spatial abilities (e.g., Thom & McGarvey, 2014). As spatial training transfers to other spatial skills that have not been directly trained (Uttal et al., 2013), it is imperative to consider specific training that enhances children's spatial abilities.

Recent studies have shifted their focus to classroom-based learning, with teachers opting for shorter interventions rather than semester-long training sessions (e.g., Sinclair & Bruce, 2014). These studies suggest that young children who engage in diverse activities, particularly those who initially perform poorly on spatial skills tests, may benefit from training (Baenninger & Newcombe., 1989). However, it is important to note that performance ratings in these studies are often based on children's mathematics assessments, which do not fully reflect their individual spatial skills. Tailoring classroom instruction to each child's spatial ability has been proposed as a more effective approach to increasing students' overall spatial ability.

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Development of the mental imaging in young children

One facet of spatial reasoning involves mental imaging, which entails visualizing and mentally transforming objects, often transitioning between two-dimensional (2-D) and three-dimensional (3-D) representations. Children acquire this skill through physical experiences and subsequently progress to mental manipulation (Clements et al., 2004). The memory of physically manipulating figures observed through hand movements and visual perception contributes to reducing working memory demands for spatial tasks. However, the developmental progression of children's physical manipulation of 2-D figures and 3-D solids is crucial, as without such advancement, children may struggle with mental manipulation.

In this study, the author explored the developmental stages of 3-D construction leading to mental imaging, drawing upon Clements' developmental framework for the composition of 2-D plane figures. Clements delineated four stages in children's developmental progression for the composition of 2-D shapes (Clements et al., 2004, p. 168). The first stage, termed "Precomposer" denotes a phase where children are unable to combine shapes to form a larger shape. The second stage, "Piece Assembler," involves children placing shapes contiguously to create pictures. The third stage, "Picture Maker," signifies the ability of children to concatenate shapes to form pictures where multiple shapes serve a unified role. The fourth stage, "Shape Composer," indicates children's capacity to combine shapes to generate new shapes or complete puzzles. In addition to these four stages, Clements et al. (2004) described three detailed developmental progressions in shape composition. "Substitution Composition" is the stage where children intentionally form composite units of shapes, understanding, for instance, that two trapezoidal pattern blocks can construct a hexagon. "Shape Composite Iterator" marks the stage where children construct and manipulate composition units while maintaining the integrity of their pattern shapes. Lastly, "Shape Composer with Superordinate Units" represents the stage wherein children build a unit and iterate it to create new units, showcasing an advanced level of spatial reasoning.

Preliminary observations were conducted to assess the developmental stages of 2-D and 3-D construction in four- and five-year-old children, particularly in geometric toy play with Polydrons. Polydron shapes, including equilateral triangles, isosceles triangles, squares, pentagons, and hexagons, were employed because of their ease of manipulation, even for young children. During free play at a nursery school, four-year-olds demonstrated the developmental composition stage of "Picture Maker," connecting 2-D images to represent people and flowers using Polydrons. Additionally, both four- and five-year-olds exhibited the "Shape Composure" stage, connecting the same shapes on a plane to create larger 2-D flat shapes.

The developmental stages of constructing 3-D solids were further observed in the work of five-year-olds. The initial stage involved constructing cylinders and cones surrounded by squares or isosceles triangles, representing the "Shape Composure" stage for 3-D solids. Subsequently, the five-year-olds created 3-D pyramids by connecting triangles to the sides based on squares, pentagons, and hexagons. The second stage featured the construction of solids with multiple faces at the bottom and sides, including a child producing a large composite solid by combining multiple cubes, reflecting the "Shape Composure Iterator." stage. The third stage saw the creation of complex 3-D structures combining various 3-D objects, representing the "Shape Composer with Superordinate Units" stage. To enhance a child's mental imaging, progressing through the developmental stages of 3-D solid construction through rich geometric experiences is crucial (Clements et al., 2004). Encouraging geometric play involves providing free geometric toys; however, not all children may be equally interested. Furthermore, some children may face challenges owing to insufficient finger dexterity. Therefore, it is essential to establish an environment that promotes 3-D object creation and offers support based on each child's developmental stage.

However, current spatial training programs often lack child-driven choices (Vogt et al., 2018) and may be distant from children's play-based lifestyles (Bruce & Hawes, 2015). To address this issue, research that focuses on play-based lifestyles that foster spatial abilities tailored to individual stages of geometric development is needed (Weisberg et al., 2013). Therefore, this study aimed to create an environment that motivates children to play and develop their geometric abilities, mainly focusing on the following research question:

RQ: How can teachers and classes motivate children's geometric play and develop their individual geometric abilities?

CONCEPTIONAL FRAMEWORK: GUIDED PLAY

Kindergarten educators often employ mathematics training programs to explicitly develop mathematical abilities (Sakakibara, 2004). However, few comprehensive studies have compared the effectiveness of these approaches in terms of learning outcomes for all children, including those with different abilities (Vogt et al., 2018). In preschool classrooms, teachers must create an environment in which children are free to interact with the content and participate in play according to their abilities. (Chien et al., 2010).

Two prominent pedagogical methods frequently contrasted in preschool education are direct instruction and free-play (Brant, 2013). Direct instruction involves the teacher playing an active role in instructing students, who largely assume passive roles. In contrast, free play allows children to choose their own activities and play independently. Children engaged in free play may encounter challenges in achieving learning goals as they might not be guided to focus on appropriate dimensions (Weisberg et al., 2013). Weisberg et al. (2013) emphasized that playful and child-centered approaches, incorporating some level of adult scaffolding, prove more effective than directed instruction in achieving outcomes in young children. Guided play has emerged as a balanced and effective pedagogical approach in preschool education that integrate the learning objectives of adult scaffolding while maintaining a child-directed environment (Weisberg et al., 2013). In guided play, adults initiate the learning process, set learning goals, and are responsible for maintaining focus on the objectives that guide children's discoveries. The training program demonstrates significant benefits for children with very low competency, whereas the play-based approach serves all children, irrespective of their competency levels, ranging from low to high (Vogt et al., 2018).

In early childhood education in Japan, as outlined by the Ministry of Education, Culture, Sports, Science and Technology (2018), interaction between children and teachers plays a crucial role in fostering curiosity and inquisitiveness. Teachers strive to establish an environment that encourages children to take initiative, allowing them to act independently and engage with both peers and educators to deepen their learning experiences. Therefore, guided play aligns seamlessly with the principles of Japanese early childhood education and has emerged as an effective method for nurturing individual geometric abilities. This study specifically investigated how guided play can support children in learning about geometry. As part of our approach, while guiding children to tackle more complex 3-D figures, we encouraged them to set their own guided play goals to enhance their geometric abilities. The rationale behind this approach is rooted in the belief that by allowing children to establish their own goals, they would be more motivated to undertake the construction of more intricate 3-D solids. This personalized goal-setting process was intended to empower the children and engage them in their geometric learning journey.

METHODS

The data for this study were gathered through experimental lessons on shapes conducted at a local nursery in Tokyo, Japan. The objectives of this study were to assess the efficacy of guided play in teaching and learning 3-D shapes and to identify the outcomes of children's learning experiences through guided play. A design-based research method (Bakker & Van Eerde, 2014) was employed for a detailed retrospective analysis to elucidate the educational environment and teacher guidance that contribute to the development of individual geometric ability. Three experimental classes were held on February 18, 19, and 21, 2020, at a local nursery school in Tokyo. Each lesson lasted 40 minutes and was conducted by the author with teacher support. In all, 18 children (8 girls and 10 boys) aged five and six years participated in the class. This guidance plan was approved by the nursery school. The purpose of the class was communicated in advance to parents and to the children whose parents agreed to participate. Three lessons were video recorded to understand the stages of geometric development of all the children and were used to reflect on each child's 3D-making history. All children's works were photographed and recorded by trained research assistants for use in personal development records. Excerpted photos were used for documentation.

In the first class, children played freely with the geometric toy Polydron and created their own geometric works. After the class, the author and class teacher classified and organized the objects created by the children. We then discussed and selected some three-dimensional objects that could challenge the children based on their abilities and created documentation for the next lesson.

Guided play was conducted in the second and third lessons. At the beginning of the first guided play session, we presented documentation of the previous three-dimensional work to the children and discussed the names and characteristics of the three-dimensional objects to encourage them to think about what they wanted to create next and guide their own goals. After the discussion, the author told the children that they could try new three-dimensional objects or imitate the work that their friends had made. During the class, the author and class teacher asked each student what they were making and encouraged them to create it. In addition, if a child was not motivated to work on it, we asked them what they wanted to make and helped them create it.

After the first guided play, the author and class teacher classified and organized the objects that the children created. Depending on the children's abilities, we discussed the goal of creating a three-dimensional object which was more difficult than that in the previous lesson, selected multiple three-dimensional objects, and created documentation. At the beginning of the second guided play, the author facilitated a discussion of the characteristics of the three-dimensional object that we wanted the children to try and proceeded with the discussion in a way that encouraged them to want to create that object. During the class, the author and teacher asked each child what they had created and encouraged them. The author worked with the class teacher to ask children who were not keen on playing with geometric toys what they wanted to create and then helped them create it. This collaborative approach aims to ensure a supportive and engaging learning environment for all participants.

RESULTS

The first lesson involved free play with geometric toys, specifically Polydron, in which the author gathered the children and instructed them to create something independently or collaboratively with their peers. The observations during this lesson revealed five developmental stages in the construction of 2-D and 3-D figures. The first stage saw the connection of the same shapes to create a 2-D surface, exemplified by three children creating a flat surface with hexagons, marking the “Shape Composure” stage for 2-D figures.

The second stage involved creating cones or columns by connecting the same shape to the sides of a base, resulting in four children constructing triangular and hexagonal prisms, signifying the “Shape Composer” stage for 3-D solids. Two of the four children constructed hexagonal prisms, while the remaining two constructed a triangular pyramid and a hexagonal pyramid. The third stage entailed creating solids with multiple faces at the bottom and sides, with five children developing solids, such as a spaceship and a slide, representing the “Shape Composure Iterator” stage. The fourth stage involved creating solids by connecting multiple shapes, with four children making a star and a pillar, reflecting the “Substitution Composure” stage. The fifth stage comprised complex 3-D solids, including spheres created by connecting pentagons and hexagons, illustrating the “Shape Composer with Superordinate Units” stage.

The second lesson introduced guided play, in with the author showed documentation of the shapes created individually by the children and discussed their characteristics. Following the discussion, the children were encouraged to replicate their friends’ creations or design new 3-D objects. Peer influence was leveraged to challenge the children to create more complex 3-D objects. The children were instructed to seek assistance from the teacher if needed. Table 1 presents the 2-D and 3-D objects created during the second lesson.

The following characteristics were observed in the works created by the children during the second lesson: First, three children connected squares to create a flat 2-D surface. In the previous lesson, these three children made a flat 2-D surface by connecting hexagons. These children were still at the “Shape Composer” stage of developmental progression for the composition of 2-D shapes (Clements et al., 2004). Second, six children started to spin tops connected to eight isosceles triangles. In addition, the number of children who formed closes polyhedral (henceforth, “spheres”) by connecting pentagons and hexagons increased from two to five. After completing the second lesson, the author and classroom teacher discussed two tasks: assisting children with difficulty constructing 3-D solids and challenging children who were enthusiastic about spinning tops to create more difficult 3-D objects. For the first task, we showed the documentation of tops at the beginning of the third lesson to guide children who had difficulty making 3-D solids and to inspire them to make tops. They could connect the faces by pushing them together on the floor and easily expanding the flat 2-D faces. However, when making a top, it is necessary to connect the isosceles triangles with the fingers; therefore, we thought that making a top would develop hand dexterity. The second task was to inspire the children to construct more complex 3-D solids. For this purpose, we presented the documentation of the sphere and discussed its characteristics. In the second lesson, several friends became interested in the large sphere that one girl created by connecting pentagons and hexagons. To create a large sphere, three hexagons must be connected at one vertex and pentagons must be connected between them. This requires strength in the children's fingertips because they must connect them face-to-face in the air. Creating a sphere can assist children in developing their fingertips and understanding the characteristics of 3-D solids that can be made by connecting pentagons and hexagons.

	First lesson (Free play)	Second lesson (First guide play)	Third lesson (Second guided play)
2-D Shape Composer	3 Hexagonal tiling (3)	3 Square tiling (3)	0
3-D Shape Composer	4 Hexagonal prism (2) Triangular pyramid (1) Hexagonal pyramid (1)	6 Tops (6)	5 Tops (5)
Shape Composure Iterator	5 Spaceship (4) Slider (1)	0	1 Castle (1)
Substitution Composure	4 Star column (4)	3 Star column (3)	1 Star column (1)
Shape Composer with Superordinate Units	2 Sphere (2)	6 Sphere (5) 3-D cat face (1)	9 Sphere (8) Hexagonal composite solid (1)

Table1. Types and numbers of 2-D planes and 3-D solid figures created by children

The third lesson consisted of guided play session. Commencing the lesson, the author presented documentation of the previously constructed tops and engaged the children in a discussion of their characteristics. One boy suggested that alternately arranging triangles of two colors to create a frame would result in a visually appealing effect during spinning. Another boy recommended spinning the top as flat as possible to prolong the spin duration. Following this, the author showcased the documentation of the sphere and flat surfaces, prompting the children to identify the differences. One boy noted that the sphere was round, and the surface was flat.

Concluding the discussion, the author asked the children what they wanted to construct next, guiding them toward the challenge of creating spheres. Eight children formed three groups and collaborated to construct the spheres. Simultaneously, five children chose to construct tops and derive enjoyment from playing with them. Notably, two of the children created a 2-D square tiling during the second lesson. Additionally, a child who had constructed a 2-D square tiling in the preceding lesson evolved in geometric development, progressing from the “2-D Shape Composure” stage to the “Substitution Composure” stage. This advancement involved using composite units of shapes to construct 3-D solids, as exemplified by the creation of a star-shaped cylinder while referencing the documentation. The child ingeniously used five equilateral triangles to form a pentagon, showcasing a more sophisticated level of geometric composition.

DISCUSSION

The objective of this study was to explore how teachers and classes can enhance the motivation for individual geometric play and foster unique geometric abilities through guided play using documentation. Preliminary discussions using documentation before the two guided play sessions facilitated the children’s contemplation of the challenge of creating more intricate 3-D solids.

An analysis of the changes in the 3-D works created by the children, as depicted in Table 1, reveals a noteworthy increase in the number of children engaging in 3-D creations at more advanced developmental stages in each successive round. Furthermore, guided play allows us to offer personalized guidance based on individual geometric abilities. For instance, a girl who had previously created a 2-D flat plane twice in a row expressed uncertainty about what to make the third time, but when prompted by the classroom teacher, she chose to make a top. While connecting the triangles with the teacher’s assistance, the girl gained confidence and successfully connected the last five triangles, creating a top independently. This illustrates how environmental settings, including playing with geometric toys and engaging in discussions using documentation, have the potential to motivate children and propel them through various developmental stages of geometry.

In future research, we will examine the broader impact of guided play on children’s everyday play and the ongoing development of their geometric skills.

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TOPIC B

Curricular and methodological approaches

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CLASSROOM SIMULATORS: A NEW TRAINING APPROACH TO INVESTIGATE TEACHERS' PROFESSIONAL KNOWLEDGE AND SUPPORT ITS DEVELOPMENT

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This paper offers an empirical study for supporting teachers' educators. It presents a new form of professional learning based on the use of a classroom simulator in training courses. Among the various research questions that this training approach makes it possible to tackle, we choose to focus in this paper on the way in which teachers' beliefs and geometry knowledge can be revealed by means of the simulation.

INTRODUCTION

This paper offers an empirical study for supporting teachers' educators. It presents a new form of professional learning based on the use of a classroom simulator in training courses. The design of the simulator is guided both by the outcomes of research into the characteristics and needs of geometric learning at lower secondary school level, and by the effective difficulties experienced by these students in solving geometric tasks. The simulated class situation consists of solving a problem within a Dynamic Geometry Environment (DGE) starting by drawing and manipulation, to pass afterwards to conjecturing and analytic visualization and then to proving.

Among the various research questions that this training approach makes it possible to tackle, we choose to focus in this paper on the way in which teachers' beliefs and geometry knowledge can be revealed by means of the simulation. Indeed, Thomas and Palmer (2014) highlight that teachers not only need a special kind of knowledge for technology implementation, but that beliefs play a crucial role since they frame, guide and filter situations, actions and intentions. Training courses that directly address teachers' competencies are therefore regarded as important in supporting the development of knowledge, beliefs and practices (Hegedus et al. 2017). The specific research question we address is: what do trainee teachers' simulated practices reveal about their professional knowledge and beliefs about geometry teaching and particularly about the role of DGEs in solving geometric problems?

The rationale behind our questioning is informed by a range of factors and contextual elements. First, in French curriculum, students are expected to know how to draw figures before engaging in geometrical reasoning. Constructing with instruments is compulsory and is expected to help students passing from tangible to abstract and understanding geometric concepts. DGEs could be the appropriate place to make this passage. Constructing tasks play thus a key role in curricula, as they help students to move from spatial to geometrical thinking from "seeing on a drawing" to proving (Mithalal & Balacheff, 2019). Second, it is nowadays widely shared that DGEs could contribute in developing geometrical learning, particularly in supporting solving geometrical problems. It is acknowledged that they have the potential to "encourage both exploration and proof, because it makes it so easy to pose and test conjectures" (Hanna, 2000, p. 13). Nevertheless, researchers, such as

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Mariotti (2006) underline that even if dragging within DGEs provides the students with strong perceptual evidence that a certain property is true, this may reinforce a critical point in the teaching of geometry concerning the relationship between empirical evidence and theoretical reasons. Finally, one of the challenges that research has attempted to meet is to design and/or analyze situations where the connection between spatial and geometric reasoning is effective. Fischbein (1993) has emphasized that one of the main tasks of mathematics education (in the domain of geometry) is "to create types of didactical situations which would systematically ask for a strict cooperation between the two aspects [image and concepts], up to their fusion in unitary mental objects" (p 161). However, many researchers highlighted that although the situations designed and analyzed were very promising, they have been little used in teaching. The problem of the usability of these situations and "their communication to teachers has not been yet solved" (Bloch & Pressiat, 2009, p 66 [personal translation]).

To address these issues in the context of teacher training, we designed and used a computerized classroom simulator (SIC). The advantage of using such simulator is that it enables teachers to experiment classroom practices in an environment that is both safe - all types of experiments are allowed (no impact on real students) - and allows experiments to be repeated. Indeed, simulation is often used in professional training because it is less risky and speeds up the process of acquiring experience (Pastré, 2005). Moreover, the designed training course includes an analysis of the geometric problem, as well as a collective discussion of the concepts involved and the role of DGEs.

LEARNING GEOMETRY IN THE TRANSITION FROM PRIMARY TO SECONDARY SCHOOL

This section is far from being exhaustive, its aim is rather to highlight some of the research findings on which our work is based, firstly with regard to the learning of geometry in general, and secondly in the specific context of DGEs. A significant amount of research has been devoted to studying students' learning in the transition from the tangible geometry of elementary school to the reasoning geometry of early secondary school; characteristics and difficulties have been identified.

Within French research, two main conceptual perspectives inform the field of geometry didactics. Houdement and Kuzniak (1998) distinguish three paradigms of geometry: natural geometry (GI), natural axiomatic geometry (GII) and formalist axiomatic geometry (GIII). The passage from GI to GII is identified as difficult to implement in the classroom and even if students entering secondary school have certain knowledge within GI, they don't necessarily link geometric constructions to the properties and theorems that justify them; their geometry tends to amalgamate GI and GII. Berthelot & Salin (1998) differentiate between spatial and geometric knowledge and Parzysz (1991) shows that spatial objects and physical representations play an important role in the conceptualization of geometrical concepts. Moreover, Sinclair et al. (2019) underline the strong link between drawing, spatial reasoning and the learning of geometry. This leads us to reflect on the place of construction in geometric work. It is not the precise technical mastery of drawing processes with instruments, but rather the mental objects constructed by the student to accomplish this process, and what ensues for a better apprehension of the figure (Duval, 2006) that is at stake in the school-level transition that interests us here.

The use of DGEs in geometry learning and teaching is nowadays recognized as offering novel ways of carrying out geometrical activities in mathematics education and participating to the progress in students' conceptualization, influencing thereby different aspects of geometry learning. Within an exploratory approach, research studies stress the key role of dragging in forming a mathematical conjecture (Healy, 2000). By making it possible to drag and drop points and to multiply experiments, DGEs encourage access to conjecture and possibly reasoning to validate this conjecture. Furthermore, Hoyles and Jones (1998) claim that dynamic geometry, supported by "what if" and" what if not" questions, has the potential to promote links between empirical and deductive reasoning. Yet, teaching approaches involving a cycle of exploration-conjecturing-proving require thoughtful design; "engaging students in situations which make them aware of the constructive character of mathematical activities, especially those involving conjecture and proof, possess complex challenges" (Durand-Guerrier et al., 2012, p. 364). Even if dragging is a powerful tool for conjecturing, the resistance of the objects drawn brings a conviction that could slow down students' understanding of the need to prove by arguments. Similarly, while the above comments show the importance of manipulating geometric figures in a DGE, constructing these figures using the software's tools is a separate task from the conjecturing/proving process. Thus, the importance of designing appropriate tasks and the role the teacher should play in their implementation seem even more important in order to meet some learning aims.

SIMULATOR AND TEACHER TRAINING COURSE DESIGN

Tasks implemented in SIC: transition between spatial and geometric reasoning within DGEs

Designing a Classroom Simulator means first and foremost to find a teaching situation that is conducive to raising the professional and mathematical questions that will become the focus of training. To encourage teachers to question the role of geometric construction within DGEs in the proof process, we used the problem described in Figure 1. This task considers the epistemological aspects identified by Lesnes-Cuisiniez (2021) in his synthesis of research on the double break between physical and theoretical geometry. These include the need to distinguish figure drawing, instrumental mobilization, heuristic arguments, and theoretical validation. We also added a promising idea: using a problem in which perception is challenged to highlight the need to rely on reasoning rather than measurement or perception. For SIC, we have chosen a problem where perception is distorted (segment [AC] seems larger than segment [EG] in Figure 1), which is reinforced by the question posed which is not: "are the lengths the same?" but "Say which of the two segments is larger."

Make the figure shown in the opposite.
Say which of the two segments, [EG] or [AC], is larger.
Explain why.

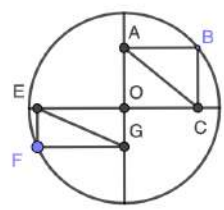

Figure 1: Problem implemented in SIC.

This problem is available in several resources for lower secondary school teachers in paper-and-pencil environment; we have adapted it to be solved in a DGE. It is considered both in these resources and by the teachers who have tested it as an open problem that is simple to understand and enables the learner to engage in an experimental approach. To begin, the student is asked to draw the figure in a DGE. This requires specific knowledge that is not the same as that required to draw the figure in paper-and-pencil, notably the order of construction steps, the commands available in the software (the software commands that the teacher can choose to authorize or not such as "perpendicular bisector") and the robustness of drawing under dragging (Healy 2000). Afterwards, the student is asked to conjecture the answer to the question: "Say which of the two segments is larger." Finally, the student is asked to explain why, which refers to the process of argumentation and/or proof. From a didactic point of view, there are several analyzable difficulties. The one we are aiming at here is that there is little connection between the construction work in the DGE and the conjecture work. In fact, the conjecture can be made by using the "measurement" command or by moving the points connected to the circle to superimpose them; the construction command adds nothing to the conjecture. The proof phase requires to have noticed that the two segments are the diagonals of two rectangles and that the second one is in fact the radius of the circle. The construction does not allow identifying the rectangles (since it rather directs towards the idea of an orthogonal projection). In conclusion, being able to manipulate the figure in the DGE is important and sufficient to conjecture and prove; constructing it step by step doesn't provide necessarily additional means to complete these two tasks.

The Design of SIC

Our goals in building SIC (available at http://www.fabien-emprin.ovh) are to get trainees to reflect on their practices, using their professional knowledge to solve a professional problem and to test several hypotheses by having the possibility to try again and again. To achieve these goals, we used part-scale simulators, which enable the user to make choices (what the teacher says, to whom he/she says it: one student or the whole classroom, what he/she does: act directly on a student's screen or projecting the same screen on the white board, etc.) and to see the effect of his/her choices on what students do (actions on the DGE, verbal responses, etc.). According to Pastré (2005), it is a way of "impoverishing the situation to make it more accessible to learning" (p. 27). We use a non-random model in which an action always has the same effect each time it is repeated, with no introduction of random phenomena, enabling to carry out analyses between trials by the same user and between users. In this sense, it's an experience-building accelerator.

To design a simulation, combined with a tool for computing these interactions, we needed to define the possible choices of the teacher and the effects of these choices for students in real classrooms. Our method was based on recording enough classroom settings. To do this, we set up a learning situation and a support-resource accessible to the teacher to implement in his/her classroom (which could have been found in an educational resource (textbook, website)). We carried out an a priori analysis of the situation, based on didactic knowledge of geometry, highlighting possible choices in terms of tasks and class management. We then provided the situation to teachers, who were free to interpret it according to how they plan to carry it out in their classrooms. By observing these classrooms' settings, we gathered information that enabled us to compare the a priori analysis with actual implementations, and to identify the training knowledge that can be updated in the simulation. We observed around ten classroom settings, then built a simulation reproducing the teachers' choices and the students' reactions, while respecting the average proportion observed.

The feedback provided by the simulator is twofold: during the simulated session, the user can access the student's work, either by observing it (on the DGE screen, for example) or by questioning it; at the end of the session, the user can access what each student found (his/her construction, his/her answer to the question) and what the student remembers one week after the session. The latter could help the user measure the effect on students' learning.

A TRAINING COURSE USING SIC

First, we emphasize that we do not consider that SIC has any intrinsic value; playing the simulation doesn't inherently develop professional skills. It's the debriefing that follows the simulation that allows questions and analyses to emerge and brings out the trainees' professional knowledge and thus possibly enables effective training to take place (Pastré, 2005). Secondly, we consider that the trainee teacher's professional knowledge develops by going back and forth between the deployment of existing knowledge and the information derived from the interactions first with SIC, and afterwards with other trainees (with and without the trainer's interventions). The training course centered on the simulator, as described below, aims both to immerse teachers in a (simulated) professional context and to provide reflective feedback on their choices and actions with the simulator. An alternation of moments of experimentation and discussions is therefore supposed to develop the teachers' professional knowledge.

The course, designed for in-service secondary school mathematics teachers, is organized in five phases following the more general training approach proposed by Abboud et al. (2022). This process is presented to trainees at the beginning of the course. The aim of the first phase is to make trainees analyze the mathematical task possibly by carrying it out themselves and anticipating possible students' difficulties. The second phase takes the form of a group discussion on what the trainees were able to anticipate throughout the task analysis: what problems and difficulties the students are likely to encounter, what objective should be assigned to the teaching/learning situation… During the third phase, the trainees use SIC. Thanks to the fact that the time is simulated (independent of real time), they can make several trials, but without having to complete the simulation each time. At the end of each simulation, they obtain summary information on the students' learning, which enables them to adapt their choices and test new hypotheses during the next trial. The fourth phase is again a collective discussion on the performance of the virtual students and the comparison with the real students (their own) and ways of helping them to learn geometry when accomplishing the task. Finally, during the fifth phase, the trainees are asked to design a class session based on the geometric task they had tried out and its implications in terms of promoting students' geometric reasoning.

It is during phases 2 and 4 that the trainer can pick up on elements of the teachers' practices and can intervene by supplying/sharing knowledge for professional development. It is these two phases that interest us in this article. In what follows we propose to analyze some exchanges from phase 2 and others from phase 4. In addition, as SIC makes it possible to keep track, in the form of a chronological table, of the actions undertaken by the trainee teacher during the various trails made during phase 3, we provide examples of these traces in order to gain an overall view of how the training session unfolded.

Analysis of interactions from Phase 2

After clarifying, at the request of the trainees, that the virtual students had already experienced the DGE, the discussion turns to the difference between construction within a DGE and within paper-and-pencil environment; the trainer provides then several clarifications on this point. Given the risk of multiple difficulties, one of the trainees suggests that the drawing should be carried out on paper-and-pencil instead. This first exchange informs us that the teachers are primarily concerned with the students' ability to construct the figure correctly so that they can then engage in conjecture. They have the feeling that using the DGE could generate several difficulties. This leads them to favor to abandon its use for the construction phase. Another exchange takes place with the trainer on the construction process, which is different in paper-and-pencil, particularly about constraints on the order of construction. We can notice here that, although the task was chosen for its potential in the DG environment in terms of two aspects - construction and manipulation - the teachers seemed doubtful about the usefulness of the first and preferred to start with paper-and-pencil to ensure less difficulty in the construction phase and favorable conditions to engage in the manipulation one. In the course of the ensuing discussion, we observe an exchange relating to the conjecture and the usefulness of using a DGE to "see it" in the sense highlighted by Mariotti (2006) where dragging provides the students with strong perceptual evidence that a certain property is true. The trainees emphasis the dynamic nature of the software and the fact that it allows them to visualize an infinite number of figures (thus permitting the generalizing the observed property).

The whole discussion in phase 2 thus seems to show teachers' beliefs that distinguish between the role of the DGE in the drawing process and its role in the conjecturing own. In the first case, it seems more efficient to go back to the usual paper-and-pencil tools to make sure that the students are constructing the figure correctly (by using skills they have already acquired) before engaging in a more complex (for this class level) conjecturing task, where the dynamic aspect of DGEs seems to be unequaled in paper-and-pencil and is therefore very useful.

Analysis of interactions from Phase 4

Following the same thread, we observe an exchange in phase 4 initiated by a trainee who suggests giving the students the "right" figure from the start, so that they can move on to conjecture. The trainer then informs him that there were three variations of this proposal within SIC to help students make the right figure, without giving them everything straight away. This gives rise to a discussion about how to help the students make the conjecture. In the ensuing discussion, we observe the trainer attempting to provoke a reflection on the fact that giving the students the correct figure without them having constructed it themselves doesn't guarantee that they can engage in the conjecturing process. This attempt fails to achieve its objective, and the trainees prefer to direct the discussion towards ways of finding and/or validating the conjecture. They suggest providing "instrumental" help to students: make the software display the measurements so that equality can be seen. Another possibility of instrumental help was raised in the discussion: asking students to superimpose the points but recognizing it as a special case that doesn't lead to generalizing the conjecture.

Using the simulator seems thus to make it possible to question the trainees' professional knowledge and make them aware of the link that can exist between the construction phase and the conjecture phase and the role that the DGE's functionalities can play in the latter.

Analysis of traces from Phase 3

The analysis of these traces aims first to identify the moments of transition from the construction phase to the conjecture phase in the various trials and second to examine how trainees use SIC as a professional tool for experimentation. We give here the example of the traces of trials of two trainee teachers. *Greg* carried out 3 trials. On the first one, the transition to the conjecture took place at the 38th minute. Noticing that this does not leave enough time within an average duration of a classroom session (55 min) to complete the conjecture phase, he suggests that the task has to be carried out in two sessions. In the second trial, he thus took the time needed for the construction (43 min with session duration of 109 min). He then questioned the need for students to engage in the construction, and rather chose at the third trial to give the figure that had already been constructed switching to the conjecture phase after 12 minutes. *Chris* carried out 4 trials. The first two trials were used to explore the possibilities of SIC, in particular by going to check on all the virtual students during the construction phase and asking some of them whether the figure resists or constructing the figure for them. On trial 3, he accelerated the construction phase by showing the correct figure to the class (at the 33rd min) before asking for the conjecture and giving two minutes later the indication to display the measurements of the two segments. He didn't move on to the proof phase until the 75th minute and he accelerated this phase by immediately asking students to put OB in the radius position and measure it. On trial 4, he first did much the same thing before moving to a pooling of the conjecture at the 62nd minute and then writing down the results at the 73rd minute to close the session.

In broad terms, the analyses show that the trainees who appropriate the simulator as a tool for testing and experimenting, carry out trials that evolve into shorter construction phase and realize that it is the manipulation of the figure and not the drawing that is useful for the proof.

CONCLUSION

This exploratory study highlights the potential of simulation-based training approach to reveal teachers' professional knowledge of geometry and particularly about supporting students to link geometrical construction and geometrical reasoning.

The aim of using SIC was also to engage teachers in collective reflection on their classroom practices in geometry, with SIC acting as a revealer of these practices. We showed how it has revealed to the trainer some of the trainees' geometric beliefs and practices. We also looked at the knowledge that the trainer considered relevant to contribute in situ to promote professional learning on key elements to geometry teaching enhanced by dynamic geometry.

While geometric construction work, under certain conditions, is conducive to the transition from GI to GII, the technical skill of drawing, whether on paper-and-pencil environment or in a DGE, is a specific skill. However, simulated practices show that teachers remain attached to this activity, with certain confusion between constructing and conjecturing. Feedback from the simulator may lead them to reconsider this role.

This study shows the potential of simulated practices for both the trainer and the researcher. For the moment, the data extracted from the software can only be used by the researcher because of its complex format, but it is possible to imagine transforming it into learning analytics so that the trainer can also use it directly in his/her training sessions.

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EXPERT JUDGMENT FOR CONTENT VALIDATION OF A QUESTIONNAIRE ON THE LEVEL 5 DEFINITION PROCESS WITHIN THE VAN HIELE FRAMEWORK

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This paper presents the process of designing a questionnaire aimed at measuring the degree of acquisition in the definition process in Geometry among individuals at advanced levels of the Van Hiele model. Our theoretical framework encompasses not only the seminal references of the Van Hiele model but also prior work by the authors in which indicators were designed to characterize the abilities of a high-level Geometry student. In conjunction with the questionnaire, we elucidate the content validation process, conducted through expert judgment. This process entailed semi-structured interviews to refine the task design and closed-ended questions regarding the agreement between the questionnaire items and the indicators describing the definition processes at Van Hiele level 5. As a result, the questionnaire is presented along with its corresponding agreement indices as assessed by Fleiss's Kappa.

THEORETICAL FRAMEWORK

In this section, we will briefly introduce the fundamental aspects of the Van Hiele model (Van Hiele, 1957) and the prior research that underpins the questionnaire's design. Our primary focus lies on the fifth level of the model, as an in-depth understanding of this level is of particular interest for analyzing geometry learning at the university level. Given the limited existing research on this level, we initially had to design and describe its indicators for each process (Arnal-Bailera & Manero, 2023). Subsequently, our task is to design a questionnaire that facilitates an assessment of the acquisition of this level among undergraduate mathematics students. The specific objective here is to delineate the ongoing validation process of a questionnaire designed for assessing the definition process.

The Van Hiele model (Van Hiele, 1957) stands as one of the most significant theoretical frameworks pertaining to the teaching and learning of geometry across all educational tiers. This model posits the existence of five distinct levels of geometric reasoning. The highest level (fifth-rigor) can be succinctly summarized as follows: individuals at this stage can compare systems grounded in different axioms and can investigate various geometries in the absence of concrete models. What is essential is not merely the ability to work within different geometries or with varying metrics, but rather the capacity to establish relationships among them. It is imperative to comprehend that shifting geometric contexts (by changing the geometry or metric) yields objects of distinct natures, propositions with differing demonstrations, and varying mathematical facts that are transferable between contexts, while others are not.

In previous works (Arnal-Bailera & Manero, 2023), we have developed specific indicators delineating how individuals operating at a Van Hiele level 5 should manifest these abilities when mobilizing skills associated with the processes of definition, proof, and classification. Specifically, when delving into the process of definition at level 5, we have identified five indicators (see Table 1): indicators Def1 and Def2 constitute the initial set of indicators that pertain to the influence of working within diverse geometrical contexts, whereas indicators Def3, Def4, and Def5 are associated with the rationale underlying the formulation of new definitions or the selection between existing ones. In the context of the definition process, the utilization and formulation of definitions are typically investigated as distinct aspects. At this level, the indicators that have emerged are predominantly linked to the formulation of definitions.

Indicator	Description
Def1	Constructs and uses definitions in different axiomatic systems.
Def2	Understands that defining a given mathematical object is not absolute, but is an action relative to the geometric context in which one works, implying for example that the defined object may have different properties in each context.
Def3	Defines new objects, for example, because it may be necessary to generalize existing ones or to prove a statement.
Def4	Understands that a definition arises out of the necessity to introduce a new mathematical object or to emphasize a property.
Def5	Compares equivalent definitions to choose the most interesting one, depending on the work to be done.

Table 1: Indicators for the definition process (Arnal-Bailera & Manero, 2023)

METHODOLOGY

The methodology employed in the design and validation of the questionnaire consisted of two phases. The first phase involved conducting three semi-structured interviews with experts who possessed teaching and/or research experience in the field of Geometry at the university level. During these interviews, the questionnaire was presented to the experts, and their opinions were sought regarding i) the mathematical difficulty of the proposed activities, considering Mathematics undergraduate students as future respondents, and ii) the correspondence between items and indicators in the process of definition. Based on the insights gathered from each of these interviews, certain aspects of the questionnaire were reformulated to ensure a clearer evaluation of certain items or to enhance the comprehensibility of the wording. This revised version was then presented to the following expert.

In the second phase, closed questionnaires regarding the agreement between indicators and items were sent to five Ph.D. holders in Mathematics with research experience. These questionnaires required a dichotomous response (Yes/No) concerning this agreement. Subsequently, using the SPSS software, the Fleiss's Kappa coefficient was analyzed to assess the content validity of the questionnaire (Meyer & Booker, 2001). Future validation processes require the administration and analysis of the questionnaire to undergraduate students.

RESULTS

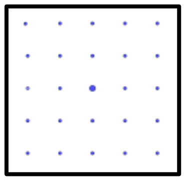
We present below some of the results obtained in each of the two phases. Due to space constraints, we summarize the most significant findings derived from the semi-structured interviews with experts, which contributed to the improvement of the initial versions of the questionnaire. After that, we provide an interpretation of a summary of the output obtained using SPSS about the Fleiss Kappa.

Semi-structured interviews

The first interview (conducted with Expert 1) reviewed the initial version of the questionnaire, revealing that the indicator def4 was not clearly presented. Consequently, the questionnaire would not be able to accurately assess the level of acquisition of this indicator.

Item 3
Normally to measure distances in the plane we use the Euclidean metric, which is defined as follows: Given two points, the distance between them is the length of the segment joining them. However, we can define other distances, such as the so-called postman's (or Taxicab) distance, which is defined as follows: the distance between two points is given by the shortest route joining those using only horizontal and vertical lines.
Item 3.1: If we define the circumference as the set of points that are equidistant (at the same distance) from another point, what is the shape of the circumferences with the Taxicab metric? Justify your answer.

Item 3.2: If we define a circumference as a set of points that are equidistant from another point, what shape do they have with the distance from the postman? Justify your answer. You can use the grid to draw the points that are equidistant from the point indicated.



Item 3.3: If we define a circle as a set of points that are equidistant (at the same distance) from another point. Look at the different drawings below:

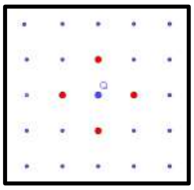
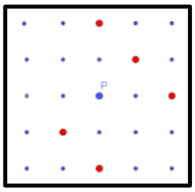
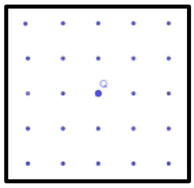
All the marked points are, with the Taxicab metric, at distance __ from point Q. 	All the marked points are, with the Taxicab metric, at distance __ from point P. 	What is the shape of the circles with the Taxicab metric? Justify your answer. 

Figure 1. Item 3 as presented to the first expert.

We analyze the moment when Expert 1 was reviewing the third item (see Figure 1): a discussion ensued regarding whether someone who correctly answers item 3.1 demonstrates an understanding that a definition arises from the need to introduce a new mathematical object or emphasize a property. The interview excerpt illustrates the utility of the item, with the expert suggesting a reversal of the ideas presented in this item. It was proposed to start by presenting specific cases and then inquire about the property that the red points have in relation to the blue, with the (imprecise at that moment) idea that asking about the property would prompt the need to define distance.

Expert 1: I don't quite see it. This task is similar to the previous one. It's essentially the same question with a different distance. Maybe we can approach it differently and start by asking what the red points in the middle square have in common with the blue ones. Let's see if someone mentions that they are at the same distance, without defining distance. Let's see if the need to define distance arises.

Interviewer: Yes, we could use diagrams like the one in the center and ask for a distance so that the red points are equidistant from P.

Expert 1: Or, without using the word "distance", what do the red points have in common with the blue ones? Or, define something that all these points have in relation to the blue ones.

Between the interview with the first expert and the second one (Expert 2), the idea explained earlier was implemented. However, the researchers found it more suitable to implement it in item 2, whose initial formulation can be seen in Figure 2, and its subsequent formulation in Figure 3. This decision was made to maintain the structure of the item 3 super-item (Jaime & Gutiérrez, 1994), which is convenient if the test would aim to measure the acquisition of lower levels than level 5.

Item 2
Normally, to measure distances in the plane, we use the Euclidean distance, which is defined as follows: given points A=(x_1,y_1) and B=(x_2,y_2), the distance between them is given by the length of the line segment that connects them:

$$d_E(A,B):=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

However, we can define other different distances, such as the **radial distance**, which is defined as follows:

$$d_R(A,B):=\begin{cases} d_E(A,B) \; if \; A,B \; and \; O \; are \; aligned \\ d_E(A,O)+d_E(O,B) \; otherwise \end{cases}$$

Here, O represents the origin of coordinates, i.e. , the point with coordinates O=(0,0). If we define a circle as the set of points equidistant (at the same distance) from another point, **what shape do the circles have with the radial distance**?

Figure 2. Item 2 as presented to the first expert.

The modifications to item 2 (see Figure 3) involved adding images with discrete sets of points that have a relationship with another point. Firstly, in Sub-item 2.1, the question asked was about identifying that relationship (i.e., what property these points satisfy). Then, in Sub-item 2.2, respondents were asked to construct the definition of the (continuous) set of points that satisfy that property. The continuous set was presented based on a single image.

On the other hand, Expert 1 pointed out that def5 was not reflected in item 2 because equivalent definitions were not being compared. Since it was also necessary to reinforce this indicator in the questionnaire, it was decided to expand this item with a Sub-item 2.3, including a new formulation of the definition of radial distance for comparison with the original. This new formulation is of a verbal nature and provides a more descriptive explanation of the process by which radial distance between two points is determined: radial distance between two points is the minimum length of the segment or segments that connect them and lie on lines passing through the origin.

In the following, we present an excerpt from the interview with the second expert (Expert 2) when discussing the revised version of item 2. On one hand, the expert finds it challenging to identify a specific relationship among the discrete set of points based on the examples presented, as multiple different relationships appear to exist. Consequently, the expert considers it appropriate to increase the number of images for both discrete and continuous cases. On the other hand, the expert acknowledges the presence of indicator def4 in question 1 but expresses doubts about its accurate evaluation in the form it had in that version. Additionally, the expert agrees that question 2 in the Sub-item 2.3 assesses indicator def5.

Expert 2: When I first thought about it, I gave an answer that does not say much: all points are at a positive distance... then I thought of something else... that the absolute values of the coordinates of the other points are larger... (...)

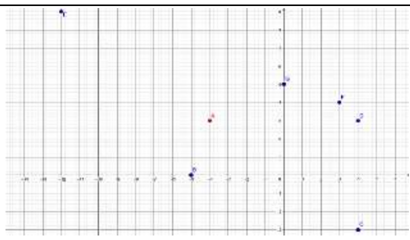
Expert 2: This one does not really evaluate anything [item 2.2]; it puts you in a situation to do this one [item 2.3], prepares you for the next one... It's like good marketing; they've created the need to construct a definition... now this one [item 2.3] will really evaluate something... def4 is important... you understand... it's a complicated thing, understanding... (...)

Interviewer: We could give item 2.2 a super-item structure with several images...

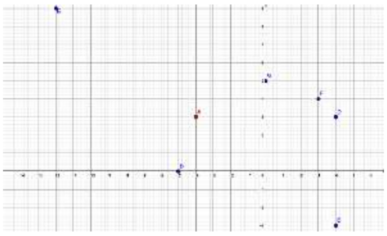
Expert 2: Yes, I could have four examples, and I want a definition for all four cases. That way, it generalizes, going from the concrete to the abstract... It's about understanding... you guys will know about that. About understanding, I don't know how to answer that. (...)

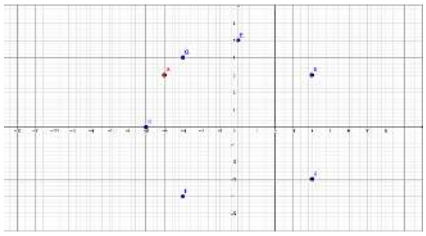
Interviewer: So, here we could put 4 images, ask for the property, and then here, another 4 images, and ask for the definition. That way, we would convince more of the presence of def4.

Item 2.1.A: Could you describe a property that all the blue points (B-G) have in relation to the red point (A) in the given image?

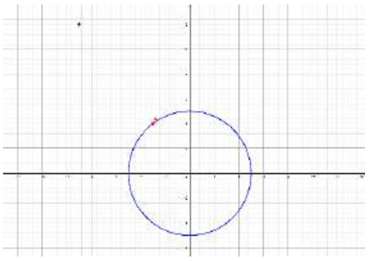


Item 2.1.B: Could you describe a property that all the blue points (B-G) have in relation to the red point (A) in the given images?





Item 2.2: In the given image, the blue points share a property with respect to the red point (A). Construct a definition for the set of blue points with respect to the red point (A). We will call this set S_A.



Item 2.3: Normally, to measure distances in the plane, we use the Euclidean distance, which is defined as follows. Given two points $A=(x_1,y_1)$ y $B=(x_2,y_2)$, the distance between them is given by the length of the line segment that connects them, that is,

$$d_E(A,B) := \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

However, we can define other different distances, such as the **radial distance**, which can be defined as follows:

Definition 1:

$$d_R(A,B) := \begin{cases} d_E(A,B) \text{ if } A, B \text{ and } O \text{ are aligned} \\ d_E(A,O)+d_E(O,B) \text{ otherwise} \end{cases}$$

Here, *O* represents the origin of coordinates, that is, the point with coordinates *O*=(0,0).

Question 1. Could you describe a situation/moment/context that can be modeled using this metric or in which this metric may be necessary/useful?

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Similarly, the radial distance can be described as follows:

Definition 2: The radial distance between two points is the minimum length of the segment or segments that connect them, with these segments contained in lines that pass through the origin. As you can see, these two definitions, one algebraic and the other verbal, are of different natures.

Question 2: Compare both definitions (algebraic and verbal) by describing in which contexts you would use one and when you would use the other.

Figure 3. Item 2 as presented to the second expert.

It should be noticed that a page break has been included in Sub-item 2.3 since the second part of this Sub-item could produce answers to the previous part, thus the page break indicates that when the second part is reached the previous responses cannot be changed.

Through the questions and answers in the interview with the second expert, the conclusion was reached to reformulate item 2 to facilitate its resolution. In Sub-item 2.1, four discrete sets of points were presented (in addition to the ones shown in Figure 3-Sub-item 2.1), and participants were asked to infer, for each set, the property that connects them to another point marked in a different color. Then, in Sub-item 2.2, another four sets were presented (in addition to the one shown in Figure 3-Sub-item 2.2), this time continuous, and participants were asked to construct the definition for all of them. The objective was for the first group of sets to prompt reflection on the existence of a common property among all blue points with respect to the red one. Subsequently, a definition would be constructed based on this property, which could be related to the presence of def4. No changes were made in Sub-item 2.3.

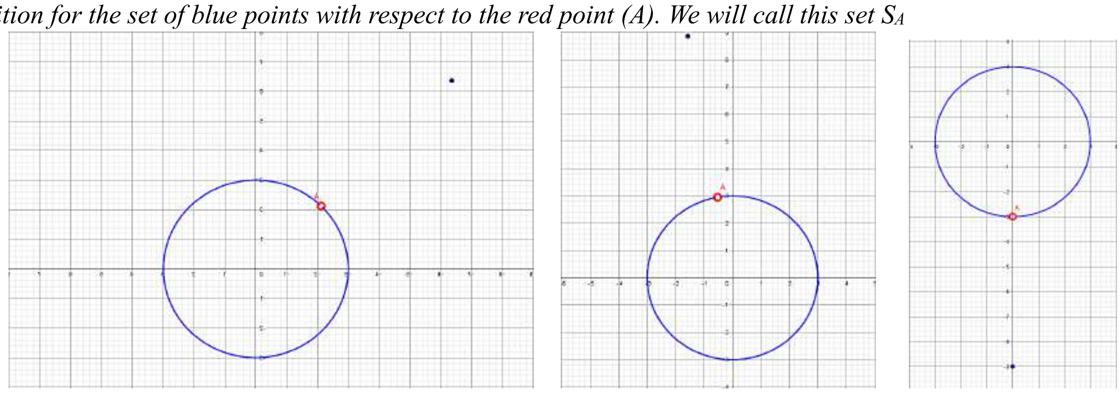
This new version of item 2 was presented to the third expert who was interviewed (Expert 3). Despite the changes made, this expert concurred with expert 2 regarding the difficulty in eliciting a relevant property when discrete sets of points were presented in Sub-item 2.1.

Expert 3: The first Sub-item is exceedingly challenging to solve; it is conceivable that some students may provide an unexpected property, in the sense that a discrete set of points always possesses a connecting property, even if it takes the form of a 27th-degree equation. This approach would not be beneficial for addressing the following Sub-item.

Interviewer: Would you consider a version without the first Sub-item (the one concerning discrete sets) to be better?

Expert 3: Yes, as the first Sub-item does not contribute anything, and no indicators from the ones you propose can be identified.

Item 2.1: In the given images, the blue points share a property with respect to the red point (A). *Construct a definition for the set of blue points with respect to the red point (A). We will call this set S_A*



Item 2.2: Normally, to measure distances in the plane, we use the Euclidean distance, which is defined as follows. Given two points A=(x_1,y_1) y B=(x_2,y_2), the distance between them is given by the length of the line segment that connects them, that is,

$$d_E(A,B) := \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

However, we can define other different distances, such as the **radial distance**, which can be defined as follows:

Definition 1:

$$d_R(A,B) := \begin{cases} d_E(A,B) \text{ } if \text{ } A, B \text{ } and \text{ } O \text{ } are \text{ } aligned \\ d_E(A,O) + d_E(O,B) \text{ } otherwise \end{cases}$$

Here, *O* represents the origin of coordinates, that is, the point with coordinates *O*=(0,0).

Question 1. Could you describe a situation/moment/context that can be modeled using this metric or in which this metric may be necessary/useful?

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Similarly, the radial distance can be described as follows:

Definition 2: The radial distance between two points is the minimum length of the segment or segments that connect them, with these segments contained in lines that pass through the origin.

Question 2: Compare both definitions (algebraic and verbal) by describing in which contexts you would use one and when you would use the other.

Figure 4. Final version of Item 2

Furthermore, Expert 3 suggested that both definitions were not truly of distinct natures, as they were different only in terms of the representation but similar in terms of the aspects of the concept used to construct them. This suggestion was incorporated by removing the phrase "are of different natures" in the final version of item 2 (see Figure 4), once Sub-item 2.1 had been eliminated, and the following items had been renumbered.

Based on the discussions with the experts during the interviews, we wish to emphasize that when an expert validates an indicator in an item, they are not necessarily thinking about the possibility of finding written evidence. Rather, they infer that the indicator is necessary to address the question posed by the item. We bring this up with the understanding that when students in the Mathematics Bachelor's program respond to this questionnaire, they may not necessarily provide written evidence for all these indicators in every item. Nevertheless, we will consider that an item serves to assess specific indicators because this is the expert's opinion, rather than relying on the presence of specific sentences that can be associated with each indicator.

Fleiss Kappa Indicator

After the interviews, the questionnaire was dispatched to four experts in the field of Geometry, who provided responses concerning the presence of each indicator across various Sub-items. Consequently, each expert provided 35 dichotomous (Yes/No) responses, which were subsequently analyzed using the SPSS software to calculate Fleiss's Kappa statistic, both for the entire questionnaire and for each individual item (see Table 2). This statistic expresses the level of consensus among experts regarding which process definition indicators are measured by the entire questionnaire or each individual item.

Analysis type	Kappa and agreement strength (Landis & Koch, 1977)	p-value
Questionnaire	0,533 – Moderate	0,000
Item 1	0,306 – Fair	0,018
Item 2	0,231 – Fair	0,073
Item 3	0,864 – Almost perfect	0,000

Table 2: Results for the Fleiss Kappa statistic

Notably, there is a statistically significant moderate agreement concerning the entire questionnaire. However, when analyzing the items separately, almost perfect and significant agreement is observed for Item 3, while only acceptable agreement is found for the other two items.

From the descriptive analysis of expert responses, it can be inferred that there is unanimity among experts that indicator def1 is measured in all three items, while def2 is measured in Item 3 and def5 is measured in Items 1 and 2. Additionally, a majority of experts believe that def2 is also measured by items 1 and 2, and def3 is measured by Item 2.

CONCLUSIONS

In the initial phase of questionnaire design, three interviews were conducted proving cautious optimism regarding the potential completion of item validation. The analysis of the interviews contributed to the refinement of the questionnaire: some items were clarified in their wording, others were removed, and some others were adapted to encompass a greater number of indicators.

Secondly, the analysis of the Fleiss Kappa indicator suggests that the work is not yet entirely finished. Item 3 shows nearly perfect agreement in its ability to measure indicators def1 and def2. Nevertheless, experts did not express a high level of consensus about Item 2 despite the careful reflection process undertaken during its design. Specifically, our intention was for Sub-item 2.1 to measure def3 as it introduces a new object (radial distance) through graphic language and requests the construction of its definition. Similarly, in Sub-item 2.2, we expected experts to recognize that def4 is being evaluated, as we understood that the first definition of radial distance would be necessary to describe the objects appearing in Sub-item 2.1 more simply. Another expected response for this Sub-item was consider this distance as modeling real-life situations, such as the operation of a train network passing through a station. Only 2 out of 4 experts agreed with this interpretation, indicating that a more explicit wording of Sub-item 2.2 and others, could enhance expert agreement. For instance, revising Sub-item 2.2 as follows: "What is the need for introducing this new distance? What intra-mathematical or extra-mathematical situations or contexts can it model?" may improve consensus.

In addition to deliberating on the content of the items, there appears to be a need for enhancing certain methodological aspects, such as offering a more thorough elucidation of the meaning of each item to enhance experts' comprehension of the questionnaire they are assessing. This recommendation is rooted in informal feedback received from some experts and is substantiated by existing literature (Meyer & Booker, 2001).

We understand that the process we have presented in this contribution corresponds only to content validation via experts in the area. Subsequently, a second validation process is requisite, including the analysis of the responses of actual students answering the questionnaire and analyzing if in such responses the corresponding indicators appear.

Acknowledgements

We want to thank the group of experts for their patience during the study. Their valuable contributions and dedication have made this research possible.

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TEACHING AND LEARNING GEOMETRY OF PRE-SERVICE PRIMARY EDUCATION TEACHERS BASED ON THE VISUAL-ANALYTICAL METHOD OF DIRECTED OBSERVATION

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This paper presents the results of a quasi-experimental research study conducted with pre-service primary education teachers in Croatia aged 19 to 23. The purpose of the research was to evaluate an alternative approach to learning and teaching geometry to develop visualization skills and geometric thinking and achieve better learning outcomes. The alternative teaching approach was based on van Hiele's theoretical framework, on the visual-analytical method of directed observation, and on the balancing of visual, linguistic, and symbolic expressions. The control group was taught in the traditional way. In order to collect data on the research participants, the three tests were used before and after teaching: the first to measure the level of geometric thinking, the second to gain insight into geometric knowledge and visual skills, and the third to measure special visual-spatial ability. At the beginning of teaching, there were no statistically significant differences among the research participants. After teaching, the progress of the participants in the experimental group was statistically significant compared to the participants of the control group.

INTRODUCTION

The results of research in education indicate that students at all levels of mathematics education have the most difficulties when learning geometry (e.g., PISA, 2012; TIMMS 2017), especially when establishing connections between geometric concepts (e.g., Baranović, 2019). This problem comes to the fore especially when students transition from school to higher mathematics education, because the discourses of learning mathematics at school and at the university level are quite different. At the university level for students is difficult to resolve cognitive conflicts that they develop during pre-tertiary education which they are often not even aware of (e.g., Thoma & Nardi, 2018).

Mastering the geometry program at the university level requires at least the fourth level of geometric thinking according to van Hiele, but practice and various research in education show that students are not ready for this (e.g., Usiskin; 1982; Crowley 1987; De Villiers, 2009). To learn geometric concepts with understanding and their efficient application in solving problems and proving statements, students should have appropriate visual-spatial skills, since perception alone is not sufficient for mathematical processing of visual representations (Duval 1995; 1998). The causes of weak visualization skills of students of all ages, which are indicated by practice and various research in education, are primarily a dislike of the mathematical community to visualization as well as the difficulties that arise during visual processing (Presmeg, 1986; 2014). In the last few decades, the need for continuous and systematic development and use of visual-spatial abilities in teaching of mathematics throughout the entire educational vertical has been recognized and intensive work is being done to build a theoretical framework that would enable this (Presmeg, 2014).

As a result of the above, geometric concepts are largely mastered procedurally by memorizing appropriate definitions, rules and formulas without conceptual understanding. Hence, students are usually not ready to continue studying geometry at the university level. Therefore, it is possible to ask: How can we teach geometry to future teachers and prepare them to effectively teach geometry in primary education if they are not ready to continue learning according to the university discourse? In order to answer this question, an alternative teaching approach was prepared and implemented.

THEORETICAL BACKGROUND

For effective learning and teaching of Euclidean geometry, it is necessary to know: the characteristics of the axiomatic construction of the deductive system, possible difficulties in learning and teaching geometric concepts, the appropriate theoretical framework for the development of geometric thinking as well as the characteristics of visualization elements in the context of geometry.

Learning and teaching geometry as a deductive system

In the process of axiomatic construction of a deductive system, the processes of defining terms and the classifications based on them, the processes of setting statements, testing and proving their truth are important. The results of mathematics education research indicate the multiple benefits of learning formal definitions through the defining process, because in the case of giving ready-made definitions as they are usually learned by heart and without understanding (e.g., de Villiers, 2009). In addition, one should also take into account the duality of the process of acquiring a mathematical (as well as geometrical) concepts, especially the mutuality and completeness of these two processes (Viner, 1983) as well as the duality of the nature of geometric concepts and the balancing between its abstract and concrete properties (Fischbein & Nachlieli, 1998). In particular, in working with concrete visual representations, which are used to gain intuitive insight into abstract geometric concepts, various difficulties arise due to the particularity of the representation, the use of prototypes or the impossibility of "seeing" the representation in different ways (Yerushalmy & Chazan, 1990).

The defining process naturally continues by establishing connections between properties, of the same or different concepts, and ends with assertion (de Villiers, 2003). Making statements is based on basic logical operations, principles and reasoning, which serve as "glue" to connect the two essential parts of the statement (the assumption and the conclusion) and to ensure its exact meaning (Hammack, 2013). The skill of recognizing and parsing the assumption and the conclusion of a statement is a necessary prerequisite in setting up a converse statement, contrapositive and negation of a statement and proving or disproving its truth. Given that proven statements are most often used in mathematics classes, proof is used more for the purpose of persuasion, explanation, discovery, systematization, communication or intellectual challenge instead of determining the truth of the statement. In addition, a proof is a means of transmitting mathematical knowledge because a whole series of mathematical concepts, strategies, methods, established connections, systematization woven into the proof (de Villiers, 2003).

Finally, in the process of learning and teaching mathematics, one of the important activities is solving problems. Therefore, every teacher should know the characteristics of different types of tasks, different strategies and solving skills in order to provide his students with a suitable environment in which everyone will develop the necessary knowledge and skills according to their abilities. In this paper, three tasks classifications are considered: according to goal, demandingness and purposefulness. According to the goal to be achieved there are: 'tasks to find', 'tasks to prove' and ‘tasks to construct' (Polya, 1966). According to the level of thinking required for solving there are: tasks with lower cognitive demands that serve to develop procedural knowledge and skills and tasks with higher cognitive demands that serve to develop conceptual knowledge (Smit & Stein, 1998). According to purpose there are tasks that facilitate the transition from school to university level mathematics education, also known as "unusual tasks" (Breen et al., 2013), which are a type of tasks with higher cognitive demands. For learning and teaching the process of solving tasks, as well as for the process of proving Polya (1966) proposed four phases: (1) understanding the task, (2) devising a plan, (3) carrying out of the plan and (4) looking back.

Numerous studies in education confirm the multiple benefits of solving tasks with higher cognitive demands. For example, solving unusual tasks requires relying on previous experience, higher mental work, more patience and perseverance, searching for and establishing relationships between different concepts, etc., which results in flexibility in the thinking process and the development of mathematical thinking and reasoning. At the same time, they can serve as a real detector of difficulties and misunderstandings because when solving them, typical mistakes of students coming to the fore as well as their creativity (Leikin & Lev, 2007; Breen et al., 2013).

Development of geometric thinking

Van Hiele's theoretical framework is widely accepted to monitor and ensure the development of geometric thinking, through three aspects. The first aspect refers to the process of developing abstract thinking, hierarchically through five levels: recognition, analysis, informal deduction, formal deduction and level of rigor. The second aspect refers to considering the characteristics of the five-part model: structure, principle of object duality and methods of thinking, communication, mutual (mis)understanding, age. The third aspect refers to the learning process in five stages that contribute to progress from one level of thinking to another: inquiry/information, directed orientation, explication, free orientation and integration (Van Hiele, 1986; Crowley, 1987). The results of education research confirm the effectiveness of the teaching strategy according to van Hiele's stages of learning: the stages provide an effective framework for clearly structuring teaching units and guiding students from one level to another (e.g. Dongawi, 2014), and their application improves students' geometric thinking, understanding geometric concepts, and consequently learning outcomes (e.g., Crowley, 1987; Teppo, 1991).

Development of visual-spatial abilities

In recent years, the Cattell-Horn-Carroll (CHC) theory of cognitive abilities, which represents the most comprehensive structural model of intelligence, has been increasingly used. According to the CHC model, spatial ability is referred to visual processing (labeled Gv), and implies "the ability to generate, perceive, analyze, synthesize, store, retrieve, manipulate, transform, and think with visual patterns and stimuli" (Flanagan & Shauna, 2014, p. 7). According to Lohman (1993), individual differences in spatial abilities are realized in the speed of performing transformations, especially rotation, then in the skill of creating and retaining mental images, the amount of visual-spatial information that a person can maintain in an active state, and finally in the sophistication and flexibility of available strategies to solve certain tasks.

Through various types of research, a strong mutual connection between visual-spatial abilities and mathematical (especially geometric) achievements has been confirmed (e.g., Clements et al., 1997; Bruce & Hawes, 2015). In mathematics education research, the definition of visualization proposed by Arcavi (2003), which includes products and processes, is widely accepted. Visualization products include various types of mental images (Presmeg 1986), various types of external representations (Nakahara, 2007) as well as their mutuality. Visualization processes include the processes of creation and use of products, and differ with respect to the place, manner or purpose of creation/use. For successful work in geometric figures, Duval (1998) proposes a theoretical framework that considers four types of figure apprehension: perceptual, sequential, discursive and operational. The processing of the figure begins in a pure perceptual recognition of what is shown, and then, through mutual use of sequential, discursive and operational processing of the figure, the mathematical message that the figure represents in the appropriate context is revealed. To master the transition from pure perceptual apprehension to recognizing what is mathematically important, students need to know and develop different visualization processes, first separately and then in mutual coordination (Duval, 1995). In order to successfully learn geometry, it is necessary to develop the "geometric eye" through various types of tasks in which one should first gain certain practical experiences: by experimenting, measuring, drawing, etc., then make statements based on what has been observed and finally prove these statements by connecting them with definitions, axioms and theorems which are already known (Godfey 1903 according to Fujita & Jones, 2002).

RESEARCH DESIGN AND QUESTIONS

The empirical research is a form of quasi-experimental research with two non-equivalent groups (Cohen et al., 2007) and was carried out in real-time during one semester during regular classes of Euclidean geometry with all enrolled students of the respective university. Both groups were tested at the beginning and at the end of the semester with three identical written tests. The experimental group participants were taught using an alternative approach, while the control group participants were taught in a traditional way.

The purpose of this research was to evaluate the chosen alternative approach in learning and teaching geometry at the tertiary level. The main goal was to examine whether an alternative approach leads to improvement of visual-spatial abilities and progress in geometric thinking, a greater tendency to visualize and consequently to better outcomes of learning geometry (or not). In accordance with the main goal, five groups of research tasks were set, and in this paper two groups are distinguished: (1) immediately before learning geometry, determine the relationship between the visual-spatial abilities, levels of geometric thinking and geometric prior knowledge of the participants, especially in the sense of a possible predictor progress, (2) after the teaching of geometry, determine whether the participants, who were taught with an alternative approach, made significant progress in the accumulated geometric knowledge and visualization skills and visual-spatial abilities compared to the participants who were taught in the traditional way (or not).

METHODOLOGY

To obtain answers to the questions raised, the collected data were processed by integrating quantitative and qualitative methods because complex research questions can be more fully answered using a mixed methodology than individual analyzes separately.

Sample

The research includes all students of teacher studies at two universities in Croatia, who attended the geometry classes in the summer semester of 2015/2016. A total of 90 students participated: 52 students (average age 21.8) who attended geometry classes at one University made up the experimental group, and 38 students (average age 19.4) who attended geometry classes at another University made up the control group. These two groups were chosen because of the greatest congruence of the curriculum for learning geometry.

Instrument

To collect data on the knowledge and skills of the participants immediately before learning and after learning geometry, three tests were used. To measure levels of geometric thinking according to van Hiele's model (VH test), the test from the CDASSG project was used with permission (Usiskin, 1982). A test designed for this research (GEO test) was used to measure knowledge of geometry and visualization skills. The final design of the GEO test was achieved after the results of a pilot study and consultation with three experts. Within 28 tasks, formed into four parts, elementary geometric concepts, their definitions, visual representations, and applications were tested. The SPAC test (Smith & Whetton, 1998) was used to measure a special factor of spatial abilities.

Data collection and analysis

All of the three tests were conducted at the beginning of the summer semester in the first week and at the end of the semester in the last week. The evaluation of the VH test was carried out according to the proposal of Usiskin (1982): first, each correct answer was evaluated by levels, and then the entire test using mild (3 out of 5) and more strict (4 out of 5) criteria within the classic (C) and modified scale (M)). The evaluation of the GEO test was carried out first for each of the four parts of which is composed, and then collectively: each correct solution 2 points, incorrect or incomplete 1 point, no answer 0 points. Due to space limitations, it is not possible to provide a detailed description of the GEO test in this paper. The SPAC test was conducted and evaluated by a psychologist according to strictly defined rules of psychological testing.

Intervention

During the intervention in experimental group, two main teaching strategies dominated: (1) structuring teaching topics according to van Hiele's stages of learning and (2) teaching based on the visual-analytical method of directed observation. Both strategies were permeated by three-layered teaching by balancing three ways of expression: visual, linguistic and symbolic (VLS system and teaching for short; figure 1). The teaching activities included: tangram activities, drawing, constructing, defining, making statements, testing the truth with proving or disproving, and solving problems. The selected activities were intertwined and built on each other through different topics, and were chosen with the aim of developing visual-spatial abilities, thinking processes and learning geometric concepts with understanding.

Traditionally, teaching in a control group means: first, the teacher teaches frontally, emphasizing theoretical description and symbolic writing, and then the participants do the tasks individually, usually imitating their teacher. Visualization is used, but to a lesser extent and only as an intermediary.

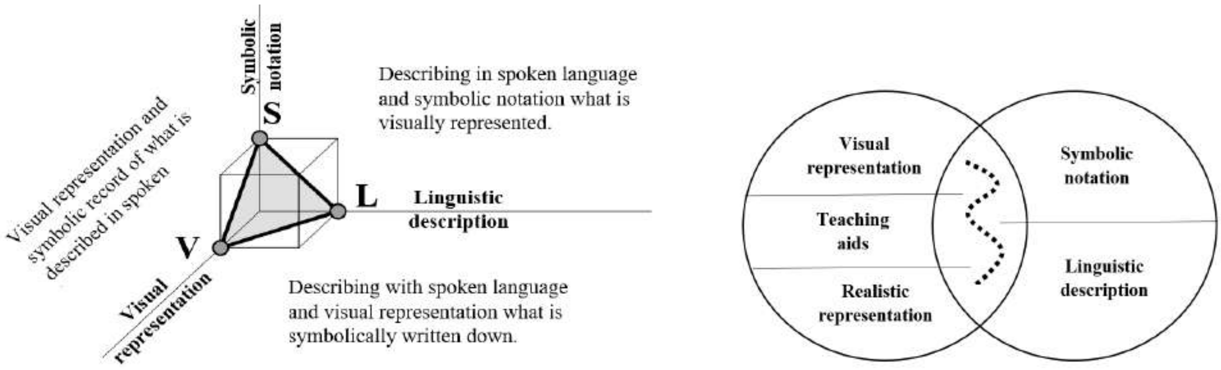

Figure 1. VLS system and VLS teaching

RESULTS AND DISCUSSION

Given that the data on the research participants was collected through three tests of different formd and purposes, mixed data processing was used and the analysis of the results and discussion were done through different aspects. Due to space limitations, this paper presents only some summative results that indicate the outcomes of learning geometry, achieved within the experimental and control groups and between them.

After the characteristics of the participants of the experimental and control groups were determined by analyzing the results of the three tests, a t-test was performed with the aim of determining the statistical significance of the difference between these two groups before the intervention (table 1).

	Experimental group			Control group			t-test		
Measuring	N	*M*	*SD*	N	*M*	*SD*	*t*	df	*p*
1CVH4	37	1.486	.768	30	1.567	.679	-.447	65	.656
1MVH4	38	1.500	.762	31	1.548	.675	-.276	67	.783
1GEO	52	5.962	2.368	34	5.676	2.889	.500	84	.618
1SPAC	46	55.609	10.697	31	58.419	9.113	-1.198	75	.235

Table 1. Results of the t-test for the experimental and control groups before the intervention

Based on the results (Table 1) there is no statistically significant difference ($p > 0.05$) between the groups in any of the considered categories. Considering some parameters, the mean values of the control group are slightly better compared to the mean values of the experimental group (1CVH4, 1MVH4 and 1SPAC). Considering that there was no statistically significant difference between the research participants before the intervention, reliability was tested for each test on the whole data of both groups using Cronbach's alpha coefficient (α). According to the coefficient values the tests can be considered reliable measuring instruments: the VH test has satisfactory internal consistency ($\alpha = 0.576$), the GEO test is really reliable ($\alpha = 0.921$) and the SPAC test is a highly reliable instrument ($\alpha = 0.754$) (Novak, 2020).

The statistical significance of the difference in achievements between the experimental and control groups after teaching, for all of the three tests, was tested in the t-test (Table 2). Based on the results the progress in the two tests was statistically significant in favor of the experimental group. The statistical significance of 1% was on the GEO test ($p < 0.001$) and on all parameters of the VH test, except for the VH3 criterion where the statistical significance is 5%. The difference in achievements on the SPAC test is not statistically significant between the groups ($p = 0.287$). From the analysis by groups is evident that progress was achieved within each group: in the experimental group it was statistically significant ($p < 0.01$), while in the control group was not ($p = 0.465$).

	Experimental group			Control group			t-test		
Test	N	*M*	*SD*	N	*M*	*SD*	*t*	df	*p*
2CVH4	38	2.026	1.078	30	1.267	.640	3.412	66	.001
2MVH4	38	2.026	1.078	30	1.267	.640	3.412	66	.001
2GEO	45	29.378	6.994	33	16.485	6.690	8.191	76	.000
2SPAC	44	59.659	9.906	26	57.077	9.398	1.074	68	.287

Table 2. Comparison of achievement in all three tests between groups, after the intervention

The presented results are in line with the results of other research in education: by mutual use of visual and analytical methods; by adapting teaching strategies to the participants' prior knowledge; by respecting the process of axiomatic construction of the deductive system; and by using different representations and their interconnections geometric achievements can be improved (e.g., Presmeg 1986; van Hiele, 1986; Bruce & Hawes, 2015).

CONCLUSION

These results indicate that the geometry learning outcomes of pre-service primary education teachers and, thus, their preparation for work can be better when the teaching is adapted to their prior knowledge, and geometry is learned and taught through the systematic and mutual development of geometric knowledge, thinking and visual-spatial skills. In addition, these results can be an incentive for teachers at all levels of education to pay more attention to the three-layer teaching method, mutually and in harmony. It would be important to investigate what teaching outcomes would be achieved by applying this teaching strategy to groups with other characteristics.

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GEOMETRY TEACHING FROM BABYLON TO THE COMPUTER ERA

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Geometry has been taught for millennia and we have rather detailed references of the geometry taught in Scribal Schools in Babylon, 4000 years ago. Today, with the computer revolution and Artificial Intelligence available, nobody seems to be proposing we stop teaching Geometry. But which Geometry should we teach, why and how? Which are the driving forces that influence each curriculum? How the new computing tools open new pathways for Geometry teaching? We will mention in some detail the controversies around the official Syllabus in Portugal in the last decades to try to identify which are the prevalent ideas in the curriculum and which are the different cultures involved in the decision process, from a purely abstract approach to a more applied approach, that answer to different societal and cultural views and mix the proposals of mathematicians, mathematics educators and teachers in the classroom. We observe that amid societal changes and advances in knowledge, Geometry is always focused on Problem Solving, use and exploration of diagrams to conjecture and achieve some kind of proof. We need to take intelligent paths to make the most of technology and not be a slave to it.

In order to mention the main ideas in some detail we have included diagrams, tables and the longer quotations in a web page available here: ***https://www.geogebra.org/m/cfnj6cb2***

THE SYLLABUS IN PORTUGAL

In the years 2021-2023 a new syllabus has been approved by the Portuguese Ministry of Education for the Mathematics courses of Basic Education (years 1 to 9) and Secondary Education (years 10 to 12), and this was done only for Mathematics. In Portugal the compulsory education includes all 12 years of school but there were no changes in other areas, as the Ministry of Education recognized there were unacceptable failure rates in Mathematics in Portugal. The Ministry began commissioning a special report to receive proposals to change the situation. The report *Recommendations for improving student learning in Mathematics* was published in 2020 (Carvalho e Silva, 2020) and was based on all the reports produced in Mathematics Education in Portugal in the last 30 years as well as on most international reports produced in this period, including international reports from OECD and UNESCO, as well as international assessments. It recommended a global revision of the mathematics curriculum in Portugal, that should be based on a certain number of principles, namely:

i) Universality, internal coherence, relevance, focus and higher cognitive level.

ii) Fulfill all the goals that justify the universality of the access to this course, taking into account the cultural, social and political dimensions of mathematics learning;

iii) Value understanding and look for an equilibrium of problem solving, mathematical reasoning, communication, connections, multiple representations, procedural fluency, creativity, digital literacy, reflection, resilience and individual and work group.

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 157-164). ICMI.

These are general principles but particular topics like Geometry are not discussed. New syllabuses were produced in 2021 for Basic Education (Canavarro, 2021) and in 2023 for Secondary Education (Carvalho e Silva, 2023). Other details that do not relate to Geometry will not be discussed here.

Geometry in Basic Education

Geometry occupies a substantial space in the new curriculum for Basic Schools. In the first four years the students begin their "development of spatial reasoning, with an emphasis on visualization and spatial orientation" (Canavarro, 2021); they use various types of materials and also technology (namely visual programming with Scratch and robots). In the second cycle of studies (years 5 and 6) the measure of angles and the study of triangles is introduced and dynamic geometry environments, such as GeoGebra are recommended. In the last three years of Basic Education (3rd cycle, years 7 to 9) the goal for Geometry is to "continue developing students' spatial reasoning, expanding their understanding of space", as well as "the establishment of algebraic relationships from the study of geometric objects (…) accompanied by experience (where technology plays a fundamental role)" (Canavarro, 2021). Also geometric transformations are studied in a progressively more abstract and formal way.

Geometry in Secondary Education

Secondary Education in Portugal offers several paths of study that lead to further studies in Higher Education or offer a possibility of looking for a job immediately after school with a more profession oriented study. Students encounter six possible paths and in all of them we find some Geometry. The path *Mathematics Applied to Social Sciences* includes a chapter on Graph Theory and we will not discuss the details here (Carvalho e Silva, 2018). The Science and Technology path and the Economics path share the same Mathematics course spanning 3 years and 4.5 hours of class a week (Carvalho e Silva, 2023). This Mathematics course is called Mathematics A. The geometry topics studied in Mathematics A go from the notable points on the triangle with a synthetic perspective to matrices and geometric transformations, and include analytical geometry and trigonometry.

The chapter "Synthetic geometry in the plane" is a classical one but is new as such in the Portuguese curriculum. It is intended as a crossroad between classical geometry and experimentation with dynamical geometry systems, in order to "develop in students a taste for argumentation in general and proof as a central element of mathematics, for example with regard to the inscribed circle and the circumscribed circle" (Carvalho e Silva, 2023).

Analytic geometric is classic also but has a more algebraic flavor; in the curriculum students are encouraged to make software explorations (namely with *Geogebra 3D*) and work with mathematical modeling problems.

Implementation of the curricular change

We all know that an official document (intended curriculum) does not guarantee its implementation (enacted curriculum) even with good orientations (planned curriculum). Lots of countries have devised ways of connecting the intended curriculum to the enacted curriculum. We can quote the example of Costa Rica whose Reform has been very well documented by its mentor Angel Ruiz; this was a curricular change in mathematics only, that began in 2012. It developed original and innovative ideas and instruments, including blended courses for teachers, fully virtual courses with MOOC modality for high school teachers and students, and shorter virtual courses called Mini-MOOCs (Ruiz, 2015).

In Portugal several curricular changes were accompanied by a number of measures to work towards the coherence of the curriculum. Since 2021 several actions were taken: some classes (pilots) to anticipate the implementation of the new syllabus were run, tested materials were afterwards made available to the rest of the schools including comments from pilot classes, tasks were published online, continuous professional development courses were made in two phases: the first one to prepare schools leaders that in the sequence propose themselves continuous professional development courses to other teachers in all schools.

Using the history of education tool

The options taken in the new syllabus were publicly discussed, there was a preliminary version for an open discussion and public sessions were held to discuss with teachers, associations and other partners. There was no consensus on a number of issues so the team writing the curriculum (including mathematicians, mathematics educators and experienced teachers) had to choose paths in a number of issues. Why these options were ultimately held? Are they promising? Will the global situation in mathematics education improve? The tools used to make choices in the curriculum were the experience of different countries, the most successful ones including international schools with its own curriculum, and past history of mathematics education, namely in Portugal. The educational system in Portugal was structured in 1772 and greatly revamped in 1836, so there is a long history of mathematics education in Portugal. Other countries and regions have an even longer history. What can we learn from our collective past experience teaching Geometry?

HISTORY FLASHBACK ON GEOMETRY TEACHING

It is always useful to study which were the guiding principles and the school practice in past times. The Portuguese document *Recommendations for improving student learning in Mathematics* (Carvalho e Silva, 2020), produced before the last rewriting of the curriculum, made a summary of the main documents produced in Portugal in the last 30 years and discussed what happened in France, Finland, Estonia and Singapore. In the end a lot of experience was gained from the emphasis on problem solving, applications and mathematical modeling these countries have, and also on the use of technology namely with computational thinking with a large practical work in schools particularly in France (numerous classroom materials are available on the web also through the work of the well-known IREM).

Let’s go back here some millennia.

Teaching in Babylon

Scribal Schools from the Old Babylonian period are fascinating. Some 4000 years ago, we can find that “education went hand in hand with creative activity, supported by a very active milieu. (…) The most active centers were able to influence other, less creative ones, where the educational framework was less institutional. (…) These scribes conferred together and traveled.” (Proust, 2014). A lot of researchers have studied scribal schools and it is clear that Geometry was present from the beginning. As Friberg puts it: “to divide given parcels of land into shares according to some intricate set of rules, or dividing given amounts of food stuff into rations of various sizes according to some other intricate set of rules, must have been an important part of the metro-mathematical education given to the young scribes in the scribe schools" (Friberg, 2014). It is clear that Geometry, Arithmetic and Algebra were studied together. Geometrical diagrams were frequent in clay tablets.

Euclid, the *Elements* and *Pseudaria*

As Robin Hartshorne points out "Throughout most of its history, Euclid's *Elements* has been the principal manual of geometry and indeed the required introduction to any of the sciences." (Hartshorne, 2000). More than one thousand editions and translations are known. In the great reformation of the studies in Portugal, around 1772, Mathematics took center place at the University and so all students at the University had to study Geometry, Euclidian style, and the textbook used was a translation of the *Elements* in Portuguese.

A lost work attributed to Euclid, with a more marked pedagogical character, *Pseudaria*, never made into mainstream teaching and the knowledge of its content is almost completely lost because, possibly, as points out Fabio Acerbi, "the tradition of its pedagogical use was soon broken (…) as a consequence of a likely increase of the dogmatic character of mathematical teaching in post-Hellenistic times" (Acerbi, 2007).

Multiple translations and uses of the *Elements*

As was already mentioned, more than one thousand editions and translations of Euclid's *Element*s are known. It was used at almost all schools in the world at some point. The first Chinese translation was made by the Jesuit Matteo Ricci (1552–1610), in 1607, when he was assigned to work in China (Joseph, 2011).

The contents and style of Euclid's *Element*s were imitated by numerous school textbooks for Basic and Secondary schools. The most used textbook in Geometry in Portugal for some 30 years, from 1944 to 1974, was the first in Portugal to use "the treatment of demonstrations in justified steps, as Americans and English use it, [and] it seems to us very advisable" (Paulo, 1944). The price to pay with this more formal approach, as the reviewer quoted wrote, is "that despite everything, sometimes the author makes an appeal to students' intuition, it is a shame that it is no longer done often".

Modern Mathematics

The so called *Modern Mathematics Movement* changed completely the teaching of Mathematics in numerous countries including Portugal. The Portuguese mathematics curriculum for Basic and Secondary School saw a considerable reduction of the study of classical Geometry and some emphasis on the study of geometric transformations. Euclid's approach completely disappeared.

The coordinator of the modernization of Mathematics teaching in Portugal, mathematician José Sebastião e Silva (1914-1972), wrote that the teaching of Geometry should find an equilibrium between abstractions and the real world. He was very much influenced by Emma Castelnuovo (1913-2014).

As the 24th ICMI Study volume puts it:

> Classical synthetic geometry was completely eliminated and the main aim was not to study geometrical figures but to construct an algebraic tool to describe first the affine, then the Euclidian plane and space. Principal notions were projections, vectors, frames, transformations, etc. (Gosztonyi et al., 2023, p. 53)

Tensions in the XXIst Century – Basic School

From 1990 to 2023, there were several changes in the mathematics curriculum for Basic School and Secondary School in Portugal. The reform of the educational system of 1990 introduced new syllabuses at all levels for all courses, but other changes were mainly concentrated on changes in Mathematics, in part because in the first TIMMS and PISA Portugal was in a very modest position.

The 1990 syllabus for mathematics in Basic School initiated an extensive study of geometric transformations in a rather intuitive and balanced way; but, with an overcrowded curriculum (reformers were very ambitious) geometric transformations tended to be omitted. The next change of the curriculum in basic school in 2007 took a different approach, more visual and connected to the real world, insisting in the development of the spatial sense of students, based on exploration, manipulation and experimentation with concrete materials. Small chains of deductions should be introduced in years 7 to 9 in topics like Parallel Lines, Similar Triangles, Pythagoras Theorem and Geometric Transformations (Breda et al, 2011).

But there were radical changes again in 2012-2013, and a highly structured approach, based on logic and set theory was introduced in Basic School with an extensive study of axiomatics in the 9th grade; numerous proofs became compulsory along the official "correct" use of the vocabulary of the axiomatic method and a rather detailed knowledge of the history of the axiomatization of Geometry. The axiomatization of Geometry with points, relation "a point is between two other points" and "pairs of points are equidistant" are mentioned as objects of study, as well as Hyperbolic Geometry. This syllabus was a huge failure, some minor modifications were introduced and in 2021 a new syllabus was approved by the Ministry of Education, somehow recovering the syllabus from 2007, that had been very well received by mathematics teachers at the time.

Tensions in the XXIst Century – Secondary School

We will consider here the official documents for mathematics teaching at secondary school in Portugal on the course called "Mathematics A". There are big contrasts between the documents of 2013 and 2023. In 2013 the main goal was "the comprehension and hierarchization of mathematical concepts, the systematic study of their properties and the clear and precise argumentation, typical of this discipline" (Bivar et al. 2013). Mathematical modeling could "transmit to students a distorted vision of how one can, in fact, correctly apply Mathematics to the real world" (Bivar et al., 2013) and the use of intuition was discouraged as "conjectures formulated but not demonstrated are of limited interest" (Bivar et al., 2013). In contrast to this vision, the new documents of 2023 present the work in mathematics at secondary school as following:

> (…) develop in students the ability to identify relevant mathematical concepts to solve real problems, apply appropriate mathematical procedures and interpret results in different contexts. Mathematical reasoning is the basis of the processes of understanding mathematical concepts and objects, which can and should be analyzed, represented and related in different ways. The formulation of hypotheses, testing of conjectures, deduction, generalization and abstraction are equally important, in the construction of logical arguments and conclusions, whose communication in an appropriate way is increasingly important in today's world. (Carvalho e Silva, 2023).

Which contents and methods are the most suitable for our century?

XXIst Century: a world full of smart technology

As some people said (no quote necessary) in a world of technology that draws nice graphs and images, why study so much Mathematics (Functions or Geometry)? In fact we are in a world that already produces algorithms that can make automatic deductions in Geometry and this has the potential of changing completely the way Geometry is taught. We face three types of essentially new challenges:

a) Dynamic Geometry Software (DGS);
b) Automated proof assistants, that can be Interactive Theorem Provers (TP) or Automated Theorem provers (ATP);
c) Large language models (LLM) like ChatGPT.

DGS has been around for some time and has always been considered "rather efficient in the learning of geometry" (Balacheff, de La Tour, 2019); it is being continuously upgraded, and a surprising challenge is *Geogebra* that can run on smartphones with Augmented Reality (AR) tools; we are in a world where all students (at least from 12 years old) have a smartphone at a rate that surpasses 90%; which are the implications?

Beyond mechanizing computation, these technologies are now mechanizing mathematical reasoning and proofs with unprecedented consequences. DGS and ATP have been combined in a software with AR and used in an educational setting outdoors (Botana, Kovács, Martínez-Sevilla & Recio, 2019), (Botana, Kovács & Recio, 2020). In a recent paper Nuno Baeta and Pedro Quaresma describe an algorithm (a "Geometry Automated-Theorem-Prover") that will "be able to efficiently prove a large set of geometric conjectures, producing readable proofs" and "will open its use by third-party programs, e.g. the dynamic geometry systems" (Baeta and Quaresma, 2023). These methods began being tested in secondary schools (Teles, Baeta and Quaresma, 2022) and, even if practical difficulties are enormous, one day they will surely be used regularly in classrooms. How?

Large language models are a recent acquisition to the world of pedagogical tools and have become very controversial because they produce mathematical errors frequently: the outputs a LLM gives are just words that have a high probability of showing up together in a sentence that was produced millions of times in the texts the LLM "read". But LLM are already being used in school environments, associated to sophisticated mathematical software like Mathematica or as just a tool to give better feedback like in Geogebra as an *AI Math Assistant* (Hohenwarter, 2023) and *Khan Academy* as a text builder assistant. Will they help us teach Geometry better?

Po-Shen Loh, the coach of the American Math Olympiad team, calls the use of ChatGPT the "invasion" and thinks "the key to survival is knowing how to solve problems - and knowing which problems to solve" (Loh, 2023). He urges school to focus on "creativity, emotion and the stuff that distinguishes man from machine" (Loh, 2023).

LESSONS FOR TODAY

In this extremely brief historical excursion, we see that in each period of time, Geometry had a different character but it always had some kind of relevance related to the reasoning that it conveyed and to the real life situations it was connected to; diagrams were always used from Babylon to Euclid and the Computer Era. Somehow, connecting diagrams to its abstract representation can lead to errors (like in *Pseudaria*) but this also stimulates a healthy or corrected reasoning, enabling students to be able to deal with new problems, understanding them, looking for strategies, producing and criticizing conjectures and trying to prove them, learning how to deal with similar problems or problems that might in part benefit from the new strategies found.

The necessity to produce diagrams is a constant and our computer era is the most fertile in the availability of tools that enable us to produce a wide range of diagrams, static or dynamic. Pedagogical tools like *Geogebra 3D*, *GeoGebra AR* or *GeoGebra Discovery* (ATP) (Kovács et al., 2022) are powerful and simple to use and very promising for the classroom.

Teaching and learning Geometry did not change much for millennia. Archimedes, in his *Method*, tells Eratosthenes how to investigate Geometry problems with Mechanics and then prove the results obtained using Geometry and he said the investigative part is also useful for the proof of these results (Heath, 1912). Tools and scope and interplay with other areas have changed a lot in the present century but we must see what is now important in Geometry and find new ideas to continue teaching and learning Geometry in a meaningful way. As Gila Hanna and Xiaoheng Yan (2021) put it "there is a need for new approaches to teaching proof, ones that capitalize not only on newly-available technology but also on modern theories of teaching and learning".

In the words of late President of ICMI, Miguel de Guzmán, when speaking about the impacts of new technology on mathematics teaching: "What is truly important will be the preparation [of students] for an **intelligent dialogue with the tools that already exist**, which some already have and others will have in a future that is almost present" (De Guzmán, 1992) (my emphasis).

We certainly need to reflect intensely on new and significant ways to teach Geometry in Basic and Secondary School, that use technology in an intelligent and meaningful way.

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DEFINITIONS OF PROSPECTIVE PRIMARY TEACHERS CONCERNING THE AREA OF 2D FIGURES

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The aim of the present study is to characterize the personal definitions about the area of 2D figures of a group of prospective primary teachers. For this purpose, tools of the Ontosemiotic Approach to Mathematical Knowledge and Instruction are used. A content analysis is carried out to identify the emergence of different objects and processes in the definitions of prospective primary teachers. The results show that, for the most part, personal definitions of prospective primary teachers match institutionalized definitions of area as an object of reference. Likewise, the results show that the structure of the definitions provided is given by different objects and processes that allow describing the way in which prospective primary teachers use the different area partial meanings in their definitions. Finally, the results show tendencies of prospective primary teachers in the use of geometric, numerical, and algebraic representations to define area.

INTRODUCTION

Definitions of geometric concepts are a fundamental component of teachers who teach mathematics (Ball et al., 2008; Zazkis & Leikin, 2008), as they reveal a series of logical relationships, for example, between propositions, didactic learning sequences, mathematical connections, or mathematical communication (Leikin & Winicki-Landman, 2001). The variety of ideas about the structure of a mathematical definition and the associated challenges, has led researchers to suggest that definitions should be explicitly addressed as part of initial and continuing teacher education (Leikin & Winicki-Landman, 2001; Zazkis & Leikin, 2008). This is because teachers play a determining role in the definitions used in the classroom and in the role given to these definitions in mathematical activity. However, research that addresses the knowledge that teachers have about mathematical definitions is scarce, pointing mostly to the difficulties associated with this process (Leikin & Winicki-Landman, 2001). Particularly, some studies show the difficulties of prospective and in-service teachers in defining geometric concepts, as quadrilaterals or polygons (e.g., Avcu, 2022; Fujita & Jones, 2007; Leikin & Zazkis, 2010; Miller, 2018). In explicit relation to the area of 2D figures, studies are more scarce and suggest that prospective teachers tend to define area as "length x width" (Livy et al., 2012), coinciding with students' definitions (e.g., Zacharos, 2006). Understanding the personal definitions of prospective primary teachers is a crucial step before designing targeted interventions, as it provides insights into the specific areas of difficulty and misconceptions, and therefore, allows to address the root causes of the difficulties outlined above. In this context, the aim of this paper is to characterize the personal definitions about the area of 2D figures of a group of prospective primary teachers (PPT). Personal definitions are understood as those interpretations made by PPTs of a formal definition of the area in the sense of Tall and Viner (1981). For this purpose, we take into consideration tools from the Ontosemiotic Approach to Mathematical Knowledge and Instruction, which we describe in the following section.

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Definitions in the teaching of geometry

According to Tall and Vinner (1981) and Vinner (1983) the definition of a concept corresponds to a verbal statement used to specify such concept, explaining it precisely and in a non-circular way. The latter implies that the term to be defined should not be used in the definition (Zaslavsky & Shir, 2005). Tall and Vinner (1981) make a classification of concept definitions, identifying a formal definition of the concept and a personal definition of the concept. The formal definition of the concept corresponds to that which is accepted by the mathematical community in general (Bingölbali & Monaghan, 2008; Moore, 1994). To understand this type of definition, PPTs must generate their own interpretations of the definition (Viholainen, 2008), which may change from one person to another and from one context to another (Pinto & Tall, 2002). In this context of change, Tall and Vinner (1981) point out that personal definitions correspond to a "discursive description that the learner uses for his own explanation of his evoked concept image" (p. 152), this evoked image being the one that is activated at a given moment. The authors point out that the personal definition of a concept may be equivalent to the formal definition, or it may disagree and, consequently, be incomplete or erroneous (Tall & Vinner, 1981). Thus, both definition itself and concept image come into play in the process of defining, and the understanding of a geometric concept is determined by the link between the concept image and the concept definition (Gutiérrez & Jaime, 2012). The concept image is often in conflict with the formal definition of a concept (Tall & Vinner, 1981). For example, even if a student has been taught, and is able to give a formal definition of a parallelogram as a quadrilateral with two pairs of parallel opposite sides, he or she may not consider rectangles, squares, and rhombuses as parallelograms, because the concept image they have of these is that not all angles ("amplitude") or sides ("length") can be equal. In the negotiation of this type of conflicts, the role of definitions can be decisive (Vinner, 1991), since they allow establishing relationships between the conditions (e.g., parallelism) that geometric figures must meet to be classified as parallelograms.

In line with the above, this paper delves into personal definitions that PPTs have about the area of 2D figures. To do so, we use some of the tools of the Ontosemiotic Approach which attributes an essential role to language and categorizes types of objects that emerge in mathematical activity.

Ontosemiotic Approach (OSA)

Godino et al. (2007) define mathematical practice as any action or manifestation performed by someone to solve mathematical problems and communicate the solution to others. The set of such practices is referred to as a system of practices, composed by operative facet (e.g., problems) and discursive facet (e.g., arguments), which are mutually interrelated (Wilhemi et al., 2007). Since defining mathematical objects involves "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner, 1991, p.65), mathematical definitions represent a discursive component of mathematical system of practices (e.g., the formalized language used in defining mathematical concepts), which comes after the initial practical engagement or operative practices (Wilhemi et al., 2007). In this sense, language is the articulating axis of the operative-discursive facets, allowing the communication of procedures and meanings involved in mathematical activity (Wilhemi et al., 2007). Operational and discursive practices can be attributed to an individual or shared within an institution (Godino et al., 2007). In the first case, reference is made to the personal meaning of the object. In the second case, reference is made to the corresponding institutional meaning. The personal meaning of the object is conditioned by its relation/interpretation of the institutional or formal meaning of such object (Tall & Vinner, 1981). A personal object is then an emergent of the system of personal practices linked to a given problem situation. The description and analysis of mathematical activity is conditioned by six primary objects (Godino et al., 2007): linguistic elements; problem-situations; concepts; properties/propositions; procedures; arguments. The primary objects detailed in the OSA emerge from systems of practices which underlines their complexity and the need to articulate them. The later can be done through epistemic configurations (networks of institutional objects) or cognitive configurations (networks of personal objects including cognitive constructs such as conceptual images). Moreover, the relationships that can be established between primary objects take place through different processes, such as definition, which can be considered from different dual facets giving rise to other fundamental epistemic/cognitive processes in mathematical activity (Godino et al., 2007): materialization-idealization (ostensive-non-ostensive), particularization-generalization (intensive-extensive), decomposition-reification (unitary-systemic) and representation-meaning (expression-content).

From the above-mentioned background, in this paper definition is understood as a process that involves the cognitive configuration of PPTs and, thus, their personal meanings. Hence, it is accepted that the personal definition of area constitutes the personal meaning of such an object from the institutionalization-personalization duality (Godino et al., 2007).

METHOD

The study is situated in an interpretive paradigm and follows a qualitative approach (Cohen et al., 2007). A content analysis involving deductive coding is performed, being the unit of analysis the written protocols of PPTs. An epistemic configuration of area (Caviedes et al., 2021) is used to identify how the cognitive configuration that is activated in the definitions of PPTs matches up the institutional definition of area (in the sense of Tall and Vinner, 1981). The institutionalized definition of area is based on three partial meanings presented in the epistemic configuration mentioned above, which can be summarized as follows: *two-dimensional surface that can be covered with units of measurement and/or calculated by means of formulas*. Deductive coding makes it possible to identify the primary objects that account for the cognitive configuration of PPTs, and thus the personal meanings associated with area and the processes in their dual facets. The process of defining is then characterized by the gradual, systematic, and progressive emergence of different objects, processes, and meanings (Godino et al., 2007).

A semi-structured open-ended questionnaire was designed to be solved individually and in writing by a group of 70 PPT in the third year of the Primary Education degree, in the 2020-21 academic year at a public University in Spain. The questionnaire was part of an evaluated activity, had a total of 8 tasks and sought to problematize mathematical and didactic knowledge about area of PPTs (see Caviedes et al., 2023a, 2023b). Although PPTs had worked on activities related to area (e.g., use of different contexts and procedures to solve tasks, analysis of students' responses), they had not worked on its definition. The questionnaire was applied by the professor in charge of the subject, in online format due to the COVID-19 health contingency, and the PPTs had one week to answer it. Due to the objective of this paper, evidence of the written protocols corresponding to Task 7 is presented, posing the following problem situation: *If you had to introduce the geometric content of area in 5th grade, how would you define area?* It's worth mentioning that PPTs were told to focus on *how* to define.

Analysis

The analysis is carried out with the support of MAXQDAplus software. A system of a priori categories is established, given by the epistemic configuration presented in Caviedes et al. (2021) and by the processes of the OSA. The definitions provided by the PPTs allow inferring the emergence of primary objects associated with three partial meanings of area (Caviedes et al., 2021): (1) Sp1-area as space delimited by a closed line; (2) Sp2-area as two-dimensional units covering a surface; (3) Sp3-area as a product of two linear dimensions. The emergence of partial meanings allows us to identify the way in which personal definitions of PPTs enable the emergence of processes of representation/meaning and idealization/materialization. The first is associated with the expression-content duality, where a given representation informs about the structure of that object (Font & Rubio, 2017). The second process is associated with the ostensive-non-ostensive duality, since mathematical objects are in general not perceptible, but are used in mathematical practices through their associated ostensive (e.g., notations, graphs, etc.) (Font & Rubio, 2017). We identify 3 types of definitions: (1) definitions that involve two dual processes, (2) definitions that involve one dual process; (3) definitions that do not involve dual processes.

The definition of PPT 55 (Figure 1) allows inferring the emergence of a representation/meaning process, since PPT 55 attributes a content (area) to a rectangular surface, granting multiplicative meaning to the structure of rows and columns of the area model.

PPT 55: *First, I would define it as the amount of space or surface area covered by a flat (two-dimensional) figure. The area is measured in square units of a fixed size, such as square centimeters, square inches, square miles, etc. Thus, if we delimit on the plan any figure, to find the area we count how many squares of a certain size will cover the region inside the polygon. So, an example would be the following, where we find a square formed by the following units (5x4):*

In this case, you can count the squares and get 20, so the area is 20 square units. However, this method can be ineffective if the rectangle is of larger dimensions or the units are smaller... in this case you can use multiplication, 5 x 4, since there are 4 rows of 5 squares.

Figure 1: Definition of PPT 55

The emergence of the concept of the square unit in the two-dimensional treatment of area (which can be expressed as the product of two lengths) is inferred, as PPT 55 mentions, superficially, that any figure (polygonal) can be measured using squares. Similarly, a process of idealization/materialization is observed, since PPT 55 evokes the properties of the units of measurement, which materializes the idea of area as a space that can be measured in square units and, subsequently, the algorithm that represents the area formula. In this sense, the idealization/materialization process allows giving a material sense to the area formula (the squares that cover the figure and the structure in rows and columns). From this process we infer the emergence of the concept of spatial structuring and the procedure of obtaining the area in an additive way, by means of graphic and symbolic representations.

Thus, a personal meaning associated with Sp1, area as space delimited by a closed line; Sp2, area as the number of two-dimensional units that cover a surface; and Sp3, area as the product of two linear dimensions, is inferred.

Moreover, we infer the emergence of two concepts, the surface as an attribute that allows objects to be measured, which has length and width (two-dimensionality), and the square units of measurement, as a region of the plane delimited by a square and as a fixed quantity of a physical magnitude.

The definition of PPT 65 (Figure 2) allows inferring the emergence of a representation/meaning process. PPT 65 attributes a content (area) to a polygonal figure, giving meaning to the property of being a bounded/enclosed space with a surface extension that can be calculated. To exemplify this, PPT 65 makes use of a counterexample (non-delimited/enclosed surface). It is possible to infer that the personal definition of PPT 65 reduces the area to the surface of polygons, without contemplating, for example, concave or irregular surfaces. In this way, a bias that relates area to conventional geometric representations is evident. In addition, it is possible to infer a personal meaning associated with Sp1, area as space delimited by a closed line, through the emergence of the concept of surface as a quality that allows objects to be measured, which has length and width (two-dimensionality).

PPT 65*:* *The area is a surface that we find within limits, therefore, we could say that it is the surface of a closed space. Moreover, this surface is flat (two-dimensional). We can calculate the area of polygons because they are a closed space, for example:*

But we CANNOT calculate the area of surfaces that are not bounded, for example:

Figure 2: Definition of PPT 65

The definition of PPT 56 (Figure 3) allows us to exemplify those definitions that do not show evidence of dual processes. It is possible to infer a personal meaning associated with Sp1, area as space delimited by a closed line and, likewise, the emergence of the concept of surface as a quality of two-dimensional objects. However, it is not possible to infer the emergence of dual processes, since the definition of PPT 56 does not make explicit the notion of closed or internal space of a two-dimensional figure or object, that is, there is no reference to the two-dimensionality of the area (beyond the use of the concept of surface), since a "thing" can have one, two or three dimensions.

PPT 56: *To introduce the concept of area in the fifth-grade classroom, I would define it as follows: area is the surface occupied by something.*

Figure 3: Definition of PPT 56

RESULTS

Table 1 presents the two dual processes derived from the personal definitions of PPTs.

Emerging processes	Descriptor (Personal meaning)	Partial meanings	Frequency
Idealization-materialization	(1) The emergence of the iteration procedure of units of measurement (standard and non-standard), and their respective properties, materializes the idea of area as a space that can be measured in square units.	Sp1, Sp2 Sp1, Sp2, Sp3	1** 2**
	(2) The emergence of the product formula procedure, through the rectangular model of multiplication, materializes the rectangle formula by structuring rows and columns.	Sp2, Sp3	1**
Representation-meaning	(1) A content is attributed to the square polygonal surface, giving meaning to the property of being a delimited/closed space that can be decomposed into other figures.	Sp1	1*
	(2) A content is attributed to a polygonal figure, giving meaning to the property of being a delimited/enclosed space with a surface area that can be measured.	Sp1 Sp3 Sp1, Sp2 Sp1, Sp3 Sp1, Sp2, Sp3	26* 2* 6* 17* 1**
	(3) A content is attributed to the rectangular surface, giving multiplicative meaning to the row and column structure of the area model.	Sp1, Sp2, Sp3	3**

Table 1: Meanings and processes emerging from the definitions of PPTs (N=69). [**indicates the number of PPTs mobilizing two dual processes in their personal definitions; *indicates the number of PPTs mobilizing one dual process in their personal definitions]

Each process is linked to one or more partial meanings of area (which constitute the institutional definition), and thus to specific primary objects which can be seen in detail in Caviedes et al. (2021).Table 1 shows that a majority of PPTs use the second representation/meaning process (attributing a content to a polygonal figure, giving meaning to the property of being a delimited/enclosed space with a measurable surface extension) based on Sp1. On the contrary, a minority of PPTs use the other processes of representation/ meaning, or the processes of idealization/materialization, which would allow them to make more complete personal definitions through the joint mobilization of the three partial meanings of the area. In addition, Table 1 also shows the relationship between meanings and processes. It is observed that the idealization/materialization processes occur only when two or three partial meanings of the area are connected. In the case of the representation/ meaning processes, it is observed that the first one (attributing a content to a square polygonal surface by decomposing it into other figures) occurs when mobilizing only Sp1. The second representation/ meaning process (attributing a content to a polygonal surface by measuring) occurs both by mobilizing the different partial meanings and by mobilizing only Sp1 and Sp3. The third representation/ meaning process (attributing a content to a rectangular polygonal surface by structuring rows and columns) occurs when Sp1, Sp2 and Sp3 are mobilized together.

The definitions of PPTs present a structure that differs in the emergence of the mathematical primary objects corresponding to the three partial meanings of the area, and in the processes that these meanings make possible. This way, the personal meaning evidenced, and therefore the concept image, can be equivalent to the institutionalized meaning if Sp1, Sp2 and Sp3 are mobilized in the process of defining. The joint mobilization of these meanings would account for robust personal definitions. Isolated mobilization of partial meanings would account for incomplete personal definitions.

CONCLUSIONS

The aim of this study was to characterize the personal definitions of the area of 2D figures of a group of PPT. The use of the configuration of objects and processes as a theoretical/analytical tool makes explicit the complexity underlying the process of defining (De Villiers et al., 2009), showing that this process involves describing objects and concepts, identifying properties of mathematical objects (Zaslavsky & Shir, 2005), communicating mathematical ideas, and coordinating different representations (Sinclair et al., 2012). The mobilization of different partial meanings makes it possible the emergence of certain processes that strengthen and complexify the definitions of PPTs. Hence, primary objects and processes account for the concept image that underlies personal definitions of PPTs (Tall & Vinner, 1981). Moreover, it is evidenced that PPTs concept image about the area of 2D figures can be put in correspondence with three partial meanings of the area (Caviedes et al., 2021), whose primary objects vary according to the intra-mathematical context in which they are framed. Therefore, this study considers the complexity of the process of defining (in terms of objects and processes) as a differentiating aspect of it and, at the same time, unifying, in the sense that it allows characterizing robust definitions through partial meanings of reference. In this sense, we agree that learning the process of defining should be an essential part of the learning process of mathematics in teacher training (Zaslavsky et al., 2003), since personal definition can have an impact on how they interpret the curriculum, what activities to propose and how to interpret students' mathematical practices. However, studies involving the design of teaching experiments based on these results are needed to strengthen the definitions and address the respective difficulties of PPTs.

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PROBLEM TYPES IN GEOMETRY TEXTBOOKS: RUSSIA'S EXPERIENCE

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This short paper is devoted to the analysis of problems offered in Russian geometry textbooks. These problems determine the types of activity prevalent in school work, which constitutes one of its important characteristics. We demonstrate that in Russian teaching practice, problems involving proofs and calculations continue to occupy an absolutely dominant position, and moreover, that students who are especially interested in mathematics are offered more problems involving proofs than "ordinary" students. We argue that the number of problem types (and consequently the types of activity in geometry classes) should be substantially increased in Russia, which requires specific work both on the part of textbook authors and on the part of teacher educators.

INTRODUCTION: FORMULATING THE QUESTION

Much has been written in recent years about textbooks and about geometry textbooks in particular. We might mention, for example, the historical overview by Schubring (2023) or the proceedings of special conferences devoted to textbooks, such as Rezat et. al. (2019). At the same time, studies devoted to problems in textbooks are rare. As an example of what might be characterized as a typical study, we should mention the paper by Usiskin (2014), which addresses the investigation of some topic and mainly discusses why this topic is useful, to what extent it is present in textbooks, and so on. Of course, there exists a vast literature about problems (for example, the recent book edited by Leikin, 2023, to name just one study), but in this case the focus is usually specifically on the process of problem solving or problem posing, and furthermore, problems are often examined without any connection to the specific textbook. Studies about problems in textbooks also exist, of course: for example (to offer a list that in no way aspires to completeness), there are studies that to one or another degree address proofs in problems from school textbooks (Stylianides, 2009), or the organization of sets of problems and their interconnections (Karp, 2023), or the differential cognitive loads associated with problem solving, for example, in connection with the number of problems for which the textbook provides no direct recommendations (Jäder et al., 2020), and so on. In other words, what such studies usually focus on are characteristics that are important specifically for solving problems, not characteristics that are specific to their subject (in particular, geometry). Meanwhile, it is interesting to understand what it is that the hypothetical student must actually do in class, and what it is that this student must learn to do by studying the subject – and this is something that finds expression first and foremost in problems.

The classification of problems is quite a complicated matter, not least because the same problem can be solved in different ways. Below, in discussing problem types, we will take our cue from the assignment: from what exactly the student is asked to do, for example, to prove, to calculate, or to construct using compass and straightedge. It is clear that by limiting ourselves to such an examination,

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we significantly simplify the picture – a problem that asks a student to prove something can be very simple or very difficult. Conversely, problems that ask students to construct something have usually included proofs as well. In general, we must acknowledge at once that we make no claim to address the full scope of the topic, if only because we will not analyze all textbooks or even all topics in selected textbooks. The present study is a preliminary one, and we hope to continue it in several directions, the most important of which is an international one: in the present study, we will confine ourselves to textbooks from one country, but it would be interesting to analyze and compare different countries from this point of view, as we believe the results might turn out to be very different.

Let us note that for quite a long time the study of geometry, as well as other mathematical disciplines, practically did not involve independent problem solving at all. Høyrup (2014), a historian of medieval mathematics, asks why medieval universities did not prepare major mathematicians, and sees the explanation for this in their form of instruction, which was aimed at attaining wisdom, not making it. They prepared not mathematicians, but religious thinkers. This tradition endured for a long time beyond the Middle Ages, even if not in such an explicit form – problems began to be solved and presented in textbooks, but they were few in number, and their role was minor – and in addition, it was different than what it is today, aimed not at development, better assimilation, training, etc., but once again at acquiring wisdom, except now it was somewhat differently structured (Karp, 2020).

In general, the answer to the question, "what must a student take away from a course in geometry?" has varied in different eras and for different segments of the population. I will permit myself to quote from a review, from 1849, of Geometric Notes for Young Women by V. Mikhelson: "V. Mikhelson had made of geometry everything that could possibly be made of it by the most tender feeling for young women – he has turned it into a light art, similar to embroidery or other female pursuits, of using geometric terms in appropriate situations" (Anon, 1849). This was achieved, of course, by means of a reduction of the level of difficulty, but also by the author's understanding of what exactly young female students were supposed to do.

It is natural to pose the question formulated above not only for different periods and different segments of the population, but also in relation to presumed levels of mathematical ability. Accordingly, the present study attempt to address the following research question: What types of problems (in the sense established above) are represented in "ordinary" and in more "advanced" Russian geometry textbooks, and what is the relation between problems of different types in such textbooks.

Let us repeat that the methodological problem of determining the type to which one or another problem belongs will be solved based on the formulation given in the problem itself ("prove," "find," "draw," etc.). The identification of new types, that is, instances in which several problems can be united in one group, will be discussed separately below.

ON RUSSIAN (SOVIET) GEOMETRY TEXTBOOKS

We will be analyzing Russian geometry textbooks, and therefore we need to say a few words about their specific characteristics. In what follows we rely on the study by Karp and Werner (2011). The Russian course in geometry took shape in the nineteenth century and has not undergone too many changes: it has remained traditional in the sense that, just as it did back then, it still represents a systematic and methodical exposition of plane geometry in grades 7-9 and of solid geometry in grades

10-11. We should emphasize at once, therefore, that the Russian course is one that is taught over five years for 2-3 hours per week, rather than a course taught for only one year for five days a week, as it is in the USA, for example. The textbooks contain many proofs and it is expected that practically every assertion will be proven (in reality, of course, this is not quite the case – a number of fundamental assertions are accepted without proofs, and the list of axioms, which are referred to by precisely this word and are simply stated or “swallowed,” is more extensive than it needs to be).

Our investigation is made easier by the fact that the USSR followed a single textbook policy, in other words, the same textbook was used across the country. Subsequently, even already a couple of years before the collapse of the USSR, different textbooks began to be permitted, but in any case their number was and remains small (textbooks go through state expert review and must be approved in order to be admitted to the schools; the teachers are expected to teach using the textbook, although the extent to which what happens in class corresponds to the textbook has hardly been monitored in any way in recent decades). Consequently, in analyzing, say, the textbooks by Atanasyan, et al. (2004), we can be confident that they provide a realistic understanding about what actually goes on in Russian schools – it is a certainty that they are used by substantially over one half of the students in the country.

We will focus on textbooks that are used today or that were used very recently. Even so, to convey a sense of the tradition, we should say that from the 1930s until the mid-1970s, schools used Andrey Kiselev’s textbooks, which were first published in 1892, and Nikolai Rybkin’s problem book. In his introduction to his textbook, Kiselev himself explained that: “The book is supplied with a considerable number of exercises, comprised in part of certain theorems that did not make it into the text but are of interest, but mainly of compass and straightedge problems and calculation problems.” This had been even more the case with the textbook by Avgust Davidov, which dominated the market during the pre-Kiselev period and remained in use in schools for over a half-century. It is noteworthy that initially, this textbook contained only problems involving proofs and construction. Only later were problems involving calculations added. And both the former and the latter remained relatively few in number: in the entire course on plane and solid geometry, there were 256 problems involving proofs and construction, and 337 problems involving calculations (Karp, 2015). Note that later books, written under the influence of the international reform movement, were usually characterized by far greater variety. For example, the geometry textbook by Alexander Kulisher from 1922 also contains problems that require students to draw something, or to cut something out, or even to describe what they feel with their hands when touching one or another geometric object – in this case, the influence of the idea of the informal study of geometry is evident. However, it is clear that the success of such innovative approaches at that time was very limited (Karp, 2012). But let us turn to the present.

TWO CONTEMPORARY TEXTBOOKS

We will first discuss two widely used geometry textbooks – the already mentioned textbook by Atanasyan, et al. and the textbook by Pogorelov (for example, Pogorelov, 1985). Both textbooks appeared in the late 1970s and during a certain period practically divided the whole Russian market between them. Subsequently, the textbooks of A.V. Pogorelov, who died in 2002, became less popular, but continued to be actively used later as well. Over the years, certain changes appeared in these textbooks (for example, the numeration of grade levels changed, and grades 6-10 became grades 7-11), but in terms of their approaches to problems, there were, in our view, no noticeable changes.

Below, we will limit ourselves to analyzing one section, Quadrilaterals. The topics covered in both textbooks in this case are practically identical, apart from the fact that this section in Pogorelov's textbook includes a paragraph devoted to the so-called Thales's theorem about equal line segments intercepted by parallel lines on the rays of an angle. In Atanasyan's textbook, this theorem is given as a problem (although its solution is immediately given as well). The results of the analysis are shown in the table below.

Assignment	Atanasyan	Pogorelov
Prove	22	17
Calculate	22	40
Construct	10	8
Draw	0	1
Analyze a drawing	3	1
Investigate	1	1
Count the variants	3	1
Total	61	69

Table 1: Problem types in textbooks by Atanasyan, et al. and Pogorelov

Let us explain the categories introduced above. While the category "Construct" includes problems that require students to carry out a rigorous straightedge and compass construction which is usually unique, under the category "Draw" we include assignments that require students, for example, to construct any parallelogram. Under the category "Count the variants" we include problems that require students, for example, to determine how many parallelograms can be constructed with vertices in three given points that are not on the same straight line. "Analyze a drawing" includes, for example, a problem that requires students to select that letter in a picture which has an axis of symmetry. Lastly, "Investigate" may include quite different assignments, which require students to verify (sometimes by proving, sometimes by providing an example) one or another assertion, for example, that any quadrilateral whose diagonals are equal and perpendicular is a square.

TWO OTHER, LESS POPULAR TEXTBOOKS

We will now analyze two textbooks that were published in much smaller editions – one of them, Atanasyan, et al. (1996), consisted of supplementary chapters to the textbook by the same authors cited above, intended for schools and classes with an advanced course of study in mathematics (sometimes loosely called schools for the mathematically gifted). This textbook included certain additional theoretical topics and many additional problems. Another textbook, Smirnova and Smirnov (2001), was intended for use in "ordinary" schools, but represents an attempt to make a more "humanities-friendly" textbook, that is, to take into account the varied interests and inclinations of the students. This textbook, too, was published in relatively small editions. In both cases, as above, we will analyze the section devoted to quadrilaterals – in the textbook by Atanasyan, et al. (1996), this section constitutes a separate chapter; in the textbook by Smirnova and Smirnov (2001), it consists of several paragraphs.

Assignment	Atanasyan, et al	Smirnovs
Prove	66	24
Calculate	6	37

Construct	6	9
The World around us	0	1
Investigate	20	6
Count the variants	2	1
Total	100	78

Table 2: Problem types in textbooks by Atanasyan, et al. and Smirnova and Smirnov.

Note that the problem we included in the category "The World around us" requires students to explain situations from the surrounding world – specifically, why a certain axis occupies a vertical position.

TWO SOLID GEOMETRY TEXTBOOKS

Let us analyze two more textbooks, in solid geometry, by Atanasyan et al. (2000) and Sharygin (1999). The former remains the most widely used solid geometry textbook in the country, while the latter is distinguished by the author's methodological approach. We will analyze the chapters devoted to straight lines and planes in space, which are similar in content (without taking into account additional problems given separately in the textbooks).

Assignment	Atanasyan et al.	Sharygin
Prove	83	37
Calculate	44	40
Construct	13	5
The World around us	2	0
Investigate	10	6
Analyze the Drawing	1	0
Count the variants	0	2
Total	153	90

Table 3: Problem types in textbooks by Atanasyan et al. and Sharygin.

DISCUSSION

Naturally, each textbook possesses individual traits that are characteristic specifically of its authors, and there is no reason to attribute them to all textbooks that are addressed to the corresponding group of students. Textbooks in Russia go through a many-stage review process, and if an author today decides to stop at something easy and pleasant, like embroidery, their book will simply not be allowed in the schools; but within the bounds of the permissible, sufficiently many possibilities remain. Nonetheless, an analysis of textbooks used by many millions of students makes it possible to draw certain conclusions about the existing reality – about what exactly educators want to teach students in the course in geometry.

Evidently, two things are demanded: the ability to perform geometric calculations and the ability to carry out proofs; everything else is less important. Moreover, the great majority of the supplementary problems for students interested in mathematics are devoted specifically to proofs; but "ordinary" students are very often given such problems as well. In this study, it is not our aim to address comparisons with other countries, but clearly the frequently encountered pessimism with regard to the "ordinary" student's ability to carry out proofs is unjustified. (We might recall a similar discussion when Krutetskii refuted Thorndike, who had claimed that the standard course in algebra was not accessible to a great proportion of students: Krutetskii (1976) showed that in the USSR it was successfully assimilated, if not by everyone, then by a far greater percentage of students than was to be expected according to Thorndike.)

Straightedge and compass construction problems have clearly receded into the background – they are noticeably fewer in number. Moreover, going into a deeper analysis of the problems, we should point out that an assignment to construct a parallelogram based on its sides and angle is by no means the same thing as the classic straightedge and compass construction problems with their many steps, analysis, and unobvious proof. This has disappeared from the course in geometry.

On the other hand, what has clearly appeared and is more noticeably represented in textbooks for schools with an advanced course of study in mathematics are problems that we characterized as investigative, which require students to determine whether something can be the case, or to verify the correctness of various assertions. In "ordinary" textbooks, such "open" problems exist, but in very small numbers (which in our view is explained by a certain conservatism on the part of the authors and by the fact that such problems remain unfamiliar to teachers). At the same time, in the textbook by Smirnova and Smirnov (2001), which is aimed at a more "humanities-oriented" student, the number of such problems is somewhat higher – informal geometry and the ability to imagine one or another construction (or to understand that it might not exist) are necessary and accessible to everyone.

We should mention one other characteristic of the materials we have analyzed. They contain almost no problems involving the application of geometry or real-world problems. It may be objected, of course, that this is not the case in all sections – to repeat, we did not analyze all sections, or all textbooks – but on the whole, their absence is telling. In general, we believe that the analysis we have carried out makes it possible to discern important characteristics of the teaching of geometry in Russia with its positive and negative aspects – an unquestionable conservatism, but at the same time the preservation of a high level of substantiation-by-proof and reasoning. To repeat, it would be interesting to investigate the geometry textbooks of other countries as well, and to compare the obtained results.

SOME PRACTICAL CONCLUSIONS

Textbook authors are usually aware of what specific types of problems they offer in their textbooks. Below are several types of problems identified by Valery Ryzhik, one of the authors of the textbooks in geometry written under the supervision of the well-known geometer and academician Alexander Alexandrov (citation based on Karp, & Werner, 2011):

- *Looking.* Students are taught to interpret information presented in visual form, and students' spatial (2-dimensional and 3-dimensional) imaginations are developed.
- *Drawing.* Students develop their spatial thinking skills.
- *Investigating.* Problems whose conditions or possible results may contain some uncertainty, incompleteness, and ambiguity.
- *Applying geometry*. Problems from outside mathematics that must be translated into mathematical language.

As we can see, this list includes problem types that we have already encountered as well as new ones. For example, the type that is labeled "Investigating" differs somewhat from the type that was called so by us. The author quoted expressed the view that, for example, it is useful to give students "impossible" problems, such as computing the area of a right triangle, given one of its legs and the projections of both of its legs onto the hypotenuse (it is evident that there are too many givens here). This, in his view, develops their capacity for observation and for working in the real world, where problems are not necessarily correctly posed. The type "Looking" lends itself especially to the study of three-dimensional geometry, in which it is important to be able to reach a conclusion by looking at a drawing.

Note that in addition to the types listed above, there are also others, and combinations of the types represented above are possible as well. As an example, consider the following problem (Werner & Karp, 2019, p. 336):

> Consider four points in space, A, B, C, D. We would like to construct all planes that contain at least three of these points. It is not true that these four points could be chosen such that it would be possible to construct:
>
> a) only one plane;
>
> b) only two planes;
>
> c) only four planes;
>
> d) infinitely many planes.

This is a kind of multiple-choice problem, but what it requires of the student is not only the answer, but also the reasoning behind it. In other words, the student must imagine (draw!) such positions for the four points for which it will be possible to construct one, four, or infinitely many planes, and then to prove that it will be impossible to construct only two planes. This is a kind of investigation problem, involving various kinds of activity.

Working teachers, in our view, must be familiar with different types of activities in which students are involved in the process of studying geometry, without limiting themselves to the classical types. Our conception of geometry today is broader than it was two thousand years ago. Without giving up classical problems involving proofs and calculations, we should broaden the palette of possibilities – which poses new problems both for textbook authors and for teacher educators.

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CASUAL AND GEOMETRIC PRAXEOLOGIES IN A STUDY OF SYMMETRIC FIGURES

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In this paper we collect various solutions produced by in-service teachers in response to problems involving regular polygons. These solutions allow us to identify the most popular ideas found in the contemporary K-12 geometry curriculum. In line with previous research, we found that while the figures in question possess axial symmetry, this property is very rarely used by teachers. Despite the fact that the concept of symmetry is familiar to teachers at the intuitive level, comprehension of this concept is lacking the mathematical precision. Thus, this visual characteristic of the objects of study fails to be employed in teachers' geometric reasoning.

INTRODUCTION

Symmetric figures do not change when reflected, rotated or translated in a certain way. This property could be both intriguing for human visual perception and useful for our practical and theoretical needs. Symmetry "creates patterns that help us organize our world conceptually" (Knuchel, 2004). The idea of symmetry connects between various branches of mathematics and plays an important role in problem solving (Leikin et al, 2000). In addition, "symmetry is a paramount tool for bridging the gap between science and art" (Livio, 2005). Since ancient times of Aristotle and Plato symmetry has been associated with beauty as well as played a significant role in architectural design.

Psychologists investigated the relationship between symmetry and aesthetic preference by asking people to make a "visually pleasing" arrangement of objects given certain constrains. It was found that the majority of arrangements were symmetrical in nature (see e.g. Szilagyi & Baird, 1977). Birkhoff (1933) attempted to develop a mathematical theory of aesthetic value. He proposed that at a given level of complexity, the aesthetic value is higher for more symmetric objects. The role of symmetry in perception was stressed by Gestalt psychology. The principle of Prägnanz (or "good figure") states: "of several possible geometric organizations, the one that is seen is the one that possesses the best simplest and most stable shape" (Livio, 2005), which in practice oftentimes is symmetric. For example, a picture showing four dots which are vertices of a square is seen as a square - a symmetric, stable and closed shape. A perceived stability may be related to the fact that frequently physical systems possessing symmetry are stable because they furnish minimum potential energy.

Mirror-reflection symmetry is a specific form of symmetry which characterizes the animal kingdom. The prevalence of the bilateral symmetry in animal appearance for many million years (which is evidenced by both fossils and contemporary leaving creatures) could be explained by evolutionary biology and laws of physics. Gravity and locomotion produce distinctions between top and bottom or back and front. However, the animals' environment is much more homogeneous when it comes to distinguishing between left and right. Therefore, it is not surprising that the most popular interpretation of the word "symmetry" actually is reflection symmetry. According to Webster's Third

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 181-188). ICMI.

New International Dictionary, symmetry is “correspondence in size, shape and relative position of parts that are on opposite sides of a dividing line or median plane”. Mach (1914) noted that figures are perceived as symmetrical primarily as a result of reflection symmetry about a vertical axis. In particular, perception of reflection symmetry may depend on the orientation of the object; the symmetry may not be detected by an observer unless one turns the object in such a way that the axis of symmetry becomes vertical.

Based on the short summary presented above, it is reasonable to assume that humans have a natural ability to recognize and even are drawn to symmetrical objects, in particular to those possessing the vertical axis of symmetry. In this article we aim to investigate the practicality of this ability for a group of in-service teachers confronted with a geometrical question involving symmetric shapes.

THEORETICAL FRAMEWORK

We draw our theoretical framework from the Anthropological Theory of the Didactic (ATD), in particular the notions of praxeology and institutionalization (Bosch et al, 2019). Praxeology is a basic notion to describe human acts in particular situations. Praxeology consists of praxis and logos blocks. Further, the praxis block consists of types of tasks and techniques to solve them, while logos consists of rationales explaining and justifying the techniques used in praxis and theories within which these rationales exist.

ATD proposes that praxeologies undergo a transposition as they occur in different social institutions such as research communities, practice communities, or learning communities e.g. schools. Within different institutions different conditions are present that support or restrict individuals in the process of familiarization with objects of their study.

By observing representatives of a given institution one can judge about their prevalent praxeologies for dealing with certain tasks. Thus, we can talk about geometric praxeologies that could be learned in a course of study of this subject. These could be contrasted with casual praxeologies based on intuitive or visual rationales lacking a formal theory. Casual praxeologies can be present in performance of casual tasks for which a person had not been trained, such as making a visually pleasing arrangement of objects in the experiments of Szilagyi & Baird (1977). In particular, the idea of symmetry could be developed by osmosis, a process of subconscious assimilation of this idea from visual information, and be “based on experiential meanings of reflective symmetry as yet not connected with any accepted mathematical definition” (Hoyles, 1997). Alternatively, the notion of symmetry based on “socially-shared understanding of reflection from [..] everyday experiences” could be “expanded and connected to a mathematical meaning for the transformation,” (Hoyles, 1997) providing more precision. The two scenarios lead to developing different types of praxeologies for working with symmetric figures.

DATA COLLECTION

The data had been collected in a graduate course offered at Memorial University (Canada) online for in-service K-12 teachers. The course focuses on mathematical thinking (Mason et al, 2010) and includes such activities as individual problem solving, group discussions, and journal writing. One major assessment is based on the interconnecting problems, defined in (Kondratieva, 2011) as problems that (1) allow a simple formulation; (2) allow various solutions at both elementary and advanced levels; (3) may be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions, and (4) can be used in different grades and courses and understood in various contexts. The Hexagon Problem is an example of an interconnecting problem used in the course and served as a generating question for this study: *Show that in a regular hexagon there is a diagonal that is perpendicular to one of the sides.*

Students discussed the given interconnecting problem in randomly formed groups (normally) composed of 2 elementary and 2 secondary school teachers. There were no restrictions on using any resources or technology. Each group was required to construct at least 3 different solutions at various levels of sophistication. Each problem was discussed by students online during two weeks. Students submitted their groups' reports in both written format and as a short (20-30 minutes) video where they presented and discussed these solutions within their group.

In addition, students were asked to report in their individual journals regarding an extension problem: *Describe all regular polygons that have a diagonal perpendicular to one of the sides.*

The data collection took place in 2022 and 2023 in the school terms when the author was teaching the course. There was a total of 29 students enrolled in the course during this period. Participants' video, written group reports, and individual journals reflecting student's experiences with problem solving in the course had been analyzed. The most popular solutions are discussed in the next section.

The research questions

(1) What are the common techniques proposed by K-12 teachers to solve the Hexagon Problem?

(2) What is the role of symmetry in teachers' reasoning about this problem and its extension?

RESULTS AND DISCUSSION

As instructed in the textbook for the course, students started their investigations from listing the facts they know about regular hexagons such as "all six sides are equal", "all six interior angles are equal" and the sum of interior angles is (6 - 2) * 180° = 4 * 180° = 720°. They also clarified the terminology, for example, "*perpendicular* means intersecting at a right (90°) angle" or "a *diagonal* is a line joining two non-adjacent corners of a straight-sided shape." Then they aimed to connect these facts to the statement to be confirmed.

> I began solving this problem by identifying what I knew about the question and what I wanted to know. From prior knowledge, I knew that the sum of a hexagon's interior angles is 720°, which means that each interior angle must be 120°. Initially, I visualized the perpendicular diagonal I wanted to attack, but I wasn't quite sure how to get there.

Since it was required to find three solutions, students were searching for several useful ideas and possibilities.

> I drew a hexagon and quickly jotted down previous knowledge about this shape that I believed could aid in finding a solution, which included its sum of angles, the degree (120°) of each interior angle, and the possible division of this shape into six equilateral triangles. […] Yet, I decided that using right-angle triangles would be better suited to aid in this goal since I knew that applying the Pythagorean theorem and the slope formula could easily be applied to this specific triangle.

The most popular solution of the Hexagon problem was as follows (see also Figure 1).

> I drew my suspected perpendicular diagonal (the green line) from here. Further specialization created an isosceles triangle with two equal sides and, thus, two equal angles. […] I determined that these two angles must be 30° each. Knowing that the entire black dotted interior angle (as shown in the second hexagon)

was 120° and that ∠b was 30°, I determined that my green line was perpendicular due to a 90° angle. Generalizing this solution to the rest of the hexagon, I could now clearly state that any diagonal connecting two vertices one vertex apart would be perpendicular to a side of the hexagon.

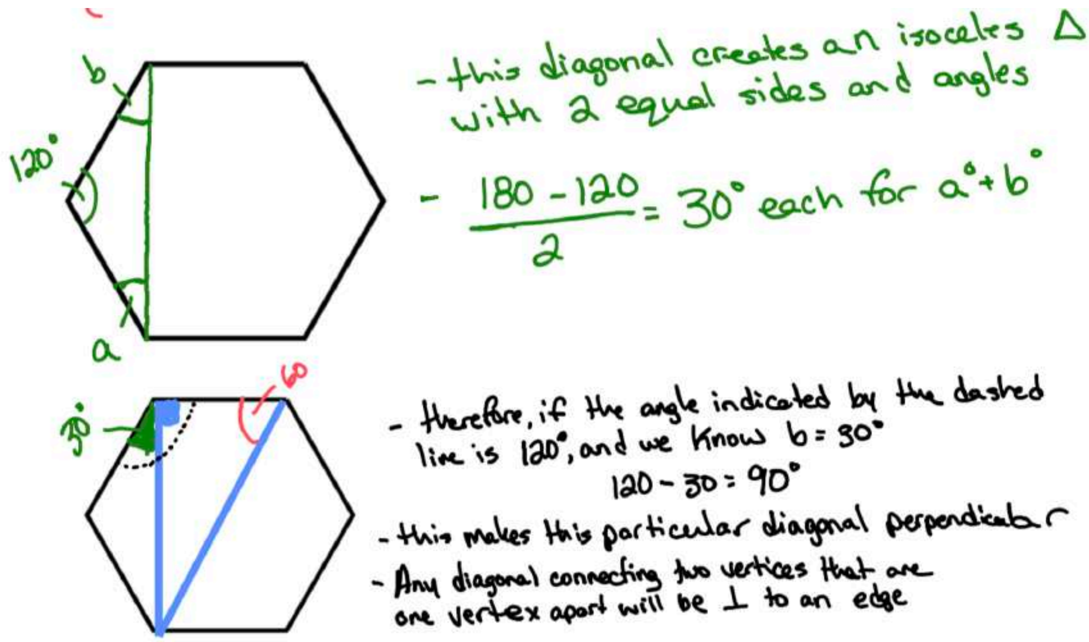

Figure 1. The most popular solution of the Hexagon Problem

Another solution based on angles consideration used triangular decomposition of a regular hexagon:

> When the diagonals are drawn, six equilateral triangles are formed (Figure 2). They are equilateral because in a regular hexagon, the radius of the hexagon (distance from the center to a vertex) is equal to the side length. In an equilateral triangle, all angles are 60°.
>
> It follows that <DAF = 60° and <DAB = 60°. The diagonals bisect the angle, making <EAF = 30°, <DAE = 30°, and <CAD = 30°. Therefore, <CAF = 30 + 30 + 30 = 90°. Since <CAF = 90°, diagonal AC is perpendicular to side AF. In the same way, diagonal AC is also perpendicular to side CD.

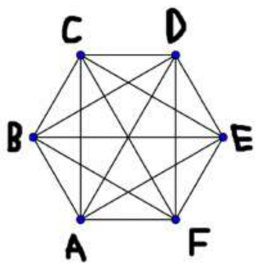

Figure 2. All diagonals and equilateral triangles in a regular hexagon

The idea that "the diagonal bisects the angle" was further explained as follows.

> We also know that when we divide an equilateral triangle in half, starting at one of the three vertices, 2 right triangles are created, resulting in angles of 90°, 60°, and 30°.

This property follows from the axial symmetry of an equilateral triangle, however students did not explicitly refer to the idea of symmetry. The idea of this approach can be developed into a rigorous proof by noticing that the angles EAF, DAE, and CAD support equal chords of the circumcircle.

The third approach employed analytic geometry, introduction of coordinates and the notion of slope. The following solution started from specializing by taking side length equal 3 units and calculating slopes of lines AB and BG (Figure 3).

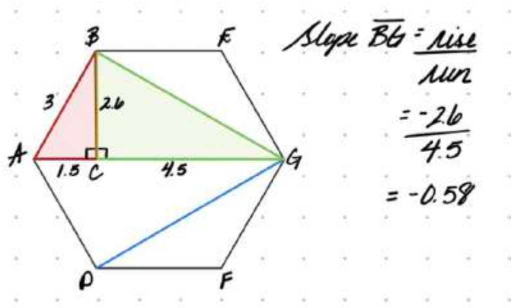

Figure 3. A solution based on introduction of coordinates and calculation of slopes of AB and BG

Note that the student made approximate calculations before moving to the general case:

> My goal was a generalization, so I let x represent any possible side length of the hexagon and set vertex G of the hexagon as the origin, $(0,0)$, of my graph since it is where I initially drew my diagonals. With the use of my previous specialization and Pythagoras' theorem, I was able to determine the coordinates of vertex A, $(-2x,0)$, and vertex B, $\left(\frac{-3x}{2},\frac{x\sqrt{3}}{2}\right)$ as shown. Next, using the slope formula, I found the slope of $\overline{AB}=\sqrt{3}$, and the slope of $\overline{BG}=-\frac{\sqrt{3}}{3}$. This was my AHA! moment. First, upon completing a quick calculation, it became evident that these slopes were negative reciprocals of one another and thus perpendicular. Furthermore, after reviewing my newfound solution, I recognized that the side length is irrelevant throughout my calculations since it is ultimately eliminated. This reveals that the slopes of these lines will always be the same regardless of the size of the hexagon.

In the fourth solution students assumed the side length of the hexagon to be x. Therefore, the long diagonal was measured as $2x$. The goal was to find the length y of the short diagonal. This can be done using the Cosine theorem as shown in Figure 4. Then, since the sides of the triangle formed by the long diagonal, the short diagonal and the side of the hexagon satisfy the Pythagorean formula, it follows that these segments form a right triangle, and so the short diagonal is perpendicular to the side of the hexagon.

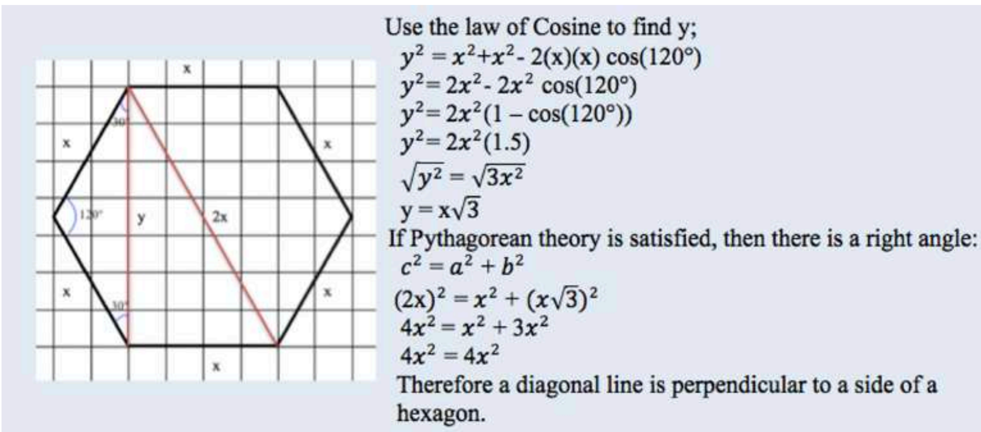

Figure 4. A solution based on the Cosine and inverse Pythagorean theorems

We note that most students availed of various technological tools helping to draw an accurate picture of a regular hexagon with its diagonals. Some of them even imitated a construction of this shape using a compass. Figure 5 gives one such example and also shows the right angle between a diagonal and a side. This conclusion could alternatively be made with reference to the Thales theorem (about three distinct points on a circle forming a right-angled triangle provided that two of the points lie on the opposite ends of a diameter).

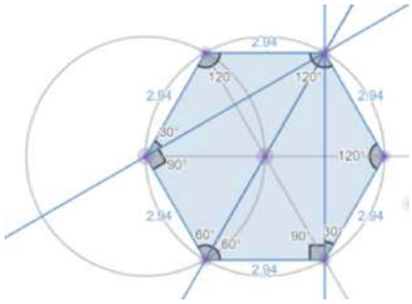

Figure 5. Construction of a regular hexagon using a compass and the Thales theorem

For some teachers using measurement tools was the first step and actually the only way of verifying the statement. One of them explained in their journal:

> Geometry is something that I am not super familiar with and is definitely something I still struggle with. When I originally solved the hexagon problem, I used measurements to help me solve the problem. The first way I did it was involving using a protractor. The second way I solve it was using GeoGebra to find the slopes of the two lines and prove that they were perpendicular because they were negative reciprocals. Both of these involved measurements and were kind of a way around using geometry.

A couple of teachers "confused the meaning of perpendicular with parallel" and many of them found this problem challenging because "these are definitely topics we do not touch on often within the current mathematics curriculum." In their journal a teacher confessed:

> Geometry is an essential area of mathematics, but its placement in the curriculum and the required activities would seem as if it's disappearing from the curriculum. In the junior high math curriculum suggested sequence, geometry is the last unit. Teachers are rushing to finish the course. The quality of work done in this unit is frequently limited, with a lower level of critical thinking.

Another teacher echoed the idea:

> It is a shame, honestly, that geometry is often pushed to the back burner because teachers find other components of math to be more important.

Nevertheless, certain geometrical ideas were well known to the teachers and allowed them to successfully approach the Hexagon Problem as is evident from the solutions presented above.

CONCLUDING REMARKS

The notion of symmetry is present in the K-12 mathematics curriculum. "Paper-folding or grids are used in early activities to assist in "the doing" of a reflection" (Hoyles & Healy, 2002). Primary and elementary students also use manipulatives such as Mira Geomirror to obtain reflections and thus creating a symmetric figure. More broadly, reflections and symmetric shapes can be produced using various technologies such as dynamic geometry software (Mackrell, 2002). Secondary school students observe reflection symmetry when they graph even functions such as parabolas or a composition with the absolute value y=f(|x|). An instrumental idea that gives precise meaning to the reflection symmetry is that the segment connecting any point with its image is perpendicular to the axis of reflection. Thus, a point and its image are at the same *perpendicular* distance from the axis.

A regular hexagon has six lines of symmetry: three lines passing through its opposite vertices and three lines passing through the midpoints of its opposite sides. In particular, a regular hexagon ABCDEF shown in Figure 6 is symmetric with respect to the line passing through point G and H, where G is the midpoint of the side BC and H is the midpoint of the side FE. The pairs of points B and C, A and D, F and E are reflections of each other with respect to the line GH. Therefore, the segments BC, AD and FE are perpendicular to GH, and thus parallel to each other. The hexagon ABCDEF is also symmetric with respect to the line passing through vertices A and D. The sides BC and EF are parallel to the line AD, where the vertex F is a reflection of B and the vertex E is a reflection of C. Therefore, the diagonals BF and CE are perpendicular AD and to the sided BC and EF.

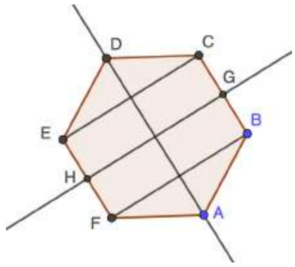

Figure 6. Two lines of symmetry in a regular hexagon and a symmetry-based solution

In my experience, the reasoning along these lines was a popular answer to the Hexagon Problem amongst my colleagues-mathematicians. At the same time, none on the 29 in-service mathematics teachers, whose solutions were collected, proposed that the symmetry could be useful. Two students stated the facts that a regular hexagon has reflection and rotational symmetries, but they did not develop any approaches based on this property.

One group of teachers noted that BCEF appears to be a rectangle (Figure 7), but during the videotaped group discussion they found that they did not know how to prove that all angles in this parallelogram are 90°.

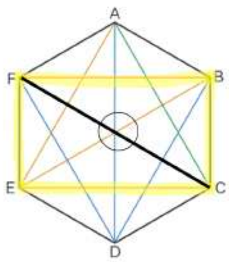

Figure 7. A group of students conjectured that BCEF was a rectangle

The thoughts about symmetry did not cross students' minds at that time as is evident from the following journal entry.

> I could clearly see a rectangle mixed into the crossings of the diagonals and sides of the hexagon. From here, I used the properties of a rectangle to prove that one of the diagonals was perpendicular to one of the sides. It was not until I read our group feedback that I thought of/considered using lines of symmetry to solve this problem and prove my solution was valid. With symmetrical lines in mind, I know that an octagon would also be another example of a regular polygon that has a diagonal that is perpendicular to one of its sides.

> It was reassuring to see that the idea of symmetry (suggested in the instructor's feedback to the group) was used for the extension problem by the above-mentioned student. However, many other students did not talk explicitly about lines of symmetry but rather noted a property that uses this fact implicitly.
>
> I concluded that in polygons with an even number of sides, the opposing sides will always be parallel, allowing for a 90 ° angle formation when a diagonal is drawn between them.

We conclude that the teachers participating in this study were familiar with several praxeologies and ideas related to geometry, such as the sum of angles in a regular n-gon, angles in an isosceles triangle, Pythagorean theorem and slopes of perpendicular lines. These ideas are clearly present in current geometrical curriculum and are well understood and employed by the teachers.

However, our results echo the observation that mathematics teachers usually do not use symmetry in problem solving unless they are trained in this direction (Leikin et al, 2000). Teachers seem to be lacking precise mathematical understanding of symmetric figures and thus perceive symmetry-based reasoning as unreliable. There was an apparent gap between teachers' intuitive sense of symmetry of an entire hexagon and a point-wise symmetry of its vertices that would allow them to develop a geometric argument resolving the Hexagon Problem. In other words, they lacked a geometric praxeology for dealing with the idea of reflection symmetry, while their casual praxeology of working with symmetric figures was not sufficiently precise for making a mathematical argument. Consequently, teachers' geometric reasoning was not based on this visual characteristic of the figures.

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LEARNING STAKES TARGETED BY TEACHERS IN A FIGURE RESTORATION ACTIVITY

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Our research focuses on teachers' practices when teaching geometry. In this text, we report on the way in which some teachers (whom we have had the opportunity to follow) take into account the stakes of geometry teaching in the 'heat of the moment'. More precisely, we use the notion of didactic vigilance, to report on the manner in which these teachers prioritize (or not) the finality of geometric learning when implementing a figure restoration situation in elementary school.

INTRODUCTION

Numerous studies conducted in the field of mathematics didactics show that geometry teaching presents specific difficulties for teachers (Mathé, Barrier & Perrin-Glorian, 2020), and that it is often focused on vocabulary acquisition, knowledge of some properties and mastery of the usual instruments (Mangiante-Orsola, 2014). More generally, our work shows that knowledge at stake are difficult for teachers to perceive and to take into account when implementing teaching-learning situations in geometry (Guille-Biel Winder, 2021; Guille-Biel Winder & Mangiante-Orsola, 2023): for example, teachers often give priority to the accuracy of plotting, to the detriment of the validity of figure construction procedures. This text is based on a larger clinical study (Mangiante-Orsola, 2023) which reports on the follow-up of nine teachers carried out as part of an in-service training program. In this program, training sessions alternate with in-class experimentations. We provided teachers with a document describing the geometry sequence *Triangles on quadrilaterals* (ibid.), and then observed their practices as they implemented the sequence. In this text, we present an analysis of the two teachers' practices, Vic (who teaches pupils aged 9 to 11) and Lea (who teaches pupils aged 9 to 10), based on the data collected: for each teacher, video recordings of the four sessions implemented and audio recording of the interview conducted after each session. Our intention is to report on the professional gestures of these teachers to better understand how, 'in the heat of the moment', they adjust their responses according to the students' proposals and consider as much as possible the stakes of geometry teaching. After introducing our theoretical framework and the methodology, we present the teaching-learning situation and analysis. This reveals three aspects of the situation that a teacher should be vigilant about implementing. We then present an analysis of the two teachers' practices regarding these three aspects. This analysis enables us to identify the teaching priorities of each teacher.

THEORETICAL FRAMEWORK

We believe that it is important to draw on the observation of actual practices to analyze the practices of schoolteachers when teaching mathematics (Butlen, Mangiante-Orsola & Masselot, 2017). To study these practices, we rely on the theoretical tools developed by Butlen, Charles-Pézard and Masselot (op. cit., 2012; Charles-Pézard, 2010; Butlen & Masselot, 2019). We take up the notion of professional gestures that Butlen et al. (2011) interpret in terms of schemes[11] (Vergnaud, 1997) and define as elementary activities built up through professional experience (for example, giving instructions, or managing the layout of what is written on the blackboard…). We use the expression 'gestures compositions' to designate sets of professional gestures with the same aim (for example, managing a clarification phase, support students during the research phase…). We also consider the notion of 'didactic vigilance' (denoted DV in the following) defined by Charles-Pézard (2010) as a permanent adjustment by the teacher to prioritize the session 'as close as possible' to the targeted mathematical learning. This concept allows us to study how teachers take teaching stakes into account (Butlen et al., 2011). Exercising some didactic vigilance requires not only the teacher's mastery of mathematical and didactic knowledge, and the implementation of gestures and gestures compositions regarding the targeted learning, but also the preponderance of mathematical learning purposes. This leads us to model these conditions as a triplet (Mathematical and Didactic Knowledge, Gestures and Gestures compositions, Mathematical Learning Purposes) and to consider the three elements of this triplet together (Guille-Biel Winder & Mangiante-Orsola, 2023). The notion of DV can be related to other notions developed in the field of mathematics didactics. For example, the concept of 'knowing-to act in the moment' (Mason & Spence, 1999) highlights the teacher's ability to draw on various types of knowledge to react to the situation, and involves a state of awareness, of readiness to 'see in the moment'. Like DV, this concept emphasizes the need for the teacher to react in the heat of the moment. The notion of 'cognitive clarity' (Downing & Fijalkow, 1984) stresses the need to help pupils recognize the object of learning and, like DV, leads us to question the teacher's ability to stand back and identify "what he or she is supposed to be leading up to". However, the dynamic nature of the concept and the need to simultaneously take into account the three conditions modeled as a triplet distinguishes the notion of DV from the notions previously mentioned. Reacting in the moment in relation to the learning targeted must be done throughout the session and supported by mathematical and didactic knowledge through the implementation of gestures and compositions of gestures.

GENERAL ANALYSIS METHOD

In the following, our intention is to describe how each teacher considers the geometric learning issues at stake. To this end, we have highlighted triplets for each of them, which model their exercise of DV and are contextualized to the selected teaching-learning situation. According to our previous work (Guille-Biel Winder & Mangiante-Orsola, 2023), our analytical approach is divided into several steps (Figure 1). We carry out an *a priori* analysis of the tasks proposed by the teaching-learning situation, and we pay particular attention to the procedures that pupils need to implement in order to achieve the expected layouts using the instruments provided for them (step 1). We then do an anticipatory analysis (step 2): that consists in questioning the task expected of the teacher, so as to identify the main difficulties he or she might encounter, as well as the aspects about which he or she should be vigilant with regard to the targeted learning – what we have termed 'vigilance points'. We analyze the mathematical knowledge at stake as well as the instrumented procedures enabling these tasks to be performed. Based on an initial *a posteriori* analysis of sessions of the same teaching-learning situation carried out in several classes (step 2bis), we design tools adapted to the leads that we have identified in the anticipatory analysis (step 3). We use these tools in specific *a posteriori* analyses to identify traces of the didactic vigilance that is exercised (step 4). Although we follow the different

[11] A scheme is an invariant form of activity organization for a specific class of situations (Vergnaud, 1997).

steps of our methodology, we do not detail all the analyses carried out here, but only highlight the most striking elements regarding the presented data.

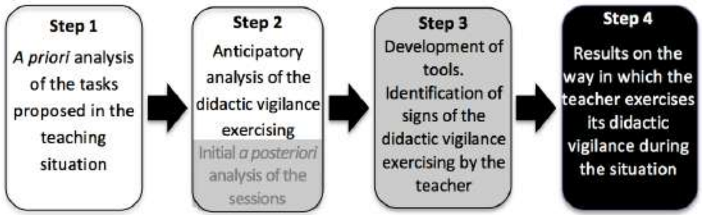

Figure 1: General analysis method

PRESENTATION OF THE TEACHING-LEARNING SITUATION

In the early 2000s, research was carried out with the aim of proposing a coherent progression of geometry throughout compulsory education, to better manage the transition from instrument-based tracing geometry to deductive geometry (Mathé, Barrier & Perrin-Glorian, 2020). Duval's work shows that the way a pupil perceives a geometric figure is decisive (Duval, 2008). Indeed, for them, the natural apprehension of figures is a vision of juxtaposed surfaces; in mathematics, however, geometrical figures are defined by relations linking lines and points. To help students enrich their geometric vision of figures, i.e., to help them see the relationships between the smaller elements (lines, points, circles) of a figure, a fundamental situation and its didactic variables have been identified: the figure restoration. A figure restoration is an action situation (Brousseau, 2002) that consists of reproducing a figure (full-scale or not) from a 'started figure' (part of the figure to obtain), with identified instruments (see examples figure 2). We note that "the reproduction of figures corresponds to the case where there is no started figure" (Mathé & Perrin-Glorian, 2023, p. 9). The started figure or some of the tools can be used to transfer information from the initial figure (the full figure to be reproduced, called 'model figure' in the following), but without giving all the information. The pupil can then start restoration more easily with figures in terms of surfaces. The choice of started figure and tools (templates, geometric tools, etc.) has a significant impact on the expected procedures: it can lead the pupils to apprehend the figure as being made up of smaller elements (lines or points). This also enables the teacher to support a change in the pupils' apprehension of figures (Mangiante-Orsola, 2023).

The *Triangles on Quadrilateral* sequence (*ibid*.) is organized in four successive phases. In each phase, pupils are asked to restore a figure, always the same one (an example is given in figure 2a) but using a different started figure and/or various instruments. The four-phase organization should enable pupils to enrich their analysis of the model figure: identify simple figures (quadrilaterals, triangles, etc.) that make it up and the links that exist between them (apprehend the figure as combination of surfaces); identify locate alignments, extended lines, get a point by intersection of lines…(analyze the figure by relations linking lines and point). We clarify these elements in the following. In the first phase, the started figure is the quadrilateral frame, and the instruments are the templates of the two large triangles, which we call T1 and T2 (figure 2b). To restore the figure, pupils need to distinguish the various simple figures that make it up (a quadrilateral, different triangles) and to use objects representing surfaces (templates). In the second phase, students are provided with another started figure (figure 2c) and a template of the 'nibbled' T'2 triangle. To place the template and thus trace the missing triangle, they must extend the sides of the small triangle already traced (T'1) and use the alignment of the sides of the small triangles (segments: parts of lines objects). Two further phases are planned. Phase 3 is designed to help pupils to go further in analyzing the model figure. Indeed, they must work out how to obtain one of the sides of the quadrilateral from points not already constructed, and to identify the links between the simple figures that make up the model figure (some of the sides of triangles T1 and T2 are carried by the diagonals of the quadrilateral). This involves identifying alignments. Phase 4 leads students to reinvest this knowledge. In the following section, we present the elements of the *a priori* analysis of phase 2 on which we are focusing.

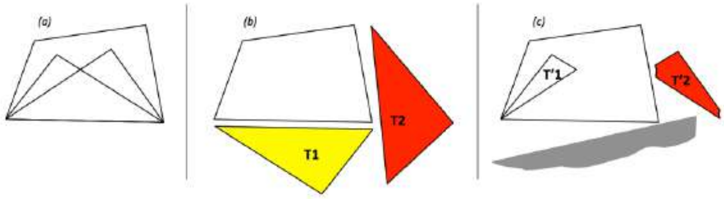

Figure 2: Model figure (a), starting figure and instruments for phases 1 (b) and 2 (c)

FIRST ANALYSIS (STEP 1 AND STEP 2 OF OUR ANALYTICAL APPROACH)

To complete the started figure of this phase (quadrilateral and triangle T'1) using a partially 'nibbled' template of triangle T'2 and a non-scale ruler (figure 2c), pupils must extend two sides of triangle T'1 to be able to place the 'nibbled' template of triangle T'2, and then use this template to trace the missing segments. This procedure can be analyzed in three different stages[12] (figure 3). It involves several difficulties in apprehending figures using instruments. For each stage, pupils must identify elements of the figure (graphic objects representing geometric objects) enabling them to position the instruments.

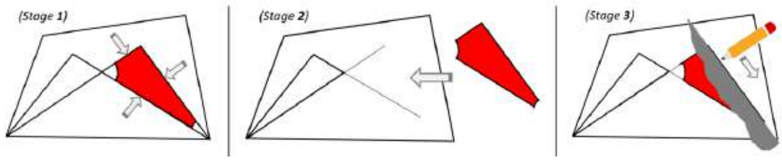

Figure 3: A three stages procedure

On Stage 1, pupils must identify the element of the model figure that corresponds to the 'nibbled' template provided (the T'2 triangle).To do this, they must match some of the edges of the 'nibbled' template (surface) to the outline of a simple figure (a triangle) on the model figure (surface outline). On Stage 2, pupils must consider that they need information to place the template on the started figure: at least two directions of straight lines are required (without using visual perception). They must then extend two sides of the T'1 triangle on the started figure. Note that this presupposes that they can use the outline of the triangle (outline of a surface) to draw straight lines with the non-scale ruler (network of straight lines). To extend the sides of T'1 triangle, pupils implicitly use the fact that some sides of T'1 and T'2 triangles are aligned. When placing the 'nibbled' template, they must then pay attention to both sides at the same time (as when positioning the square and paying attention to both sides to draw the perpendicular to a straight line passing through a specific point not belonging to this straight line): this cannot be done in a single movement. On Stage 3, pupils must finally use the template to draw the missing elements. They can either draw the outline or mark the direction of lines with graphic elements (lines of a network), that can then be extended. The alignment of the

[12] As in any research activity, pupils can grope, make mistakes and start again. For this reason, these three phases are not necessarily carried out in chronological order.

segments to be traced with one of the vertices of the quadrilateral constitutes an element of control of the directions of the obtained lines.

Anticipatory analysis of the didactic vigilance exercising. Like any figure restoration situation, the *Triangles on Quadrilateral* sequence involves tracing and consequently requires managing the transition between actions on material objects and/or various graphic representations and the geometric knowledge involved. This is a major challenge for teachers (Mangiante-Orsola, 2014; Guille-Biel Winder, 2021). With this in mind, we use the *a priori* analysis of phase 2 to identify 'points of vigilance' for the teacher in each of the three phases identified above. More specifically, we highlight the importance of getting pupils to identify the simple figure on the model that coincides with the template (phase 1), to identify and draw the geometric objects required to position the template (phase 2), and to identify and draw the missing geometric objects using the template (phase 3).

ANALYSIS OF THE DIDACTIC VIGILANCE EXERCISING

According to our methodology, we rely on a model of DV exercising in the form of a triplet. We first identify the gesture or gestures composition relating to the identification by pupils of the elements of the figure that enable them to position the instruments, and to the linking graphic objects with the corresponding geometric objects. Finally, to identify the learning purpose and evaluate the extent to which geometric learning is a priority, we relate the interventions of each teacher (her actions, her words) to the two possible priorities, situating them between justifying the actions to be performed by the precision of the tracings (graphic priority) and emphasizing to pupils the geometric knowledge involved (geometric priority). Based on *a posteriori* analyses of class sessions (step 2bis, not presented here), we have identified four points of analysis: does the teacher highlight the need to identify the elements required to place an instrument (template or non-scale ruler here)? Does she specify how many are necessary and which ones? What lexicon does she use to describe them? What type of argument does she use to convince the pupil? To find answers to these questions, we rely on our analysis of verbal interactions and actual gestures. In this text, we selected three 'remarkable events' (Leutenegger, 2000) in terms of considering the stakes of geometry teaching in the 'heat of the moment' referring to each stage of phase 2. But our analyses are also based on other data collected as part of our monitoring of these teachers (Mangiante-Orsola, 2023).

Stage 1 - Place the template on the model figure

The first remarkable event takes place during the group discussion of phase 2 in Vic's class. The teacher asked a pupil about the way he had placed the template on the model figure. She asks: 'What else did you have to do? Did you have to...? Put the triangle in the triangle's house, because it's almost his house, and it fit really well'. We identify a first element of the triplet modeling the DV exercising: the gestures composition performed by Vic is linked to the institutionalization process. It consists in explaining how to place the template on the model figure, to identify the corresponding elements of the model figure (here a triangle). The mathematical and didactic knowledge to be mobilized by the teacher concerns the fact that a triangle is defined by the contour of a surface. To associate a simple figure seen as a surface with a template, the template must be placed in such a way that its edges coincide with the contour of the surface. In order to characterize Vic's learning objective, we identify few possibilities. In this situation, Vic could have simply given a manipulative tip (e.g., explain that the template should be held firmly with the fingers, turned over if necessary or rotated). Vic does, however, make explicit the element to be considered when positioning a template on the model figure, namely an outline, but she designates it as a graphic object and even iconically (a 'little house'). Other data lead us to conclude that Vic is demanding when it comes to the precision of tracings. In fact, right from the first session of the sequence, she explains to students that, in geometry, they must be 'precise', 'clean', careful', they must not 'press too hard on the pencil', 'extend', 'go beyond', they are 'allowed to erase'. Hence, she is simply proposing an explanation of the procedure for placing the template on the model figure to obtain more precise plotting (graphic priority), without naming the involved geometric objects.

Stage 2 - Place the template on the started figure

The second remarkable event occurred in Lea's class. She intervenes with a pupil who has only extended one side of the triangle and is therefore unable to place the nibbled template correctly. Lea moves the template along the straight line drawn to show that its position cannot be determined precisely (figure 3).

Lea:	Where do you place the template on the model figure? (…) You're missing something to be more precise! (…) You need at least two directions, in other words, two straight lines to be able to place the template! Extending just one side of the triangle isn't enough!

The gestures compositions highlighted by this extract helps to regulate the student's activity. It consists of helping him to place the template on the starting figure to be able to draw a triangle. Lea's useful mathematical and didactical knowledge concerns the fact that a triangle is defined by a surface/circumference (angles, side lengths), three straight line directions or three points. Her comments suggest that she is pursuing a geometric stake. In fact, she indicates to the pupil the elements required to place the template and designates them not as graphic objects (lines) but as geometric objects ('direction'). What's more, she's able to explain the number of directions required to place the template ('you need two directions'). We can find other examples of Lea's such propositions during the sequence, as for example: 'you need two points to draw a straight line with the ruler'. In this way, Lea makes explicit the geometric proposition underpinning the appropriate use of the instrument. This relates to practical axiomatic theory (Mathé and Perrin-Glorian, 2023).

Stage 3 - Complete the tracing required to complete the started figure

The third remarkable event takes place in Vic's class. During the group discussion, Vic asked a student (Jim) who extends only one side of the triangle, about how to use the template to draw the missing elements:

Vic:	How did you manage to reproduce the same thing here? (…) Here's a first tracing. Next? [Jim misplaces the set square, which he uses as a ruler] Position your set square correctly. Which side of the triangle have you extended? (…) [Jim points to the side of the triangle to be extended]. What do you have to do to position it correctly? What are you going to do with the template because you can't go on like this? [Vic waits for Jim to draw little lines to mark the directions then speaks to the whole class] (…) Take a benchmark, do you remember? [Jim draws a small line along one edge of the template. Vic takes the setsquare in her hands, gives it to Jim and invites him to extend the line].

The gestures composition featured in this extract is part of the institutionalization process. The mathematical and didactical knowledge at stake relates to the fact that a triangle is defined by three straight line directions, and that identifying straight line directions enables pupils to enrich their analysis of the figure by considering the triangle not as defined by a contour, but by a network of straight lines. Vic, although always very attentive to the precision of the drawings, seems here to go beyond simple manipulative considerations for graphic purposes. She guides Jim more in the analysis of the figure through the clarification of elements to be identified to place the instruments (the non-graduated ruler). She doesn't use a geometric lexicon (there's no mention here of straight lines to be extended, points of intersection, aligned vertices), but it's interesting to note here her emphasis on the word ‘benchmark’, which enables her to designate the graphic element to be used to place the ruler correctly (the direction of a straight line). We also note that Vic specifies some geometric objects (the side of the triangle) and, in so doing, leads Jim to consider and draw each side of the triangle in turn, thereby segmenting its outline.

CONCLUSION

By analyzing these three remarkable events in terms of triplets, we can give an account of the way in which these two teachers exercise their didactic vigilance (these analyses are confirmed by other data not presented here). During Stage 1, Vic's DV exercising is characterized by learning stakes limited to graphic aspects, even at the knowledge institutionalization. On the other hand, Lea exercises during Stage 2 a significant DV on geometrical learning stakes: she intervenes regularly with pupils to identify the geometric objects required to position the template and designates them using geometric vocabulary. Our analysis of Stage 3 highlights an ‘in-between’ situation: it confirms Vic's particular attention to the identification of elements for positioning the instruments, since she insists on the need to ‘take reference points’. But even if the geometric objects are not explicitly described using a geometric lexicon, it's clear that Vic's often-expressed concern for precision leads her to encourage pupils to use graphic elements, and through them, the geometric objects they represent. We can conclude from this that these two teachers’ DV exercising helps to develop in the pupils a use of instruments based on a set of rules that Mathé and Perrin-Glorian (2023) refer to as practical axiomatic theory. However, our analyses reveal a gradation between two possible outcomes: justifying the actions to be performed by the precision of the lines (graphic priority) and highlighting to pupils the geometric knowledge involved (geometric priority). From this observation, we have highlighted specific difficulties for teachers who are often situated in this ‘in-between’ situation, between graphic and geometric learning goals. In training, making teachers aware of the existence of geometric propositions that underpin the proper use of instruments could, in our view, help them take better account of the learning stakes involved in geometry teaching.

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WHAT CAN THE STRUCTURE OF A GEOMETRY TASK STATEMENT BE TO PROMOTE A CERTAIN TYPE OF ARGUMENT?

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We propose the structure that we think a task statement should have to promote the production of a specific type of argument (inductive, abductive, or deductive) during the task solution process in which, preferably, a Dynamic Geometry System (DGS) must be used. To support the idea, solution strategies produced by students of a plane geometry course of the preservice program for mathematics teachers of our University, when they address tasks that involve the perpendicular bisector of a segment, are presented. Using the argument model proposed by Toulmin to analyze these strategies, the type of main argument enounced according to the type of statement was identified. We point out how the type of task can contribute to the didactic-mathematical knowledge of the geometry teacher.

PRESENTATION

In the last three decades there has been a growing interest in including argumentation in the school curriculum as a means to develop the students' mathematical proficiency (Boero et al., 2010; McNeill & Knight, 2013; Stylianides et al., 2016). It is assumed that it favors proper mathematical actions such as inducing or abducing properties from empirical evidence to formulate conjectures, providing formal deductive proofs to validate their conjectures, and communicating new ideas. Favoring argumentation in the classroom requires that the teacher use not only his knowledge (or his beliefs) about different types of arguments, but also that relative to didactical aspects since it is useful when selecting learning tasks that promote argumentation and helps to foresee (and later evaluate) students´ activity (Lin et al., 2012). Therefore, knowing about different types of arguments and recognizing the difference between them should be part of elementary and secondary school mathematics teachers' knowledge. Even more, knowing about different types of tasks related to argumentation, how to manage them, and recognizing the relation between these and the mathematical activity they spur should be another element of a teacher's knowledge (Stylianides & Ball, 2008).

To promote preservice mathematics teachers' involvement in argumentative practices in geometry, for more than a decade we have been immersed in a curriculum innovation process with the purpose of generating a favorable environment in which it can be achieved. The didactic effort of such an innovation includes the following elements: (i) mathematics tasks which require diverse mathematical activity and the discovery of interesting geometry relations, around which the corresponding theoretic system is conformed; (ii) social interaction in the classroom, among the students while jointly solving the problems, or between teacher and students, when they publicly expose their solution and argument their ideas, and (iii) the use of a Dynamic Geometry System (DGS) as an artifact that makes possible the exploration of the proposed tasks to find a solution.

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In this paper, we expose the relation between task statements and types of arguments that each statement favors. To illustrate, we use examples of class happenings in the plane geometry course of the preservice teacher program offered by our University. Concretely, we present how each type of task, characterized by the structure of its statement, principally favors a specific type of argument. Although various researchers have shown how tasks which must be solved with a DGS promote argumentation (e.g., Arzarello et al., 2012; Baccaglini-Frank & Mariotti, 2010), they do not expose how the structure of the task statement can contribute to the production of different types of arguments. Following the idea exposed by Stylianides and Stylianides (2006), the structure of the task statements that we propose, besides promoting preservice teachers' argumentation, also becomes a concrete tool when they design tasks to favor this mathematical process among their future students.

In what follows, we initially present conceptual references with respect to argumentation and argument; then we expose the methodological aspects used for the analysis. Later, we describe the type of task statements, their relation with the types of arguments and illustrate each case with an example. To conclude, we communicate some reflections.

CONCEPTUAL REFERENCES

In this section, we initially present what we conceive as argumentation, argument and types of arguments, using Toulmin's proposal (2003) as our principal reference. Then we expose our conceptualization of learning task and argumentation task; we describe the aspects that compose the task statement and elements that can be used to design statements of argumentation tasks.

Argument and argumentation

Our conceptualization of argument and argumentation comes from a sociocultural and discursive posture; for it we used the definition of argumentation proposed by Durand-Guerrier, Boero, Douek, Epp and Tanguay (2012) and the commented narration that Knipping and Reid (2019) present with respect to the functional structure of an argument proposed by Toulmin (2003). We consider an argument to be an expositive discursive expression, according to shared norms, that presents an assertion and reasons that sustain it. The assertion can be presented as one of the following: a proposition (i.e. a sentence of which it can be said it is true or false) that affirms or negates an idea; a sentence in which a posture is stated; or a physical action with which an idea or posture is expressed. Of the idea exposed it is of interest to sustain its veracity; of the posture presented, its acceptability. The reasons can be presented as sentences (propositions or not) or as actions. The set of reasons that sustain the veracity or acceptability of an assertion conform its justification. We understand argumentation as a discursive and social cultural process destined to increase the acceptability of a point of view or to determine the veracity of an exposed idea through the presentation of propositions or actions destined to justify (or refute) them. In such process arguments arise.

In its most simple version, an argument consists of three elements –data, assertion, warrant– and has a functional structure that displays the relations among these elements, according to the role they each have (Toulmin, 2003). The data sustains the assertion; it is evidence that supports the assertion. The warrant expresses the relation between the data and the assertion in a general statement that indicates why the data is evidence to sustain the assertion. The data and the warrant (which can be explicit or implicit) are the reasons with which the assertion is justified.

Types of arguments

We present our proposal of types of arguments, following an analogous idea exposed by Pedemonte (2007). As mentioned in the definition, the three basic elements of a simple argument have a functional relation that is always the same. Considering the course of a specific argumentation, more precisely, which of the elements of a simple argument is inferred and which of the elements is taken as given, it is possible to recognize three different argumentative situations, which we denominate inductive argumentation, abductive argumentation and deductive argumentation. Abusing the language, we classify arguments according to the type of argumentation in which it appears.

Deductive argumentation begins with two types of information that is accepted as true: a general rule (if p, then q) and *data* that consists of a particular case of what the antecedent of the general rule verses. Tacitly or explicitly, the general rule is used to eventually infer an *assertion* as a necessary result, that corresponds to the respective particular case of the consequent of the general rule. In short, a *simple deductive argument* can be enounced as:

> p_i (data) is given; therefore q_i (assertion) is concluded, due to if p, then q (warrant).

Inductive argumentation occurs within the framework of a reference set (i.e., a set A of objects that have a common attribute p) that constitutes information as *data*, that is, information accepted as true. It is recognized that some cases (subsets of A) share another attribute q. By considering one or several particular unexamined cases of the reference set, inductive argumentation leads to inferring a possible pattern of generality (*warrant*) and also the possible *assertion* which states that at least one of these other cases (or all) also has attribute q. In summary, a *simple inductive argument* can be stated as:

> The elements of A have attribute p, *some* elements of A have attribute q, about *other* elements of A it is not known if they have attribute q (data). Therefore, at least *another* element of A also has attribute q (assertion), because *the* elements of A have attribute q (warrant).

In *abductive argumentation*, there is an assertion that, from the beginning, is assumed to be true or acceptable; the *data* is the element that is inferred with the intention of providing a reason that supports the veracity of the assertion, based on a general rule that is believed or known to be true. The data is obtained as a plausible or probable inference. A key feature of this type of argument is the establishment of a fact or, a fact and a warrant, that could support the assertion. In summary, a *simple abductive argument* can be stated like this:

> Since we have q_i (assertion), given that if p then q (known or inferred warrant), we could have p_i (data).

Learning tasks and argumentation tasks

We conceive a learning task as an action (or actions) that the teacher asks students to perform with the intention of providing them with an opportunity to achieve the learning expectation that he has established (Gómez et al., 2018). The statement of a learning task (task statement for short) states information about what is being asked to do (*request*). It also exposes information that places the action requested (*situation*) within a specific knowledge and practice, in this case mathematics; Eventually, it exposes suggestions to support or conditions to limit the execution of the action (*indications*). The *learning expectation* is not explicit for the students; an expert should be able to unravel it from the statement. If the learning expectation of a task is the production and exposition of arguments, it is called an *argumentation task*. In this case, the statement must contribute to the generation of a scenario in which the need to take a position or expose ideas that must be supported is felt. We now present six elements that we have considered when designing the statements of argumentation tasks, based on references from the literature that, implicitly or explicitly, have been proposed for that same purpose (e.g., Ayalon & Nama, 2023; Lin et al, 2012):

E1: What is stated in the situation and/or request has the potential to generate doubt, curiosity, uncertainty, or controversy, which when resolved leads to exposing ideas or positions.

E2: The situation presents information related to a definition, theorem and/or fact that can be used in an argument.

E3: The request asks (implicitly or explicitly) for the presentation of reasons that serve to support or refute the veracity of a proposition, or the acceptability of a position raised or an action taken (exposition of arguments).

E4: The indications are a guide that supports or limits the execution of actions in order to establish positions or formulate new ideas, and to expose arguments.

E5: The short-term learning expectation is that students engage in argumentation and formulate and ennounce arguments, relative to a specific mathematical proposition, in accordance with established classroom norms.

E6: The elements of the main argument (data, assertion, warrant) that may arise, when carrying out the task, are explicitly stated, suggested or requested in the task statement.

METHODOLOGY

The data in this report is part of a research study that had as a goal designing a sequence of tasks to promote preservice mathematics teachers' learning about argumentation. The methodological strategy used in the study was "the development of the mathematics curriculum as a scientific effort" (Battista and Clements, 2000). The experimental context was a plane Euclidean geometry course, located in the second semester of the program in our University, one of its objectives being learning to prove. In previous courses, what an argument as an object is was not addressed. The methodological strategy used in the course is characterized by the use of tasks that satisfy the aforementioned elements that engage students (working in groups of two or three) to solve them (with the use of a DGS) and requires the formulation of conjectures that must be validated with elements of the theoretical system that is being conformed. That is, the tasks proposed mainly favor abductive or inductive argumentation because conjectures must be formulated.

Each group of students must present their productions in writing, responding to specific *requests* given in the task statement: i) a narration of how they proceeded to establish the solution to the problem (e.g., what was constructed to obtain the configuration obtained in a DGS, what empirical exploration was done, what type of invariant was determined); ii) the conjecture (solution of the problem) stated as a conditional proposition that reports the relationship between what was constructed and what was discovered; and iii) an explanation of what they use to establish that the conjecture is true. The teacher allows each group to present their productions to the class, discuss which conjectures solve the problem, and adjust them, if necessary, so that they are mathematically correct; they then proceed to collectively build the proof.

The data for this report are the written productions of the students in which they report the solution to the two specific tasks. The statements include the situation and the main request; the other requests included in the original statement, mentioned in the above section, are not reported.

Task 1: Consider three non-collinear points A, B and C. Let m be the perpendicular line to $\overline{AB}$ through its midpoint and n the perpendicular line to $\overline{BC}$ through its midpoint. Let T be the intersection point of m and n. ¿What geometric characteristic does point T have as point B is moved?

Task 2: $\overline{AB}$ and $\overline{CD}$ are congruent. ¿Does there exist a point E so that ΔABE y ΔCDE are congruent?

The statements are representative of the typology that will be described; they involve the same geometric object (*perpendicular bisector of a segment*[13]), reason why we chose them for this report. We had two other sources: i) class notes prepared after each class by groups of two or three students and delivered to the teacher, in which the main aspects covered in the class are reconstructed (specifying the solution presented by each group of students, and the teacher's or students' comments about them that arise); ii) a *retrospective recall* (Sherin et al, 2011) done by the teacher after each class session, which made it possible to complement the information recorded in the class notes.

Data analysis

The data analysis consists of two parts. A preliminary analysis (of a theoretical nature), which briefly describes how a student should proceed to establish his solution, according to the type of problem and suggests, a priori, the type of argument that is expected. A retrospective analysis (empirical in nature) contrasted with the theoretical, where both the most common problem-solving strategies and their respective solution-conjectures are identified in the students' productions. From both analyses, we obtained input to identify the types of arguments, using the Toulmin Model, that are necessarily provoked when students solve a task with a particular type of statement. The *retrospective recall* made by the teacher with respect to the students' forms of exploration helped to complement the information. A reading of these same elements in the students' written reports, together with what is evident in the class notes written by the students, led us to identify the types of arguments that emerged from the students' activity.

SOME RESULTS

We found that the statement structure of an argumentation task may primarily promote different types of main arguments. This is because in the argumentation process that is favored by the statement, both the information available and the information that is inferred play a certain role in the functional structure of the main argument that emerges. For the purpose of this report, we summarize in Table 1 the expected learning in terms of the type of argument that is favored and the corresponding structure of the task statement that promotes it. Additionally, we include a narration of the production of groups of students for Task 1 and Task 2 that illustrate the results presented in Table 1.

Task 1 statement is an example of the type *Consequent search.* In it a reference set is provided that characterizes the *situation* (non-collinear points A, B and C, and perpendicular bisector of $\overline{AB}$ and $\overline{BC}$); the question *requests* finding an attribute for T. As a solution, some students represent the situation in GeoGebra and explore it empirically, moving B around the screen; they use the DGS "Trace" tool for T. This action leads them to see that, when B moves, the path of T is a straight line; they expose the attribute of T: for any point B in that plane the intersection (T) of lines m and n will always belong to the perpendicular bisector of $\overline{AC}$ (Figure 1).

[13] The geometric characterizations of that object are: 1) the locus of points that are equidistant from the endpoints of the segment and 2) the line perpendicular to the segment that contains its midpoint. Either one can be the definition of the perpendicular bisector; the other would be a theorem. Option 1 was the definition used in the course.

Type of argument	Type of statement	Statement structure	Solving process	Argument structure
Inductive argument	Consequent search	*Situation*: provides or insinuates the reference set of geometric figures. *Request*: Asks about another attribute that the objects in the set might have.	Representation of the objects of the reference set. Exploration, using wandering dragging (Arzarello et al., 2002), of cases to determine another attribute of those objects	For some objects in the set another attribute (data) is satisfied. Therefore, other object(s) in the set also have that attribute (assertion). The established conjecture would be the warrant.
Abductive argument	Antecedent search	*Situation*: provides or insinuates a fact. *Request*: Asks about conditions that could support the fact.	Exploration, using maintaining dragging (Baccaglini-Frank & Mariotti, 2010), of properties that can lead to the fact.	The proposition that is provided or implied is a fact (assertion), since the conditions found (data) exist. The established conjecture would be the warrant.

Table 1. Type of statement according to the type of argument that is mainly promoted

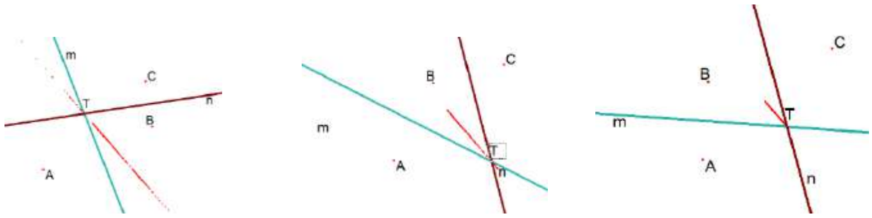

Figure 1. Graphic representation that illustrates the exploration that leads to an inductive argument

Finally, they report the following inductive argument in response to the request about what they use to be certain of the established invariant: for some points B of the plane, the corresponding points T are part of a certain subset of the line that appears to be the perpendicular bisector (data). Therefore, we can say that for the other points B [of the plane], the point T is always on that line, the perpendicular bisector [of $\overline{AC}$] (assertion). Implicitly, the warrant of the argument is the conjecture that they report: Let A, B and C be non-collinear points, m the perpendicular bisector of $\overline{AB}$ and n the perpendicular bisector of $\overline{BC}$. Let T be the intersection point of m and n. Then T belongs to the perpendicular bisector of $\overline{AC}$.

Task 2 statement is an example of the type *Antecedent search*. The *situation* corresponds to the two triangles with the aforementioned property; the *request* is to find data so that the triangles are congruent. As a solution, a group of students reported that they needed congruent corresponding sides to obtain congruent triangles. Therefore, they proceeded to determine the condition E must have so that $\overline{AE} \cong \overline{CE}$, as well as $\overline{BE} \cong \overline{DE}$. They report that their strategy consisted of taking any point E, activating its trace and moving it with the purpose of maintaining the first congruency; this trace allowed them to recognize that E should belong to the perpendicular bisector of $\overline{AC}$ (Figure 2); likewise, they determined that E should also be in the perpendicular bisector of $\overline{BD}$. They concluded that the congruency of the triangles is achieved when E is the intersection of the perpendicular bisectors (Figure 3). As an argument they reported the following: ΔABE and ΔCDE are congruent (assertion) because E is the intersection of the perpendicular bisectors of $\overline{AC}$ and $\overline{BD}$ (inferred data).

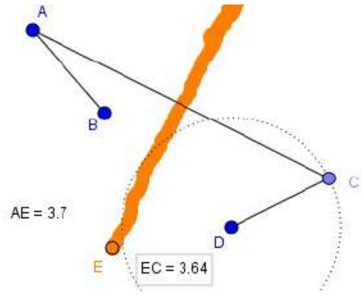

Figure 2. Exploration to determine the position of E so that $\overline{AE}$ and $\overline{CE}$ are congruent

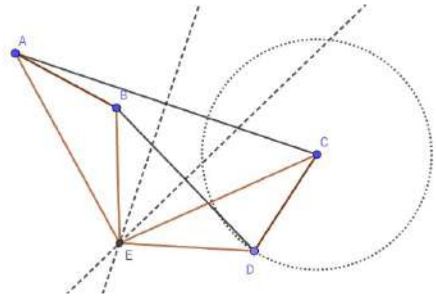

Figure 3. Representation of the solution for the second problem

FINAL REMARKS

As a result of the implementation of these tasks, our students recognize differences between types of task statements, identify the type of argument that each favor, and distinguish their role in the mathematical activity (inductive and abductive to conjecture, deductive to validate). This leads us to state that statements such as those illustrated contribute to a classroom environment in which the future teacher has the opportunity, not only to do mathematical activity close to that of a professional mathematician (in terms of exploring a situation, conjecturing, and arguing), but to experiment the potential of those types of task statements to carry out that activity. Yet we think that having experiences that lead them to argue is not enough for preservice teachers to gain specialized knowledge about argumentation. As several authors advocate (Stylianides and Ball; 2008; McNeill & Knight, 2013) to promote this process in their classes, it is necessary for this construct to become an object of study. Therefore, besides being able to establish the relationship between types of task statements and types of arguments that they favor, the students must construct their own perspective about argumentation, develop the ability to design task statements, identify and classify the arguments that are favored, identify their role in the mathematical activity, and recognize the arguments in what their students say. We would like to believe that, with what we present here, we partially respond to the concern that Stylianides et al. (2016) state about the necessity to continue thinking about features of tasks or classroom environments that promote argumentation in the mathematics classroom.

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ANALYSIS OF VISUALIZATION AS AN INDICATOR OF MATHEMATICAL GIFTEDNESS

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We present results from a research project we have conducted with primary school students who participated in an olympiad. We focus on the use of visualization as a possible characteristic of mathematically gifted students. We designed several types of problems requiring the use of visualization, to be solved in the olympiad, and observed which problems were useful for such identification and also which components of visualization, evidenced in students' solutions, were good discriminating the students with best scores, who can be considered potential mathematically gifted students. We have identified and defined seven descriptors associated with the problem-solving process, and identified the descriptors used by students in each problem. Here we show examples of the different types of problems and evidence of the descriptors of visualization evidenced in students' answers.

INTRODUCTION

There is broad agreement among researchers in mathematics education that mathematically gifted students (MG students hereafter) develop faster and use more effectively than their peers certain central abilities or skills in mathematics, such as generalization, abstraction, transfer, deductive reasoning, creativity, or flexibility (Krutetskii, 1976; Leikin, 2021; Miller, 1990). However, when analyzing the relationship between MG and visualization in mathematics, there are discrepant research results. Clements (1999) points out that "many studies have shown that children with specific spatial abilities are more mathematically competent. However, other research indicates that students who process mathematical information by verbal-logical means outperform students who process information visually" (p. 72). This diversity of findings is still present in more recent literature. It is therefore relevant for mathematics education to study this apparent contradiction and provide information to help decide whether mathematical visualization is a trait of MG students.

We present part of a research in which we have analyzed the presence of mathematical generalization, visualization, and flexibility in the solutions of students who participated in the mathematical olympiad for primary school of Costa Rica (OLCOMEP). We will focus on the ability of visualization and the problems we posed in the olympiad that required its use. One objective of our research is to identify which of these problems were more suitable for discriminating students with higher scores and, therefore, potential MG students. Another research objective, related to the previous one, is to identify which visualization features evidenced in students' solutions showed adequate power to discriminate the students with higher scores. The objective of this paper is to identify the descriptors of visualization evidenced in students' solutions and inform on their power to discriminate potential MG students. To this end, we present the different types of visualization problems and an analysis of the descriptors of visualization that some examples of students' solutions used to solve them.

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REVIEW OF LITERATURE

Several authors (Clements & Batista, 1992; Giaquinto, 2007; Gutiérrez, 1996) point out the relevance of diagrams, figures, real objects, etc., for both mathematics teaching and mathematical research. Visualization is "being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving" (Arcavi, 2003, p. 235) in school contexts and, on the other hand, "research mathematicians can think through and communicate their proofs to one another in a quite casual way, ... often indicating their intentions with diagrams" (Giaquinto, 2007, p. 81). From the cognitive point of view, Krutetskii (1976) identifies analytic and geometric (visual) types of mathematical thinking and Cain (2019) recalls that some great mathematicians showed preference for analytic thinking (e.g., Hardy or Russell) while others used geometric thinking in their research (e.g., Hadamard).

To evaluate the use of visualization by mathematics students, both psychometric tests and mathematics problem sets are used. Several researchers conclude that it is more effective and reliable to analyze solutions to problems than answers to tests (Ramírez & Flores, 2017).

Interpretations of visualization from mathematics education

Several authors have proposed different conceptualizations of visualization and its use in mathematics: Gutiérrez (1996) states that mathematical visualization is composed of three main elements: mental images (created and manipulated to solve problems or any other mathematical activity; Presmeg, 1986), processes (of converting available information into mental images and interpreting these to obtain new information; Bishop, 1983) and abilities (the skills necessary to carry out the processes and create or manipulate images; Del Grande, 1990). Clements (1999) considers that spatial thinking, or spatial sense, is composed of spatial orientation, spatial visualization, and imagery, along with skills such as creating and transforming mental images and representing them graphically. More recently, Lowrie et al. (2020) consider spatial reasoning as integrated by spatial visualization, mental rotation, spatial orientation, and spatial structuring. Miragliotta and Baccaglini-Frank (2017) call spatial reasoning "the set of cognitive processes by which mental representation for spatial objects, relationships, and transformations are constructed and manipulated" (p. 3953) and characterize it by various skills, in which the use of memory plays an important role. These interpretations of visualization are different but have as common characteristics the creation and transformation of mental images and the need to develop skills to properly use visualization.

Visualization and mathematical giftedness

There is a disparity of conclusions in the research on the use of visualization by MG students. Krutetskii (1976) recognizes the importance of visualization in mathematics by including, among "the component mathematical abilities that arise from the basic characteristics of mathematical thought ... the ability for spatial concepts" (pp. 87-88). But, when he analyzes the presence of such components in MG students' solutions, he concludes that some of them "are not obligatory in the structure of mathematical giftedness" (p. 351), among which he mentions "an ability for spatial concepts" and "an ability to visualize abstract mathematical relationships and dependencies" (p. 351). Presmeg (1986), based on Krutetskii's findings, identifies several reasons for MG students' low use of visualization, internal (natural predisposition and need to use analytical reasoning) and external (textbooks, teaching methodologies, and examination procedures). Van Garderen et al. (2014), after observing students with learning disabilities (LD) and high ability (HA), obtain similar conclusions: "a higher percentage of HA students than students with LD did not consistently utilize a diagram as a strategy to solve a problem. ..... [and] a higher percentage of students identified as HA than students with LD had difficulty using a diagram to reason with as they solved the problem" (p. 147).

On the other hand, Ramírez (2012), analyzing the strategies of secondary school students, concludes that "mathematically talented students have shown significantly higher intelligence and visual ability in tests than students in the control group" (p. 336). Other research shows that, when solving math problems in which visualization facilitates the solution, (potentially) MG students use it more consistently and effectively than average students (Escrivá et al., 2017; Ramírez & Flores, 2017). Therefore, we agree with Webb, Lubinski and Benbow (2007) that observing visualization ability, through problem solving, should be included in the processes of identifying MG students, in order to identify a type of MG students that is sometimes uncovered.

THEORETICAL FRAMEWORK

Based on Gutiérrez (1996), we define the *capacity of visualization* as the capacity to reason using spatial or visual elements, both physical (photos, diagrams, solid objects, etc.) and mental (mental images), to solve problems, perform proofs, or understand mathematical concepts and properties. Furthermore, based on this author and Del Grande (1990), we define *abilities of visualization* as skills that must be acquired and improved by students to effectively execute the necessary visualization processes with the appropriate mental images when solving problems. For our research, we have taken the definitions proposed by these authors for visual identification, mental rotation, conservation of perception, recognition of positions-in-space, recognition of spatial relationships, and visual discrimination. From them, we have defined a set of operational *specific descriptors* of the capacity of visualization that allowed us to observe evidence of the capacity and skills of visualization in students' solutions to the problems in our experiment:

- DV1. *Visual identification*: to identify a figure that is part of a complex context by isolating it from the context.
- DV2. *Mental rotation*: to produce or transform a mental image by rotating it in space. It can be based on a dynamic image or on a pair of pictorial images.
- DV3.1. *Conservation of perception of partially hidden figures*: to identify a regularity of a partially hidden figure that allows to imagine how the figure would look like if it could be fully seen.
- DV3.2. *Conservation of perception of completely hidden figures*: to identify a regularity of a structure that allows to assume that it includes hidden figures, and how the figures would look like if they could be fully seen.
- DV4.1. *Recognition of positions-in-space with respect to oneself as observer*: to recognize the positions of objects located in space in relation to oneself as observer and as the reference center.
- DV4.2. *Recognition of positions-in-space with respect to a reference object*: to recognize the positions of objects located in space in relation to another fixed object acting as a reference center.
- DV5.1. *Recognition of spatial relationships between objects*: to recognize the positions of objects located in space in relation to each other.

- DV5.2. *Recognition of spatial relationships between elements of objects*: to recognize the positions of elements of one or more objects located in space in relation to each other.
- DV6. *Visual discrimination*: to compare objects, identifying their visual similarities and differences.

As we have shown in the previous section, some authors argue that being able to choose and effectively use the most appropriate visualization skills to solve a problem is a skill that differentiates MG students.

METHODOLOGY

We gathered the solutions by a sample of 300 students in grades 2 and 4 to 6 (7 to 12 years old) who participated in the phases 2, 3, and 4 (final) of the OLCOMEP olympiad and analyzed the ways in which the students used their capacity of visualization, by identifying which descriptors were evidenced in their solutions. We created 13 problems of visualization, posed in the successive phases of the olympiad of the mentioned grades, and classified them into three types: A) given a complex structure made of simple objects, analyze its different side-views to get some information; B) given a complex structure made of simple objects, identify the quantity or types of objects that form it; C) given a structure, mentally disassemble or rearrange its elements, to fit the instructions stated in the problem. Several problems are presented in Figures 1, 3, and 5 and in Mora and Gutiérrez (2021).

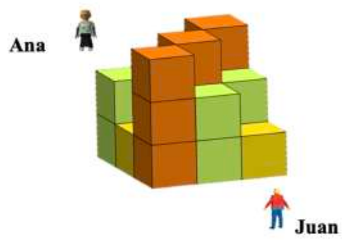
We have made a mixed analysis of the data provided by the students' solutions. The qualitative analysis has allowed us to classify the solutions according to the descriptors evidenced by each one, as shown below. The quantitative analysis has been oriented to identify the problems and descriptors that have shown the highest power of discrimination of students with better visualization skills (Mora et al., 2023).

RESULTS

In this section we analyze several problems, as examples of the types described above, and present a synthesis of the responses to each problem, analyzed in terms of the visualization descriptors evidenced by the students in each problem.

Type A problems: analyze side-views of a complex structure made of simple elements

Luis builds a city with Lego, that has buildings of three colors and the buildings of the same color have the same height. He places them in a grid of 3x3 buildings, so that in the same row and column there are no buildings of the same height. Ana and Juan are two habitants of the city that are placed as shown in the figure.



a) If Juan looks towards the city, how many buildings can he see? Justify your answer.
b) If Anna looks towards the city, how many buildings can she see? Justify your answer.
c) In what position, different from Juan's, could Ana stand to see the same number of buildings that Juan sees?

Figure 1: Problem of phase 3, grade 4

Figure 2 shows a student's solution to the problem in Figure 1. To solve it, students needed to use *visual identification* (DV1), to correctly interpret the information, to identify the polygonal shapes that form the visible faces of the buildings, the complete buildings (with their walls and roofs), and the complete structure of the city. All students scoring high used DV1 to solve the problem, while only 20% of the students scoring low did it. Therefore, this problem allowed to discriminate very well the use of DV1 by the best visualizers, who are thus potential MG students.

To answer questions a) and b), students had to use the *conservation of perception* to, considering the regularity of the structure, identify the buildings that, from John or Anne's position, are *partially hidden* (DV3.1) and those that are *completely hidden* (DV3.2). In this problem, DV3.1 was observed in 75% of the students and DV3.2 in only 62,5% of them, so using DV3.2 was more difficult for students than DV3.1. In the solutions to this problem, all students scoring high used both descriptors to solve the questions, but only 20% of students scoring low did the same. Then both descriptors may well differentiate students with high visualization ability and thus potential MG students.

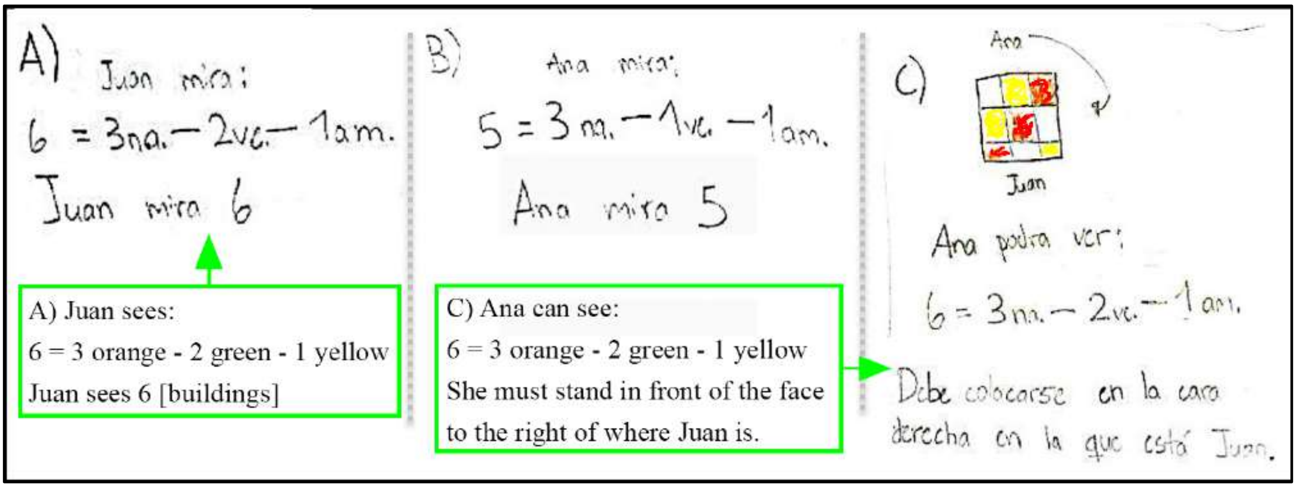
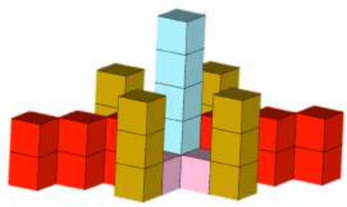


Figure 2: Student E.E63031's solution to the problem of phase 3, grade 4

To answer question c), students had to do recognition of *positions-in-space*, to choose a position taking as the reference either *themselves as an observer* (DV4.1) or *another fixed object* in the figure (DV4.2). The first option was observed in 18,7% of the students and the second one in 37,5% of them. In the solutions to this problem, all students scoring high used some of these descriptors to solve the questions, but none of the students scoring low did it. Then, this pair of descriptors may well differentiate students with high visualization ability and thus potential MG students. The student in Figure 2 used the descriptors DV1, DV3.1, DV3.2, and DV4.2.

Type B problems: identify the quantity and/or type of objects that form a complex structure

In the park there is a sculpture made of columns of cubes, the columns of the same height have the same color. If someone stands at the top of the light blue column, which is the center of the sculpture, they will see the sculpture symmetrical both comparing front and behind and comparing right and left.



a) How many cubes in total were used to build the sculpture?
b) What is the difference between the number of pink cubes and blue cubes?
c) What is the difference between the number of red cubes and brown cubes?

Figure 3: Problem of phase 2, grade 5

When solving the problem of Figure 3, students used *visual identification* (DV1) to extract information from the figure, identify the polygonal shapes that form the visible faces of each cube, and identify the cubes as simple objects and as parts of the columns. This descriptor was used by almost all students (94.4%), as it was not difficult for them to interpret the information. Once the elements of the figure were identified, the student used the *conservation of perception* to identify the cubes and count them. All the students scoring high used DV3.1 and DV3.2 to count the cubes and solve the problem. Also, 80% of the students scoring low evidenced DV3.1, so it was easy for them to identify the cubes that have a visible part or are deformed by the perspective. On the contrary, none of the students with low scores was able to identify the non-visible cubes (DV3.2). Figure 4 shows the solution by a student who drew separate representations of each element (columns) of the structure. Thus, the answer showed the three descriptors mentioned, because, after having identified the elements of the sculpture (DV1), the student used DV3.1 and DV3.2 to identify the partially and completely hidden cubes and count them.

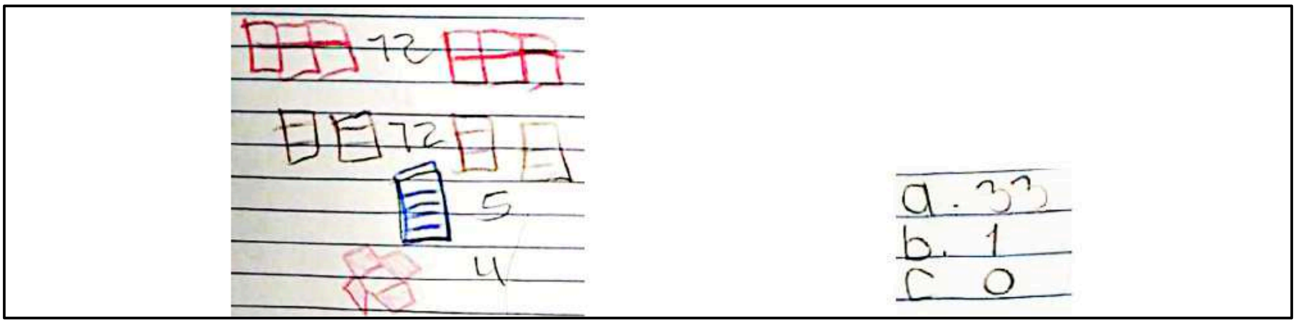


Figure 4: Student E.E52002's solution to the problem of phase 2, grade 5

Type C problems: mentally disassemble or rearrange the elements that make up a structure

Martín's friends are playing with cubes that are held together with magnets. They have built the four colored shapes above and Martín has built the two larger shapes, A and B.

a) Is it possible, from Figure A, only by removing some cubes, to obtain each of the figures built by your friends? Justify your answer.
b) Is the above possible from Figure B? Justify your answer.

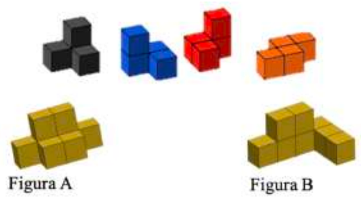


Figure 5: Problem of phase 3, grade 6

When solving the problem in Figure 5, students found it easy to identify the cubes that form each structure in the statement, with their faces and edges. Therefore, almost all students (81.25%) used *visual identification* (DV1) and, therefore, this descriptor did not help to differentiate the best visualizers of grade 6, although, in other problems of lower grades, it did discriminate them very well. To determine the shapes of the colored structures and compare them with figures A and B, it is necessary to identify some cubes of each structure that are partially or completely hidden, making use of the *conservation of perception*. In this problem, almost all high-scoring students (85.7%) showed evidence of having used DV3.1 and DV3.2. Furthermore, 57.14% of the low-scoring students evidenced the use of DV3.1. Then, DV3.1 do not help to differentiate the best visualizers in this problem. On the other hand, DV3.2 helped more to differentiate students with higher visualization capacity, because it was used by fewer low-scoring students (42.8%). When asked to compare several figures, students used *visual discrimination* (DV6) to compare some structures with others, by identifying visual similarities and differences between them. This descriptor was evidenced by almost all high-scoring students (85.7%) and only 14.28% of low-scoring students, so DV6 helped to differentiate students with higher visualization ability. Figure 6 presents the response of a student who showed these four descriptors. Consequently, among the descriptors necessary to solve this problem, DV6 is the one discriminating best the students with higher visualization capacity.

Figure A

Under cube 2 it is cube 7 and under cube 3 it is cube 8.

The black figure is made by removing cubes 3, 5, 6, and 8 from figure A.

The blue figure is made by removing cubes 1, 3, 4, and 6 from figure A.

The red figure is made by removing cubes 1, 2, 5, and 6 from figure A.

The orange figure is made by removing cubes 1, 2, 3, and 7 from figure A.

Figure B

The blue figure is made by removing cubes 1, 2, 4, and 8 from figure B.

The orange figure is made by removing cubes 5, 6, 7, and 8 from figure B.

But the black and the red figures, it is not possible to make them by removing cubes from figure B.

Figure 6: Student E.E63031's solution to the problem of phase 3, grade 6

CONCLUSIONS

Researchers in mathematics education have identified several capacities related to MG. The ability to visualize is necessary to achieve a complete mathematical education of students, although there is debate as to whether it is a characteristic ability of MG. In our study, we experimented with a set of 13 problems of three different types, administered in the OLCOMEP olympiad to participants in grades 2 and 4 to 6 of primary school. To correctly solve these problems, students had to use a variety of visualization skills. We have shown examples of the three types of problems posed, analyzed some solutions to those problems, and reported which visualization descriptors proved to be good identifiers of high visualization ability and potential MG students, as a large majority of the students who used them solved the problems requiring them well and obtained high scores in the olympiad.

Our study cannot provide generalizable answers, due to the small sample size, but it raises an important avenue for future research, consisting in analyzing: how the characteristics of different types of visualization problems influence the success of students with high and low visualization abilities; which visualization skills are more adequate to discriminate students who have high visualization abilities and, therefore, who are potential MG students.

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HOW DO PROSPECTIVE SECONDARY SCHOOL TEACHERS PROPOSE AND SOLVE GEOMETRIC TASKS?

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This study aims to characterize the way in which prospective secondary school teachers propose and solve geometric tasks. For this purpose, the mathematical objects/processes mobilized in each resolution are identified, using theoretical and methodological tools of the Ontosemiotic Approach to Mathematical Knowledge and Instruction. A content analysis is carried out that contemplates a deductive-inductive coding of the procedures and written justifications used by the prospective teachers when proposing and solving a task that involves the use of a configuration of figures. The results show that prospective teachers, for the most part, propose tasks related to the calculation of areas of 2D figures, which shows a mastery of arithmetic and the use of formulas related to known figures. In addition, they show that prospective teachers mobilize objects associated with visualization and measurement processes (e.g., properties of figures and properties of measurement) when using formulas for areas of squares and circles.

INTRODUCTION

The main objectives in teaching geometry at the secondary level include the development of knowledge and understanding of geometric properties and theorems, as well as the ability to use them and to encourage the development and use of conjecture, deductive reasoning, and proofs (Brown et al., 2004). In the chilean context, the standards of the teaching profession for secondary education establish that teachers must be able to deepen inductive and deductive reasoning, and perform demonstrations of geometric relations in the plane and in space, to use them in the design of learning activities of properties of 2D and 3D figures, so that their students can model, construct, visualize, represent, propose geometric conjectures and argue about their validity (CPEIP, 2021). Although these goals are present in most educational curricula, they remain elusive (Jones, 2001), either because of the difficulties of pre-service and in-service teachers, or because the curricular materials (e.g., textbooks) show little clarity in relation to how to promote the different processes associated with geometric reasoning, such as visualization. In this sense, numerous investigations point out that students have difficulties in recognizing shapes and geometric characteristics of objects with non-standard orientation, perceiving inclusion relationships, visualizing geometric solids in two-dimensional (2D) format, and solving measurement problems that require geometric reasoning (Fujita 2012; Levenson et al., 2011; Seah & Horne, 2019). Similarly, it has been evidenced that many prospective teachers (PT) share the difficulties and misconceptions exhibited by the students (e.g., Caviedes et al., 2023; Seah & Horne, 2020). This is relevant, because if teachers do not have the strategies to guide their students, it will be difficult for them to develop an adequate understanding of the different geometric concepts and properties. In this context, the question arises: what geometric processes do prospective teachers mobilize when proposing and solving geometric tasks? To answer this question, this paper aims to identify the mathematical objects that PTs mobilize when proposing and solving a geometric task involving a configuration of figures.

Difficulties of students and prospective teachers

The difficulties experienced by students and PTs may be due, in part, to the fact that learning geometry does not follow a linear trajectory, as opposed to learning numerical-algebraic concepts (e.g., students do not learn all three-sided shapes before moving on to learn four-sided shapes). Students' initial geometric ideas are based on prototypical figures such as triangles, squares and circles that they have already seen in their everyday context (Seah & Horne, 2019), and in order to progress in their geometric reasoning, students must understand that geometric shapes and elementary objects are not isolated entities, but correspond to a connected network of concepts, which are represented by points, lines, angles, segments, planes, space and concrete objects (Seah & Horne, 2019). Thus, for students to advance in their geometric reasoning, teachers must provide experiences that address the distinction between shapes and objects, and the exploration of the properties of geometric shapes, in order to draw on formal conceptual systems (Downton & Livy, 2021). Consequently, it is relevant to address the way in which PT resort to different processes of geometric reasoning when solving a given task. It is understood that the ability to reason geometrically goes beyond memorizing terminology and applying formulas and theorems to known situations. In this regard, Battista (2007) points out that geometric reasoning consists of "inventing" and using formal conceptual systems to investigate form and space. It involves using conjectures to deduce relationships and prove a theorem or a proposition. (Brown et al., 2004). In this way, geometric reasoning considers the logical operations involved in the resolution of a given task, where visualization is made explicit (Duval, 2017; Presmeg, 2008). Hereby, Duval (2017) points out that visualization corresponds to the sequence of operations that allows recognizing the geometric properties of a figure, being necessary to perform a visual deconstruction of the figural units that are imposed at first sight to obtain a new reconfiguration. This process constitutes a key aspect in spatial reasoning, which underlies the more general process of geometric reasoning (Battista, 2007).

Geometric Reasoning from the Ontosemiotic Approach (OSA)

Within OSA approach, understanding of a mathematical object is defined by the ability to recognize the properties and characteristics of a mathematical object, relate it to other mathematical objects, and use it in a variety of problem situations (Font et al., 2007; Godino et al., 2016). When considering the meaning of a mathematical object in terms of practices, it is possible to distinguish between sense and meaning of mathematical objects. While sense corresponds to the partial meaning of the object, meaning is reconstructed by systematically exploring the contexts of use of the object and the systems of practices involved (Godino et al., 2019). The relationship between these mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures, arguments, problem situation) is known in OSA as ontosemiotic configuration, which is epistemic or cognitive (Godino et al., 2016). Here we use the epistemic, referring to the system of practices promoted by the institution or curriculum. Considering that reasoning involves the bringing into play of different primary objects and processes, it is possible to assume reasoning as a social and epistemic macro-process, in which different primary objects (e.g., problem situations, definitions, properties, procedures, arguments and linguistic elements) are involved, and which can be analyzed from a process-product perspective. (Caviedes, 2021). OSA considers that a process refers to the "idea of a sequence of actions that is activated or developed, during a certain period of time, to achieve an objective, generally a response (output) to the proposal of a task (input), these tasks are ruled by mathematical or metamathematical rules" (Rubio, 2012, p.107).This way, both objects and processes are constituted as tools that allow directing the analysis of the mathematical activity, and providing a solution to a given problem-situation (Caviedes, 2021). In this context, this study assumes that geometric reasoning is defined by the practices performed by a person to solve different geometric problems. In these practices, primary mathematical objects and processes linked to the meaning of a given mathematical object (in a geometric context) emerge gradually, systematically and progressively. Hence, the mathematical objects associated with geometric reasoning processes, and which are put into play both in the task and in the solution proposed by prospective teachers, would account for their geometric knowledge.

METHOD

The study is situated in an interpretative paradigm and follows a qualitative approach (Cohen et al., 2007). A content analysis and a deductive-inductive coding are followed to identify, on the one hand, the primary objects mobilized by the PTs in a task involving a configuration of figures (figure composed of elementary figures); and, on the other hand, the processes linked to geometric reasoning. The following key processes for the emergence of geometric reasoning are distinguished: (1) *visualization,* linked to the recognition of geometric properties, procedures of composing-recomposing, and transformations on geometric figures and bodies; (2) *construction,* linked to procedures involving geometric instruments (ruler, compass, etc.) and/or software; (3) *measurement,* linked to procedures involving calculations and formulas (arithmetic and/or algebraic); (4) *representation*, linked to the use of figures and/or drawings to illustrate geometric elements; and, (5) *deduction,* linked to the enunciation of propositions and/or use of hypotheses. This paper deals with the processes of visualization and measurement.

The sample consisted of 19 PTs, 8 males and 11 females, who were taking the course Didactics of Geometry corresponding to the fourth year of the Pedagogy in Secondary Mathematics and Computer Science of the University of Los Lagos. The PTs were doing their professional practice, which is a part of their training, while they were taking the course. In it, the PTs are in charge of the design and planning of the mathematical tasks that they propose to the students of secondary education. As part of the course, the PTs had had previous instruction on the proposition and resolution of geometric tasks, addressing the characteristics of such tasks, the role of figures and the processes involved in geometric reasoning (e.g., visualization, deduction, construction, measurement). The task posed to the PTs is presented in Table 1. To solve this task, the PTs work in pairs, in order to complement their own proposals and resolutions and, thus, mobilize the mathematical objects associated with geometric reasoning processes.

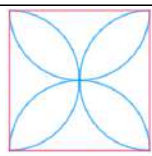
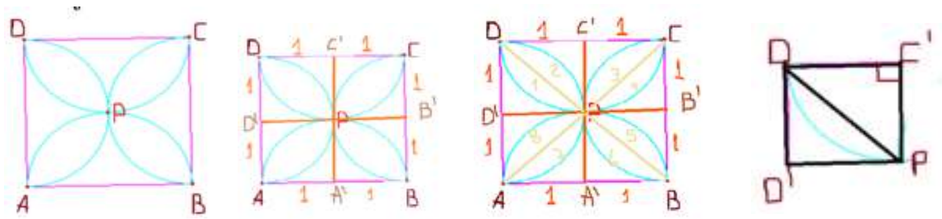
Proposal of the tasks	Figure accompanying the statement
Look at the following figure and answer: Using the Figure below**: What task would you plant to students between 12 and 15 years of age? How would you solve it? Justify your answer.**	 (Compiled by authors)

Table 1: Activity proposed to the PT group

Analysis

Figure 1 shows the solution of the task proposed by pair 2. It can be observed that the proposed task aims to determine the area of the surface enclosed by the light blue figure. For this, the PTs consider that the length of the side of the square is 2 cm and make explicit the elements that compose the configuration of figures. For example, the PTs mention elements such as arc of circumference, midpoint and side. The solution proposed by pair 2 emphasizes decomposition procedures. In this way, the PTs decompose the large square into four smaller squares and, in turn, trace the diagonals of these squares to decompose them into two right triangles. These decompositions allow the PTs to visualize that each "half petal" corresponds to a quarter of a circle whose radius is 1cm. Thus, it is inferred that pair 2 mobilizes concepts/definitions such as circumference, right angle, right triangle. In addition, among the procedures, the convenient decomposition of surfaces and the use of known formulas to calculate areas are evident. On the other hand, the property of accumulation and additivity of the area (the total surface is equal to the sum of the parts that compose it) is made explicit in the procedures that involve obtaining the area in an additive way (e.g., the area of the figure whose perimeter is delimited with light blue color is obtained by determining the value of the area of its eighth part, the sum of its parts allows obtaining the total area). Finally, it is observed that the solution of pair 2 involves the mobilization of a symbolic (arithmetic and algebraic) and geometric language, associated, respectively, to the arithmetic/algebraic procedures that allow calculating the requested area, and to the geometric representations of the figures and their respective decompositions. Thus, both the task proposed by pair 2, as well as its respective solution, show the mobilization of a measurement process involving the calculation of areas, and of a visualization process associated with the auxiliary lines tracing that allows the convenient decomposition of surfaces.

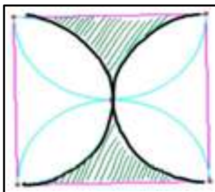
Task posed: ABCD is a square of side 2 cm, the four arcs of circumference were drawn with centre at the midpoint P of each side of the square. How long is the light blue highlighted area?



Resolution: It is known that the measure of the sides AB, BC, CD and AD are 2 cm respectively. Moreover, P is a midpoint ....
… Since each square has the same characteristics, the square D'PC'D is taken and the diagonal corresponding to DP is drawn. Therefore, the diagonal CP, AP and BP are drawn… This leaves eight figures (half of the petal), just calculate one and multiply by eight…. So, the reasoning now is that by taking the half of the petal (1) you have ¼ of the circumference, therefore, you will calculate the area ¼ of circumference, from which you subtract the area of the triangle (2).
Then: $\frac{\pi r^2}{4} - \frac{1}{2}$, where r is the radius of the circumference, whose value is $\frac{\pi(1)^2}{4} - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$
Finally, the area of the half petal (1) is $\frac{\pi}{4} - \frac{1}{2}$ … It must be multiplied by 8 times, giving the measurement of the light blue highlighted area $2\pi - 4\ cm^2$.

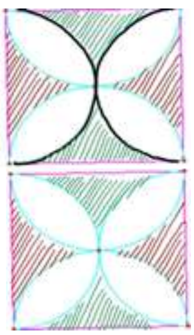
Figure 1: Solution of task proposed by pair 2 of prospective teachers

Figure 2 shows the proposed Task, as well as its respective solution, by pair 5. It can be observed that pair 5 implements solving procedures before proposing the task. Thus, the PTs start from the need to find the area of the "petals" and consider that the length of the side of the square corresponds to 6 centimeters. The PTs visualize that two of these "petals" are part of the area of a half circle, whose radius is 3 cm and whose perimeter is delimited with black color. In this way, the PTs mobilize the property of accumulation and additivity, and use known formulas, to obtain the area of the complete circle, as well as of the square of side 6 cm. The PTs subtract the area of the circle from the area of the square, obtaining the area of the green flattened area (Figure 2). Subsequently, the PTs visualize that the green flattened area is equal to the red flattened area. Therefore, they subtract this area from the area of the circle, obtaining the area of the four petals. To obtain the area of each petal, the PTs divide the area of the "flower" into four. Based on the above, it is inferred that the PTs mobilize a symbolic (arithmetic and algebraic) and geometric language, which are associated with the use of calculations and formulas that allow obtaining a certain area, and the auxiliary drawing of lines that allow the convenient decomposition of the figures, respectively. Finally, the PTs propose a task that involves the calculation of the area of the flattened area with red and green color. Since the area proposed by the PTs includes the value of each of the "petals", the solution involves only the mobilization of arithmetic procedures through a symbolic language. Thus, both the task proposed by pair 5, as well as its respective solution, account for the mobilization of a measurement process involving the calculation of areas, and of a visualization process associated with the auxiliary tracing of lines that allows the convenient decomposition of the surfaces.



Before posing the problem, **we need to know what the area of the petals of the flower is**, if one of the sides of the square is 6 centimetres. We can see two halves of a circle (highlighted in black) of radius 3 cm…joined together would form a complete circle… we will start by calculating the area of the circle: $\pi r^2 = 3{,}14 \times 3^2 = 28,26\ cm^2$…and of the flattened area: area of the square – the area of the circle is equal to the flattened area… $36 - 28{,}26 = 7{,}74 cm^2$

The green flattened area and the red flattened area have the same area, so subtracting the area



of the red flattened area from the area of the circle would give us the area of the petals... Ao – flattened area = area of the petals → $28{,}26 - 7{,}74 = 20{,}52$ → $\frac{20{,}52}{4} = 5{,}23 cm^2\ each\ petal\ldots$ so **we pose the following task**: "*We want to paint a square mosaic with a drawing of a flower, as shown in the image. We know that each petal of the flower is* $5{,}23cm^2$ *and that the perimeter of the mosaic is 24 centimetres. If we want to paint the flattened areas of the mosaic, how much surface area would be painted?*"… flattened area $= area\ of\ the\ square - area\ of\ the\ petals \rightarrow 36 - 20{,}52 = 15{,}48 cm^2\ \ldots\ the\ surface\ to\ be\ painted\ would\ be\ 15{,}48cm^2$.

Figure 2: Solution of task proposed by pair 5 of prospective teachers

RESULTS

Table 2 shows the primary objects mobilized by each pair of PTs, as well as their respective frequency. It is observed that, for the most part, PTs mobilize objects associated with the measurement process, the most recurrent being those involving arithmetic calculations and the use of formulas. Regarding the visualization process, it is observed that PTs mobilize procedures related to the decomposition of surfaces in a convenient way, in addition to direct/indirect comparisons. Although the mobilization of such procedures allows facilitating the process of measuring areas, it is necessary that the PTs can resort, in the first instance, to a reorganization of the figural units that compose the figure presented, in order to be able to identify the parts that compose it. Moreover, Table 2 shows that pair 3 is the one that manages to mobilize a greater number of objects associated with the measurement and visualization processes.

Primary objects / Pair (Pa)		**Pa. 1**	**Pa. 2**	**Pa. 3**	**Pa. 4**	**Pa. 5**	**Pa. 6**	**Pa. 7**	**Pa. 8**	**Pa. 9**	**Total**
MEASUREMENT	P - Arithmetic calculations	1	1	-	-	1	1	1	1	1	**7**
	P - Decomposition of the surface in congruent units	-	-	-	1	-	-	-	-	-	1
	P - Use of formulas	-	-	1	-	1	1	1	1	1	**6**
	L - Geometric C/D- Intersection C/D -Semicircle	1	1	1	-	1	1	1	1	1	**8**
	L - Symbolic	-	-	1	-	1	1	1	1	-	**5**
	Pp - Accumulation and additivity	-	-	1	1	1	1	1	1	1	**7**
	Pp - Elements of the square	-	-	-	-	-	-	-	-	-	-
	Pp - Elements of the circumference	-	-	-	-	-	-	-	-	-	-
	C/D- Intersection	-	-	-	-	-	-	1	-	-	1
	C/D -Semicircle	-	-	-	-	1	-	-	-	-	1
	C/D - Inversely proportional relationship between the unit of measurement and the value resulting from the measurement	-	-	-	1	-	-	-	-	-	1
	C/D – Perimeter	-	-	1	-	-	-	-	-	-	1
	C/D – Radius	-	-	1	-	-	-	-	-	1	2
	C/D - Area/surface	-	-	1	-	1	1	-	1	1	5
VISUALIZATION	P - Composition of surfaces	-	-	-	-	-	1		-	-	1
	P - Perimeter decomposition	-	-	1	-	-	-	-	-	-	1
	P - Convenient surface decomposition	-	-	1	-	1	1	1	-	-	4
	P - Direct/indirect comparisons	-	-	1	-	1	-	-	-	-	2
	TOTAL	2	2	**10**	3	9	8	7	6	6	

Note: Procedures (P); Linguistic elements (L); Concept/Definitions (C/D); Properties (Pp)

Table 2: Objects and processes mobilized by each PT pair

CONCLUSIONS

The aim of this paper was to identify the mathematical objects mobilized by PTs when proposing and solving a geometric task involving a configuration of figures. The results show that PTs have a tendency towards the use of formulas and arithmetic calculations, to the detriment of those procedures associated with the visualization process. In this sense, the results are consistent with other studies that indicate that prospective teachers have a marked tendency towards the use of formulas, like that of the students. (Caviedes et al., 2023). Although the PTs had experiences with tasks that promoted the mobilization of geometric processes (e.g., measurement, visualization, deduction), these were, for the most part, omitted by the PTs, who focused their attention on arithmetic or algebraic calculation when proposing and solving a task (Seah & Horne, 2020). However, it is inferred that the visualization process had a relevant role when working with non-prototypical figures such as the one presented, since it implies the mobilization of procedures associated with decomposition/compositions of geometric figures, in order to give support and justification to an arithmetic-algebraic procedure that allows obtaining the area (or perimeter in other cases). These results are a first approximation to those tasks that will allow the mobilization of processes associated with geometric reasoning in initial teacher training. In this sense, the results presented could have implications in the treatment given by teacher educators to the formative tasks that promote the mobilization of measurement and visualization processes, both relevant for the development of geometric reasoning. On the other hand, the chilean curriculum declares the use of spatial visualization as a tool for teaching and learning geometry, so that through different strategies students are able to recognize properties of 2D and 3D figures, taking account that visualization is understood as the simple act of seeing. However, the visualization process goes beyond just "seeing", as it is a cognitive mechanism that brings into play the distinctive recognition of shapes and the identification of objects that correspond to the recognized shapes (Duval, 2016). Therefore, it is necessary to continue exploring the way in which future teachers mobilize the different processes linked to geometric reasoning through the use of formative tasks.

Acknowledgements

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LESSON STUDY AS A CONTEXT FOR TEACHERS' LEARNING OF LANGUAGE RESPONSIVE GEOMETRY TEACHING

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Our study focuses on language responsive geometry teaching (LRGT). We illuminate the dilemmas Malawian teachers confronted and the learning opportunities opened up during different stages of lesson study (LS) professional development that focused on similar triangles. Through a socio-cultural lens and related methodology, we used data from lesson planning, lessons and lesson reflection sessions to trace the teaching dilemmas that emerged and learning opportunities opened up. We show that in their planning and reflection discussions, the teachers deliberate when and how to name and represent the concept 'similar', thus questioning their routine practice of introducing a geometric concept by a definition. The discussions during the different LS stages provided teachers learning opportunities for improving LRGT practices of linking representations to the abstract mathematical concepts and how these are talked about.

INTRODUCTION

The teaching and learning of secondary school geometry is recognised as challenging in Malawi and there are calls for enhancing teaching quality through continuous professional development (PD) (Ministry of Education, Science and Technology [MoEST], 2020). We responded to this call by initiating a Lesson Study (LS) project focused on geometry with 10 Malawian secondary mathematics teachers from two schools. Originating in Japan over 100 years ago, LS has been widely taken up in many countries as effective PD for improving teacher knowledge and practice (Huang et al., 2019). Take up of LS is relatively new in Malawi and mainly focused at elementary mathematics (Fauskanger et al., 2019). According to research, when LS is introduced in a new context, structuring frameworks, also broadly known as theory-informed approaches, can scaffold teachers' entry in LS processes and enable systematic research of teacher learning through LS (Huang et al., 2019).

Malawian secondary mathematics teachers' challenges in teaching geometry are both in content and pedagogy (MoEST, 2020). We began the LS by conducting a two-day PD workshop with the secondary mathematics teachers. In the PD, we (1) discussed with the teachers the activities for promoting geometric reasoning using the mathematics teaching framework (MTF) a theoretically informed tool for working on teaching used in LS in a similar teaching and learning context to Malawi (Adler & Alshwaikh, 2019), and (2) introduced the teachers to LS mode of PD and its stages. Through studying teachers' participation in the first cycle, we confirmed that theory-informed LS supports adapting LS in a new context and reported on how LS can support teachers' learning of exemplification, a key teaching practice within the MTF (Mwadzaangati et al., 2022). We observed little attention to explanatory communication (a connected key teaching practice in the MTF) with several mathematical terms not elaborated fully, and learner talk was largely limited to reporting measurements but not expressing their thinking (Adler et al., 2022). In cycle 2, we aimed at working with the teachers to plan, enact and reflect on both how they used words to communicate geometric

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 221-228). ICMI.

concepts, and how they engaged learners in expressing their geometric thinking, practices we associate with language responsive teaching (Prediger, 2022). Teachers from one school studied on the topic of similar triangles, a concept research has shown to be non-trivial (e.g. Seago et al., 2014).

The learners' challenges in similarity reported by the Malawian teachers are similar to those reported in literature including inability to: 1) understand and relate conditions of congruent and similar triangles; 2) identify corresponding pairs of equal angles and proportional sides in given congruent and similar figures; and 3) construct logical arguments for justifying congruency or similarity (Adler et al., 2022; Seago et al., 2014). Although use of diagrams that exemplify geometric objects (Mwadzaangati et al., 2022) and use of concrete representations is viewed as an effective approach to teaching of geometry, Seago et al. (2014) report that even with experiences of relevant diagrams and real objects, learners still struggle with meaningful geometric interpretation of observed figures. What is out of view in the studies on geometry and specifically on congruent and similar triangles are language responsive teaching (LRT) related issues and so mediation through talk that connects real or concrete objects with abstract ones. In this paper, we explore how LS became a context for the teachers' learning of LRT practices in geometry (hereafter called LRGT) through examining (1) what LRGT dilemmas did teachers confront when planning and teaching similarity? and (2) how did deliberating on these dilemmas through LS provide teachers learning opportunities in LRGT?

THE MATHEMATICS TEACHING FRAMEWORK AND LANGUAGE RESPONSIVE GEOMETRY TEACHING

We take a social cultural theoretical orientation to mathematics teaching and learning rooted in Vygotsky (1978). The MTF is the 'teaching' version of the Mathematical Discourse in Instruction analytical framework developed for describing and evaluating the quality of mathematics made available in teaching (Adler & Alshwaikh, 2019; Adler & Ronda, 2015). It draws from key tenets of socio-cultural theory that view mathematics as a network of interrelated and well-organized scientific concepts, teaching and learning as goal directed with progression towards and in disciplinary ways of thinking teaching (Adler & Alshwaikh, 2019; Adler & Ronda, 2015; Vygotsky, 1978). Its starting point is that teaching is always about something, called the lesson goal, and the work of mathematics teaching is mediational. The MTF comprises three recognisable and inter-connected teaching practices, with key mediational means (bracketed) in mathematics classroom instruction: exemplification (examples, tasks, and their representational forms), explanatory communication (word use and justifying) and learner participation (what learners are invited to do, say and write). These mediational means work together to open (or close) opportunities for mathematical activity about the lesson goal. There is attention in the framework on the connections and coherence between these practices

Considering the focus of this paper, we zoom in here on explanatory communication (EC), and so the language practices emphasised in the MTGF and the representations in use. As indicated above, these practices are critical both for geometry teaching and learning and also in LRT in mathematics in general (Prediger, 2022). EC focuses on how we use words to (1) name and mediate mathematical objects, processes and procedures, while attending to movement between colloquial/everyday and technical/formal mathematical registers, and (2) how these are justified – with attention to whether these are statements and/or assertions or whether they are elaborated and then in what way. In Vygotskian terms, word use is important and linked with the relationship between the everyday and scientific concepts. Although the MTF clarifies what teachers need to consider as regarding word use and justifications (colloquial and formal word use and different kinds of justifications) it does not clarify how these link with representations used, hence the need for further study and probably the contribution of this paper on LRGT. We distinguished representations as either verbal or visual (diagrammatic or concrete or symbolic); and elaborated word use as movement between the everyday register, meaning-related school mathematics register and technical or formal register. MTF links with the work of other scholars working with content specific language practices (eg. Prediger, 2022). To answer our research questions, we were interested in the LRGT practices the teachers interacted with during their lesson planning and reflection sessions, the dilemmas that arose from the discussions and how these became learning opportunities for them.

METHODOLOGY

Each LS cycle consisted of two initial planning sessions, followed by teaching lesson 1, reflection 1 and lesson planning 2, then lesson 2, followed by reflection 2 and lesson planning 3 (see Mwadzaangati et al., 2022 for detail). In this paper we report on the LRT issues from one school by conducting content analysis on transcripts of a) lesson planning sessions; b) lessons 1 and 2; and c) reflection 1 and 2 sessions. The lesson transcriptions capture line by line utterances together with illustrations of teacher activity in the classrooms. Our analysis involved reading the transcribed data several times and dividing it into lesson episodes related to different lesson foci. In this paper we focus on episode 2 where the notion of similarity and similar triangles was introduced, as this generated substantive dilemmas amongst the teachers related to word use and representational forms. In the lesson episode, we examined the learning opportunities opened up to the learners by analysing whether word use was in the everyday, or school mathematics, or formal register, and whether the related representations were verbal (words only) or visual (concrete manipulatives or diagrams or symbols), and how these were connected. In the lesson planning and reflection discussions, we examined what the teachers talked about in relation to word use and representations, the dilemmas implied in their talk and so opportunities opened for teachers learning.

FINDINGS

The goal in both lessons was to establish the meaning of similar triangles, in relation to congruent triangles, and the proportionality of the corresponding sides. We begin by describing the LRGT practices of word use and representations deliberated by the teachers during their initial planning and the dilemmas confronted.

LRGT practices and dilemmas discussed during the initial lesson planning sessions

The teachers began the planning session for the LS by analysing the content on ‘similarity’ from the textbooks they use in their schools. They noticed that the definition of similar triangles in some textbooks as ‘figures with the same shape’ is not complete and agreed to adapt the textbook definition as ‘figures with the same shape but different size’. In addition, the teachers analysed whether the verbal and visual diagrammatic representations used in the textbooks support understanding of the definition of similar triangles. Both defining and representing are LRT practices (Prediger, 2022) and essential aspects in geometry (Seago et al. (2014), hence called LRGT practices in this paper. While these LRTG practices became the teachers’ take-off points in both lessons, they also became a source of debate as they lead teachers to consider whether to ask learners to explore the concept of similarity before they defined it, or whether it needed to be defined first as illustrated in extract 1. Note: that Tn stands for a particular teacher in all extracts.

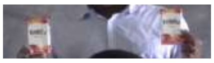
223	T4:	Okay, the first success criteria will be discussing examples of similar objects.
224	T5:	Examples?
225	T4:	Yes, examples of similar objects.
226	T3:	How can we start discussing examples of similar triangles before defining similar objects?
227	T4:	They should be defining from the examples of similar objects that they give. We can say, these are examples of similar objects, so what do you think is the definition of similar objects? The students will answer after giving examples of similar objects and seeing some of the characteristics of those similar objects.
228	T5:	For me I also think after seeing and identifying examples of similar objects then we will go to defining similar objects after defining.

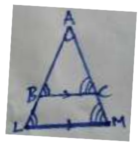
In extract 1, we notice that the teachers' discussions focus on language practices of representing similar objects (e.g. 223) and word use which include naming examples of similar objects and defining the concept 'similar' (e.g. 273, 275). The teachers had different views on whether to first ask students to define similar objects or give examples before showing them visual concrete representations. T4 and T5 suggested exploring and then naming/defining (223, 225, 227), while T3 questioned if students could give examples of similar objects before knowing its definition (226). From this discussion and what followed, we observed a LRGT dilemma between having learners experience the objects and then articulate the concept and vice versa. As we will see in teaching 1 below, the teachers finally agreed to first define the concept 'similar' before showing the examples (everyday representations).

LRGT learning opportunities opened up for learners in classroom

Teaching 1, episode 1 (which we do not show here) began with teacher asking learners to say what they know about congruent triangles (how we define these), followed by naming conditions for congruency. Episode 2 proceeded as shown in column 1, Table 1. Note that T stands for teacher, S for student, MS for many students.

As evidenced in Table 1 during teaching 1, the teacher started from colloquial language register by asking learners what they know about the word 'similar'. In the learners responses, the language register continued to move between the colloquial (same appearance, name and colour e.g. 39, 41 and 42) and school mathematics register (same shape and size e.g. 40) even with the support of everyday objects (juice boxes and plates). The definition of similar triangles given by the teacher remained in the school mathematics register too (66). We notice that the idea of first defining the concept 'similar' then showing the everyday representations did not assist to move the discussion on the definition of the concept 'similar' to formal register.

<table>
<tr><th>Teaching 1</th><th>Teaching 2</th></tr>
<tr><td>
T: Anyone who can define or explain
What you know about the word similar?
S1: Same appearance.
S2: Same shapes.
S3: Same size.
…
What do you notice about these shapes? (Two identical box juices).

S1: Same appearance.
S2: Same shapes and size.
S3: Same colour
S5: Same name
T: Okay, what about these shapes, what do you notice?

…
Are these plates similar?
MS: Yes/No
T: Can one stand and say why these plates are similar?
S1: They have same flowers.
T: Because they have same flowers.
S2: Because they have same shape and appearance.
T: Same shape and appearance.
S3: They are not similar
T: Why are you saying they are not similar?
S3: Because similar things have the same shape and size. So, if you can look at those plates, they do not have the same size.
T: Okay, similar objects are objects with the same shape…
</td><td>
T: Can we have two triangles where the corresponding angles are equal but the sizes
of the triangles are different?
MS: Yes/No!
T: What if we do something like this (Pastes diagram on board).


…
T: Let us compare angles in these two triangles ABC and ALM, do we have an angle in ALM which is equal to an angle in triangle ALM?
MS: yes!
T: Which angles?
…
139. T: Now two triangles which are the same in shape and size like these pairs of triangles we said they are congruent right?

MS: Yes
T: So how do we describe these two triangles, are there any words that you can use to describe two triangles with same shape but different size?

S4: Similar triangles
T: These are similar triangles …, triangles which have correspondingly equal angles but the sizes are different.
</td></tr>
</table>

Table 1: Lesson episode 2, introducing similar triangles

After teaching 1, the teachers met to discuss the lesson. On LRGT practice related issues, they agreed that students were able to identify characteristics of similar objects in terms of shape and that the teaching materials were relevant for building the meaning of the word 'similar'. However as also evidenced in the lesson episode, the teachers were concerned about lack of elaboration on the meaning of 'similar triangles', and lack of linking of the everyday objects and the formal concept of similar triangles. Therefore, during planning for teaching 2, the LRGT dilemma shifted to objects for illustrating congruency and similarity and how/whether to move from similar objects (the colloquial) to similar triangles (the formal) as shown in extract 2.

55	T3:	So, you can now say when we started you were saying these objects are congruent, but now what do you say about these two objects the bigger and the smaller one so maybe they can mention that they have same shape and different size.
56	T5:	Will learners be able to say that the two objects are similar?
57	T3:	That's what I said because when we are talking about congruency we talk about triangles. But this time we are bringing the objects that are not triangles, that's why I was saying can we try improvising triangles? So that we talk about triangles, because you can go and talk about objects, ask what they notice, and it will come back to you like in first lesson to say.
58	T5:	They have the same shapes, same size, same name, same colour.
59	T2:	Same height.
60	T4:	Time consuming also, it will be time consuming.
61	T3:	But still if they say they have same shape, same size you can now say mathematically when objects have the same shape and size, what do you call them? May be because the most important thing is for them to mention and understand these properties.
62.	T2:	So, in answering that question, yes, any objects can be brought but it will be convenient if we show them triangular objects first.

In extract 2, the dilemma in focus is on representations to be used for introducing similar triangles. The concern as learnt from teaching 1, we infer, is that if the teachers show everyday representations and ask learners to name them, they will not give the required name (58) and that it will be time wasting as they may end up only naming unnecessary everyday properties but not mentioning the required word 'similar' triangles (60). Their dilemma is how to move to the formal register, so they agreed to first use visual triangular representations as suggested by T2 (62) then show everyday objects later after students understand the meaning of 'similar triangles'. As teaching 2 shows in Table 1, the language register started and moved between the formal (e.g. can we have two triangles where the corresponding angles are equal) and school mathematics register (e.g. but the sizes of the triangles are different) in 62. When the students disagreed (63), he used visual diagrammatic representations of two triangles drawn on chart paper to support illustrating how it is possible to have two equiangular triangles with different sizes (66) and concluded the episode by giving definition of similar triangles which was in both formal and school mathematics register (143). In the lesson reflection on teaching 2, the teachers were happy with the way similarity was introduced and agreed that the move to concrete triangles, and engaging with whether two different triangles could be equiangular was enabling for learners.

DISCUSSION AND CONCLUSION

We have analysed extracts from teachers' lesson planning, lessons and their reflections on introducing and defining similar triangles. We notice that the teachers paid attention to LRGT practices of word use (naming and elaborating meanings) and linking these to representations, perhaps attempting to apply what they learnt in PD about LRGT. Of interest are two dilemmas for the teachers in regard to moving between naming concepts (and so the formal register) and representations to be used to bring out the mathematical talk. The first dilemma is whether to ask learners to explore a concept before they define it or define it first then ask learners to explore it. This dilemma relates to moving between the everyday and formal registers of LRT (Hardman, 2021) and confirms the Vygotskian view that language plays an important role on linking everyday concepts to scientific concepts (Vygotsky, 1978). When the teachers started from asking for the everyday meaning of the concept 'similar' then show everyday representations of similar objects, the definition of 'similar objects' remained in the everyday and school mathematics registers. This implies that the idea of starting with everyday language and everyday visual representations only opened learning opportunities for understanding the concept 'similar' in everyday and school mathematics language but did not support linking these to formal language.

The second dilemma which was observed during teaching 1 reflection was about how to link representations and formal language; how/whether to use everyday representations to link to formal language or how/whether to use geometric representations to link to formal language. The teachers had opportunity to realise that the representations used in teaching 1 did not support linking to formal language, and decided to change the representations and how to introduce the concept 'similar'. Although the teachers only thought that the everyday objects were not suitable to start with because they lead to naming unnecessary aspects, we further add that the use of the three dimensional and circular objects complicated the teaching further because these did not help to solve the highlighted challenges of clarifying the critical aspects of proportionality of sides, equal corresponding angles and scale factor in similarity (Seago et al., 2014). This was evidenced in teaching 2, whereby when the teacher introduced 'similarity' with visual geometric representations, the aspect of equal angles emerged and the discussion moved between the formal and school mathematics with clear link between the visuals and the formal language. In later episodes, the geometric objects were also used to introduce the aspect of proportional sides and the teacher ended the lesson by bringing in the everyday representations and learners were able to talk about their similarity as same shape and different sizes. Agreeing with Hardman (2021) that for some concepts, linking the abstract and the everyday is effective when the teacher starts from the formal such as naming and defining similar triangles, later relating these to the everyday, for example showing or naming examples of similar objects. The findings also confirm with literature that the teaching and learning of mathematics involves mediating the complex relationships among linguistic, symbolic, visual forms of representation of mathematical knowledge and in geometry these are crucial for learners understanding (Mwadzaangati et al., 2022: Seago et al., 2014).

The opportunity for teachers' learning that the interaction between the representations and the registers is important in geometry teaching was through LS processes of planning, teaching and reflecting. The dilemmas that the teachers experienced and discussing how to work around these during LS opened the teachers' learning opportunities about LRGT. The teachers' long debates and dilemmas in these LRGT practices of naming and representing imply that these practices are not easy to deal with in a single LS cycle and confirms the need for continuous PDs in LRT.

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MULTI-PERSPECTIVITY: A ‘RED THREAD’ THROUGH DISCUSSIONS ON GEOMETRY FOR TEACHING AND LEARNING

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What makes geometry so special in school mathematics? — This paper discusses a comprehensive and multi-perspective structure within one can speak about geometry in the context of teaching and learning. It starts with some fundamental aspects providing a general frame. Then, three dimensions organize the field: General views of geometry; approaches to geometry; geometrical activities. Finally, a few remarks are given how to probe the idea of multi-perspectivity.

The 9th ICMI Study of 1995 on “Perspectives on the Teaching of Geometry for the 21st Century” (Mammana & Villani, 1998) contains several chapters (esp. Chap. 6, Chap. 7) on evolutions, changes, trends of geometry such as changes in textbooks, curricula, technology, but also discusses deeper influences caused by the epistemology of mathematics, by the learning sciences, by social changes, etc. Constantly, the issue of the multi-perspectivity of geometry was highlighted, both, for geometry as a mathematical topic, and as a subject of school mathematics. Being so rich in perspectives seems to be characteristic for geometry (cf. Herbst et al., 2018; Graumann et al., 1996).

However, can we tap this prima facie vague idea of multi-perspectivity into a coherent, overarching, more or less systematic structure?

The “model” we propose sets a general frame for thinking about geometry in the context of teaching, learning, and education (first section), it states three dimensions to describe multi-perspectivity within a broad spectrum of geometry as a school topic (second section), and finally gives a few remarks to fields of probation of multi-perspectivity (third section).

A GENERAL FRAME FOR EDUCATIONAL THINKING ABOUT GEOMETRY: GUIDELINES AND FUNDAMENTAL ASPECTS

A still remarkable contribution is the British 2001-Report on teaching and learning geometry (The Royal Society & Joint Mathematical Council, 2001). This report does not only proceed – as to expect for a policy document – to a list of comprehensive and action related recommendations, but considers, for that reason, fundamental issues of geometry in schools, esp. for the 11-19 years old. A preface by the chairman, Adrian Oldknow, sets the tone: Being aware of the long history of the topic and of the ubiquity of geometric images, forms, models in daily live, ”… geometry should be one of the easiest branches of mathematics to teach. But this is not the case …” (RS/JMC 2001, p VII). The reasons lie in some “pitfalls”, and these root in the concept highlighted in this paper, multi-perspectivity. Thus, we must be careful. Oldknow points first to the danger of “abundance”:

> [Geometry] suffers from an embarrassment of riches in terms of theories, results, techniques and applications. […] We might refer to this, not unwelcome, problem as one of **abundance** [bold by author]. (RS/JMC 2001, p VIII).

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 229-236). ICMI.

Then, however, choices are necessary (in lessons, in the curriculum), and bring along new dangers:

> At one extreme there is a danger of choosing eclectically from this abundance in a way that leads to the teaching of a lot of apparently unconnected 'bits'. At the other extreme there is a danger of developing a tightly organised body of knowledge which addresses only a very small part of geometry. Our challenge has been to combine breadth with both educational and mathematical coherence – a problem we refer to as **coherence**. (RS/JMC 2001, p VIII)

For curriculum construction, a further problem appears. In their school career students encounter a lot of material in geometry. Besides the personal overload one seeks to avoid by establishing "coherence", there is also the mission of the educational administration to create for a certain topic a unified content with relations between the lower and the higher Grades. Oldknow says:

> We refer to this issue as one of **progression**. (RS/JMC 2001, p VIII).

The three terms abundance, coherence, and progression mark guidelines of educational thinking about geometry. They correspond to the three fundamental aspects described below: "Abundance" is an epistemological issue, since it states characteristics of geometry. "Coherence" appeals to pedagogy, since it is oriented to the personal learning, targets the construction of meaning, shows the longer chains of connections. "Progression" addresses the didactical intention of sensemaking curricula. These three aspects indicate the special focus when talking about a mathematical topic under educational viewpoints. It really makes a difference if we are speaking about geometry in the context of a sub-domain of mathematics, or about geometry as an essential part of mathematics in school, or even more general, about geometry for the purpose of building up a reflected view of "the world". It therefore deserves attention to separate the aspects to speak about geometry.

The epistemological aspect

Speaking about geometry as a mathematical topic should consider the multi-perspectivity of geometry itself. Thus, studies about historical developments, on the logical foundations, on mathematical standards in geometrical work, etc. are all welcome and fruitful, but in the educational context they always have to take into the consideration the origin of geometry as a human creation and the roots of geometry in human activity, be it cognitive, even mystic, aesthetical, technical, or in relation to the wider environment.

The pedagogical aspect

Geometry as a school subject aims at education. In Germany we call that intention "Bildung" (Neubrand & Lengnink, 2023). "Bildung" does not address the content exhibited in the curriculum alone, but targets wider connections. For any content in school, one must admit the question in how far that content can contribute to human development. The German educator Heinz-Elmar Tenorth coined it as "Kultivierung der Lernfähigkeit" (Tenorth, 1994, S. 94 ff.), i.e. *cultivating the ability to learn*, and fostering the cognitive solution to any issue in our life. In that sense, "Bildung" is a target and a corrective for teaching and learning in school, even with respect to geometry.

The didactical aspect

In the Anglo-Saxon tradition the term Curriculum means more than just the creation of syllabuses in school; in the continental tradition we prefer speaking of "Didaktik" (Westbury et al., 2000; Blum et al., 2019). In both traditions, school geometry is not only a set of contents (as important this is); rather, implementations must contain the use of that specific knowledge in contexts whatsoever (science, practice and customs in trade, craft, business, industry), a wide range of possible approaches, and the acknowledgment of the experiences of the student.

THREE DIMENSIONS TO ORGANIZE DISCOURSES OF GEOMETRY FOR TEACHING AND LEARNING

To come from these fundamental, but rather abstract aspects (as an underlying frame) to the issues of the daily agenda of geometry in school, and still not falling behind the multi-perspectivity strived for, we differentiate three dimensions to organize discourses on geometry. Each dimension should follow and explicate the three fundamental aspects, i.e. the epistemological, pedagogical, didactical foci. They are to structure multi-perspectivity.

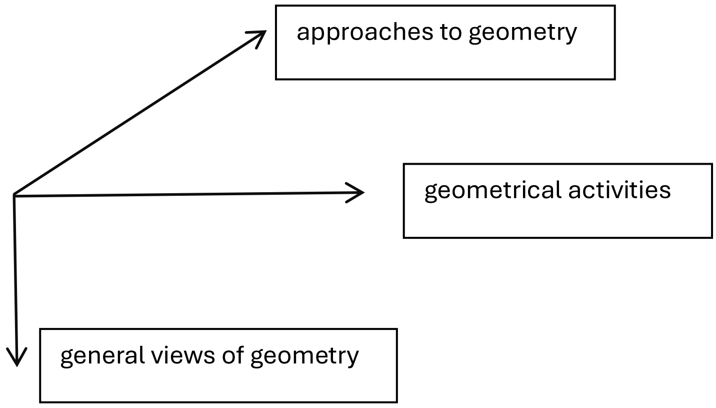

Figure 1: Three dimensions to organize discourses of geometry for teaching and learning

General views of geometry

Under the sub-domains of school mathematics geometry is unique in showing so many views. This opens a wide field of possibilities, still however, aware of the danger of abundance, and recalling that pursuing coherence is always on the agenda. What one can learn is that none of the so may views should be inferior or of less value, and equally, none of the views can stand alone. This stands behind the statement from above that the problem of abundance is "not unwelcome" (RS/JMC 2001, p VIII). The issue of multi-perspectivity of geometry itself can be traced back for long periods; an early source is Artmann (1979), a recent witness of the idea, albeit with another focus, is Kusniak (2018); cf. also the whole book of Herbst et al. (2018).

Here is a list, not claiming to be complete. Geometry can be viewed …

- … as a "ready to use" body of knowledge.

Of course, this view is, as a background, present whenever we discuss about geometry (as it would be for all other parts of school mathematics).

- … as field which gives a blueprint of "doing mathematics".

In no other field of school mathematics, experiences of doing mathematics seem as accessible as in geometry. There is the long history, but the central reason is that in geometry the way to the abstract theory is not as technically demanding, as in some other fields (say, e.g., calculus). Geometry has a wide range of theories; it could be formal, but there could be also, still strong and serious, theories keeping open the appeal to the practices (Bender & Schreiber, 1980). One even can claim that geometry is in itself a model for mathematics. Benno Artmann (1979) called it by the German word "Vorbild" (literally "preset picture", meaning something like a "template"); we list later some mathematical activities having authentical blueprints in geometry. Since one can hold the informal level for long, geometry is a good area for “speaking about mathematics” (Neubrand, 2000).

- … as a rich source of problems of a big variety.

Geometry opens a lot of possibilities of problems of different characters and wide-ranging difficulties. It goes from puzzles to severe problems. However, geometry problems often claim not only for a local solution but for embedding the problems into wider connections. This, among other reasons, is the potential of the classical (Euclidean) triangle geometry.

- … as a basis to describe, plan, construct, realize technical equipment.

This is a very specific aspect of geometry. In the real world lots of questions with geometrical background must be tackled: Streets, tunnels, ramps, bridges; buildings; gears; etc. Geometry is not only in drawing and construction but in the deeper questions like stability or directions of forces. Geometry aims at understanding, not just description. Hahn (2012) gives many examples.

- … as a basis to understand the space we live in.

Going beyond of just constructing practical things, geometry enables us to conceptualize what we see around us. In school we should use the full range of this view, from local orientation (maps, schematic plans), to the geometry which guides us through the environment we live in (the neighborhood, the earth, including weather, spreading of pollution), up to the space.

- … as a cultural achievement, as a product of the development of mankind.

This is not meant as a source of anectodical stories alone. It strives what was said before under the pedagogical aspect (“Bildung”): Human development is in no field of school mathematics as clear as in geometry. Geometry is the origin of mathematics in the cultural history. The Latin term “more geometrico” is in the Western culture a metaphor for stringent thinking. But geometry is universal: I just point to Fukagawa & Rothman (2008; Japan) or to Gerdes (2010; Africa).

- … as a rich supplier of forms for observation, interpretation, creation: visualization.

One can assume that this view of geometry is often unattended even neglected in schools, since there seem to be too less paths into the formal reign of mathematics. The opposite, however, is true. Geometry is a fundament, but also vice versa a product of visualization. Visualization as a concept strives aesthetics (Sinclair et al., 2007), it contains many far-reaching relations to arts (perspectivity, symmetries), it reaches out until the principles of the technical or even the evolutionary development (Hildebrandt & Tromba, 1996). Whatsoever, teaching and learning geometry should exploit that big stock of information, motivation, and launching platforms; see National Research Council (2006) for many suggestions.

Approaches to geometry

Multi-perspectivity in the general views opens multi-perspectivity in the approaches. No single approach fits it all or should be dominant; thus, teachers should be aware of the many possibilities. Each approach has its own dialectic. It can work for some students, or some topics; the didactical situation in the class calls for local decisions. And essentially, the term approach is too narrowly understood if we think about it only as a matter of motivation. The three dimensions influence each other. The approaches are so rich since geometry is so rich of views. The epistemological aspect triggers pedagogical orientations and didactical decisions.

Geometry can be approached (in the class, and equally by the individuals) …

- … by the relations to reality.

The condition, however, is to keep open that reality has many categories. It could start with the direct contact to things like the closer or wider environment, technical devices, facts from biology, architecture, etc. However, any approach needs goals going farther than motivation in the beginning. From the pedagogical aspect, an approach by reality should lead to deeper understanding. The German mathematics educator Heinrich Winter took the word "sublimation" to indicate that teaching geometry for "Bildung" should foster the ability to articulate the structural aspects of the world around us, it should make sensible for visual perception, transfer our observations conceptually, including a reflective attitude (Winter, 1997, p 29; my free translation).

- … by the disclosure of the inner connections, by the wish to master a certain topic.

Too often approaches to geometry are thought of as to come from outside. But approaching can also come from inside. Wishing coherence is a universal human attitude. Thus, the drive to logical order, the wish to explain, in the sequel even to prove something, is not necessarily a sign for a "ready-to-use" geometry but can be guide and generate progress. It requires a metacognitive attitude in the class (Kaune, 2006), which can indeed be well realized by geometrical topics (e.g. Neubrand, 2000: the systematization of the set of quadrilaterals). Similarly, mastering something is human as well. But sheer repetition is not enough, practicing needs connection (see many papers of Erich Wittmann, 2021). It can be quite plain in geometry. Here is an "integrated exercise": Look at all the various intersection this figure bears. What lengths of segments do appear?

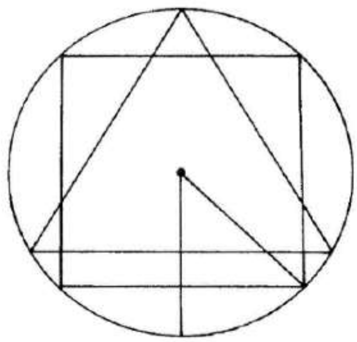

Figure 2. A circle, an equilateral triangle, a square – and many intersections

- … by using materials, by handling geometrical devices, by measuring with instruments, by reflecting digital systems.

This approach is not a plea to return to the "old" ruler-and-compass times. The essence is that these approaches come via the material manifestation of geometrical concepts. A striking example is the phenomenon of "touching", a concept reaching far into higher mathematics. Approaching geometry by devices and instruments is, furthermore, not meant as a subordination of digital approaches; rather, it claims for reflections about the differences, the advantages, and the pitfalls of each approach. The idea of "touching" is once more a good example: it calls for construction in the ruler-and-compass world, and often sticks to zoom-in / zoom-out strategies in the digital programs.

- … through curiosity, exploration, investigation, by seeking for understanding.

Since geometry is as rich in views and contexts as described, individual approaches must be valued. Nothing is more personal than curiosity. Thus, geometry is greatly welcome as an extraordinary field in which exploration and investigation can be developed within a huge variety of difficulty levels, contexts, interests, etc. This makes geometry special, and in this way one can find the strongest signals that understanding is the final aim in school.

Geometrical activities

The variety of approaches together with the pedagogical idea that learning is idiosyncratic in its nature produce the claim: Geometry becomes vivid if the students do it. There are many ways, but again no single activity marks the king's road. The more activities are done, the more options appear. One cannot do too much, rather it's dangerous to ignore or disregard possible activities.

Geometry allows activities like …

- … doing geometry by hand.

Sometimes one considers hands-on activities as the origin of thinking. At least in geometry we have a rich scale for that: folding, cutting, gluing, rolling, assembling, moving, etc. The impetus, then, to reflect these activities leads to understanding. It begins with noticing, it can continue with abbreviations, replacing an action symbolically, using a specific technique, etc., and on each stage with giving reasons for that what was done.

- … drawing (with mechanical instruments, and within digital systems).

This is geometry's specialty. There should not be no verdict that the one is the more valuable than the other. Both represent geometry on the level of creating visible products and considering their manipulation.

- … using numbers and calculation, realizing geometry with numbers.

Utilizing numbers, variables, formulas in the geometric context has many facets. It starts as early as with the geometric interpretation of numerical operations, patterns, and graphical manipulations; measurement is the next step. In this view, numbers and formulas express geometrical relations. However, there is also the other way round: Geometrical ideas contribute to the creation of analytic techniques: A "good model" of geometry requires realizations of concepts like location, distance, angles, volumes, and therefore one needs more than just the coordinates to build up Analytic Geometry (and, by that way, the door becomes open to generalize to higher dimensions). Anyway, geometry and numbers form a productive coexistence.

- … visualization.

The central human activity of seeing, i.e. using the eyes, is specific for geometry. The activity of "look at and see" should be kept open in geometry lessons as long as possible. But then, two sides can be stated: Geometry calls for assuring oneself that the seen is what really happened; from there the road is open into argumentation at various levels. On the other side, geometry provides an arsenal for the active visualizing of facts, relations, operations. I called that double nature of visualization "contemplative" vs. "active" (Neubrand, 1987). In the era of digitalization, visualization becomes ubiquitous, and hence it increasingly plays its role in geometry education. Being aware of aesthetical categories forms a background for all that. (Sinclair et al., 2007).

- … all the many typical mathematical working activities.

We already pointed to geometry as a "blueprint of mathematics". Here is a list of what can happen authentically when teaching and learning geometry: clarifying of phenomena; ordering; establishing relationships; uncovering hidden relations; going beyond what is known; broadening the horizon, generalizing, search for a thorough thought through an entire area, etc. These all are typical mathematical activities. It is up to the teacher to use that variety, and to reflect upon it (Neubrand, 2000), and geometry provides a rich substratum to let it grow.

FIELDS OF PROBATION OF MULTI-PERSPECTIVITY

Multi-perspectivity must show its value as an overarching concept in several respects. Of course, curricula should reflect the many aspects a topic provides. Today, we often construct curricula along the competencies required for that field (see Hattermann et al., 2023, for the German way to do it in geometry). Then, multi-perspectivity calls for considerations beyond the common content-process dichotomy, since it respects the deeper aspects of the content itself. Not at least, multi-perspectivity is also an issue in setting up empirical studies for assessment, in the small (Neubrand, 2021) and the large scale (Neubrand, 2013). And above all, multi-perspectivity is a concept important enough to be an essential part of teacher education.

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HOW TO TEACH GEOMETRY IN CONTINUITY ALONG SCHOOLING?

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This article reports on research, in the French context, into a possible continuum of teaching and learning plane geometry along compulsory schooling. It is based on an epistemological analysis of content, Duval's work on semiotic registers, and Brousseau's Theory of Didactic Situations, using a didactic engineering methodology.

The ICME-13 survey (Sinclair et al., 2016) shows that there is currently little research that looks at both student learning in geometry and the teaching and training of teachers from the beginning of primary school through the middle of secondary school, when the use of mathematical evidence is expected. However, the plane geometry taught in compulsory education in France, as in many other countries, includes two main types of geometry, based on very different reference mathematical practices: a "material geometry" based on construction and validation with material tools and a "theoretical geometry" based on definitions and proof. Many studies have shown that supporting students in this evolution of practice is a real difficulty for teachers (Perrin-Glorian & Salin, 2010). Over the last twenty years, research has developed in France to reflect on the teaching of geometry in a continuity from grade 1 to grade 8 and to provide tools for teachers (Mathé et al., 2020; Guille-Biel Winder & Assude, 2023). Our aim is to question what these taught geometries cover, their epistemological foundations and their didactic stakes, and to search for a possible coherent progression of geometry throughout compulsory schooling. This research takes up some of the issues raised in Perrin-Glorian and Salin (2010) in a direction indicated by Sinclair et al. (2016). We restrict ourselves here to the field of plane geometry.

THEORETICAL BACKGROUND

Material geometry and theoretical geometry

Geometry encompasses several types of practice with different epistemological foundations. In the context of plane geometry, we call "material geometry" that geometry whose objective is practical in the graphical space of 2D drawings (freehand or constructed with geometric instruments): studying or tracing material geometric figures with material tools. The object of study is a drawing on a sheet of paper, or a computer screen and the mode of validation is material, such as tracing paper, geometric tools or by dragging on the computer screen. Nevertheless, geometric knowledge can be used to guide or justify the drawing: for example, to reproduce a parallelogram, it is not enough to know the dimensions of the sides, another information is needed, an angle or a diagonal. Knowing the lengths of the sides and a diagonal, spatial knowledge can help to place the elements correctly to get a parallelogram instead of a kite shaped quadrilateral. We call "theoretical geometry" a geometry whose objects are defined in a discursive form and whose validation is based on proof. Traditionally, in France as in many other countries, these two types of practices, are taught one after the other.

However, although they use the same words, there is a rupture between these two ways of doing geometry in the nature of the objects of study, of the problems posed and in the methods of validation. Houdement and Kuzniak (2006) speak of a break between two paradigms: "Geometry I (natural)" and "Geometry II (Euclidean)". Our work is based on the idea that there are nevertheless continuity levers between these geometries.

The role of drawing in geometry. Dimensional and instrumental deconstructions

As Duval's work (1998, 2014) shows, the theoretical geometry requires the articulation of several semiotic registers, mainly the verbal register and the graphic register. In a proof, the drawing can serve various functions, such as illustrating a statement, exploring multiple configurations for conjecture, adding plots to reveal relationships or isolate sub-figures, and so on. A major difficulty is that the visualization of figures in geometry is different from the usual way of looking at and talking about drawings. Duval (2014) shows how and why the relationship between visualization and discourse in a geometric activity is done at the level of figural units, which requires a dimensional deconstruction of shapes. The visualization of figures involves cognitive issues but is also closely related to geometric knowledge. Indeed, the objects of theoretical geometry are linked by propositions that translate into visual or instrumental characteristics of material figures. Thus, the material figures used to represent geometric objects defined in the verbal register may be the same as drawings studied for themselves or drawings representing objects in space. In fact, geometric statements refer to some relationships between objects that are often represented by figural units of dimension 1 or 0 while natural vision favors figural units of dimension 2. Therefore, the use of definitions and theorems requires a *dimensional deconstruction* of shapes, "the mathematical way of seeing" drawings composed of smaller figural units, related by geometric properties. Moreover, it is often necessary to use lines that are not drawn in the figure, which is a real difficulty (Mathé & Mithalal, 2019). Let's take an example.

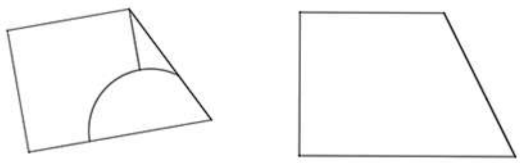

Figure 1 - Example: reproduce the model figure from the trapezoid frame

Instead of seeing a juxtaposition of three shapes, adopting a geometric vision of the figure on the left would mean deconstructing 2D shapes into sub-units and looking at the relationships between these sub-units. This requires to see networks of lines linked by intersections and angles, to extend a segment to reveal a square, to extend the arc of a circle to make a semicircle, to consider the perpendicularity of lines or the equality of lengths, to consider that the same point can be both the center of a circle and the vertex of a square, and so on. However, we hypothesize that an instrumented work in material geometry, under certain constraints that we will specify later in this text, can assist the students in constructing this mathematical relationship to drawings. Constructing or reproducing a figure with geometric tools can engage students in the instrumental deconstruction of shapes and the identification of a set of independent figural units, and of a sequence of actions performed through the use of instruments, making it possible to reconstruct either the object itself or a graphic representation of that object (Mithalal & Balacheff, 2019). Moreover, this work can allow the gradual emergence, within the graphic space, of a relationship to drawings and statements, that is close to that of theoretical geometry. For example, the reproduction of the figure, starting from the trapezoid frame (Figure 1, right) with a rectangular template and a compass makes it necessary to be aware of the geometric relationships between the trapezoid, and the square and the semicircle in order to make them appear. This fundamental learning process is not currently taught in regular classrooms.

Material geometry and spatio-geometric knowledge

Berthelot and Salin (1998) show that geometric knowledge is not spontaneously transferred to solve problems in the 3D-space in which we move. They call spatial knowledge the knowledge used to solve these spatial problems and hypothesize that developing what they call “spatio-geometric knowledge”, derived from geometric knowledge in solving spatial problems in a modeling problematic, could help students link geometric knowledge and spatial knowledge. We consider these constructs in the 2D graphic space, and we hypothesize that some problems, which we try to characterize, can help students link and distinguish spatial knowledge and geometric knowledge.

By assembling shapes or reproducing figures, students may have the opportunity, if taught, to encounter the notions of alignment, intersection, angle, perpendicularity, or parallelism. Beyond the encounter with the cultural knowledge of geometric shapes, from the very beginning of school, through the manipulation of material objects, we can propose problems of placement of material objects or of tracing, which lead children to build spatio-geometric knowledge, on which geometric knowledge can then be based. For example, a square can be made by selecting from a set of pieces two isosceles rectangle triangles with hypotenuses of equal length; a square can be drawn by using a rectangular triangle template that is rotated. The equality of lengths can be checked by juxtaposition or superposition, that of angles by superposition (Celi, in Guille-Biel Winder & Assude, 2023, Ch.3, pp. 47-72). Later, spatio-geometric knowledge can be built by reproducing geometric drawings as the example in figure 1, provided that the conditions of this reproduction are properly chosen.

Learning situations, situational and institutional knowledge

Our research is rooted in the TDS (Theory of Didactic Situations in Mathematics) (Brousseau, 1997). This theoretical framework is based on the idea that to know a mathematical concept is not only to be able to formulate a text of knowledge, but also to understand the reasons for this knowledge and at least some aspects of its usefulness and operationality in concrete situations. The aim of this theory is to identify favorable conditions for students to learn a given piece of mathematical knowledge. These conditions are seen as a *milieu* in which the subject pursues a goal. The student's interaction with this *milieu* is modeled by a *situation*.

Viewing learning as an adaptation to the constraints imposed on action in a situation leads one to consider a "double aspect of knowledge" (Margolinas & Bessot, 2021). The evolution of students' strategies in a situation reflects the evolution of their "situational knowledge". It is not always possible to express it. Learning is characterized by the stabilization of a certain amount of this knowledge as an adaptation to a particular situation. Some situational knowledge may be recognized as useful by an institution and then transformed into "institutional knowledge" in the process of institutionalization. Institutional knowledge is a text that results from the incorporation of situational knowledge into a shared institutional culture. In geometry, situational knowledge about the visualization of drawings is often not considered by school. The goal of developing this situational knowledge seems to us to be an important issue in material geometry, because we see it as a possible path from spatial knowledge to geometric knowledge. Helping teachers make connections between the strategies of constructing figures with instruments and the spatio-geometric knowledge involved and deriving the institutional knowledge of geometry from this situational knowledge also seems to us to be an important challenge.

TO TEACH GEOMETRY IN CONTINUITY ALONG COMPULSORY SCHOOLING

We now come to our proposals for the management of a continuity from spatial to geometric knowledge throughout the school. They consist in identifying a field of practices and discourses that can be carried out on material figures in order to facilitate access to theoretical geometry.

A fundamental situation in the geometric analysis of material figures: Figure reproduction

In TDS, a fundamental situation constitutes a generic situation that can be used to define several situations for didactic use (teaching situations) aimed at specific knowledge. Our work has led us to consider the reproduction of figures as a fundamental situation in the geometric analysis of figures. The generic problem consists in making a copy of a given material figure taken as a model, by means of instruments. The reproduction can be either identical, or of different size and/or orientation. Means of validation can evolve: first carried out by superimposing a solution on tracing paper on the produced figure, later by checking the properties that characterize the shape of the figure.

Geometry of tracing

The geometry of tracing (GT) is the geometry generated by this fundamental situation. It is the geometry of making or reproducing figures with drawing instruments, without measuring instruments (there are no numbers in GT), but with constraints on the use of these instruments. We have called "figure restoration" the case where we ask to reproduce the model from a given starting figure as in Figure 1. An essential point in this geometry is to consider the knowledge required to reproduce the figure in relation to the instruments available. In fact, the instruments used condition the figural units and relationships on which the reproduction is based, and thus the way the figure must be seen. The model, the starting figure, the difference between them, and the available instruments are the main didactic variables of the situation.

There is a break between material geometry and theoretical geometry. However, the objects and relations of elementary geometry (straight line, angle, parallelism, etc.) reflect spatial properties that can be perceived by our senses. Drawing instruments produce visual characteristics that correspond to the representation of geometric properties. The word "tracing" evokes instruments and material figures, but GT as we understand it is different from material geometry. It corresponds to the search for a possible link in learning between drawing with instruments and the notions of abstract geometry. It translates the fact that the objects and axioms of Euclid's geometry can largely be seen as a theorization of the construction of figures, especially those figures that can be constructed with a ruler and a compass (by drawing straight lines and circles).

In order to establish the link between drawing with instruments and geometric properties, we consider "theoretical" drawing instruments insofar as they do not have the limitations of material instruments and have only one function : a *ruler* to draw straight lines, a *"length-transfer"* to carry a length on a line already drawn from a point on that line, a *"length-bisector"* to take half a length, a *"angle-*

transfer” to reproduce an angle along a line with a point at which to place the vertex, a *compass* to draw circles.

In order to establish a geometric relationship with material figures, a precursor of that of theoretical geometry, it is necessary to respect certain rules of instrument use, which we have called the geometric use of instruments (Mathé et al., 2020; Mathé & Perrin-Glorian, in Guille-Biel Winder & Assude 2023, ch.1, pp. 3-34). To set up the ruler, you need two points, or an already drawn segment; to set up the length transfer, you need an already drawn line and a point on this line; to take the center of a segment, you transfer the length of this segment to the length bisector, then you transfer the half-length to the segment from one end; to set up the angle transfer (for example a template), you need a straight line and a point on this line: the top of the template is placed on the point and one side of the template is placed on the line, the angle is drawn on one side or the other; to draw parallel straight lines, drag one side of an angle template onto a straight line and draw on the other side. With these rules for the use of instruments, one can construct all the figures of plane geometry encountered in elementary and middle school. These functions can also be found in dynamic geometry. As the students’ knowledge progresses, the number of instruments can be reduced: the compass can soon replace the length-transfer, then the length-bisector, then the angle-transfer.

From the geometry of tracing to the theoretical geometry

To obtain proofs in theoretical geometry, we need definitions and theorems which, for the sake of overall coherence, must have an axiomatic basis. The question of the foundations of secondary school geometry was hotly debated from the 60s to the 80s. In France, the cases of congruence of triangles were abandoned and, in the 80s and 90s, geometric transformations were placed at the base. However, these tools are more accessible to students than transformations (Perrin & Perrin-Glorian, 2021). They require a less difficult dimensional deconstruction than transformations, which often require consideration of the points that define the lines (straight or circular) and the fact that a point is obtained by the intersection of two lines. Thus, the choice of an axiomatic foundation for secondary school geometry is quite important for students’ access to theoretical geometry.

Dialectics of action, formulation, and validation around figure reproduction

Of course, it would be reductive to approach the issues involved in teaching geometry solely through the instrumental dimension of geometric practice. This dimension is inseparable from other cognitive, semiotic and linguistic issues (Bulf et al., 2015). Moving from the geometric use of instruments to an initial conceptualization of spatio-geometric then geometric knowledge also requires linguistic work. Through multiple reformulations, networking, generalization (Sfard, 2001), it will also be a matter of supporting the students from a technical language (related to the use of instruments) to a geometric language (Petitfour, 2017). It is then a question of making the problems posed evolve towards the search for justifications and reasons for construction methods, and of moving the students from pragmatic proofs to intellectual proofs. Thus, we can see the emergence of different types of knowledge that can be linked to different types of situations for didactic use, derived from the fundamental situation of figure reproduction, based on the model of the dialectics of action, formulation and validation proposed by Brousseau (1997). In the next section, we briefly discuss how we have used our theoretical reflection to design learning situations that have been implemented, for more than twenty years now, in primary and early middle school classes.

TOWARDS LEARNING SITUATIONS

The progressions and situations that we are experimenting with in the classroom to organize the spatial to the geometric in school (Mathé et al., 2020; Mathé & Perrin-Glorian, in Guille-Biel Winder & Assude 2023, ch.1, pp. 3-34), articulate these different types of situations. We illustrate our argument with an *a priori* analysis[14] of an example involving the square, to show in this very simple case the dialectic between situations of action, formulation, and validation on the reproduction of figures that we used in class on situations that we cannot describe here, for lack of space.

Reproducing geometric figures as action situations

In the *action situations*, the students have a model figure, possibly a starting figure, and instruments. Problem solving is characterized by successful action. Students act directly on the *milieu*, which in turn provides them with *feedback*. They can test the validity of their reproduction strategies by comparing the material figure produced with the model to be reproduced (for example, by superimposing a solution provided on tracing paper). This feedback helps students to develop new, more operational strategies, for constructing new situational knowledge. For example, consider the problem of reproducing a square from two given consecutive sides (Figure 2). To do this, we need to draw the other two sides and the fourth vertex (point C) of the square. The knowledge required depends on the tools available. If you have a ruler and a right-angle transfer, without length transfer, you must think of the sides of the square as lines that can be extended and the vertex as the intersection of these lines. If you have a ruler and a compass, you must think of the equality of the sides and the missing point as the intersection of two circular arcs and think of the compass as a means of getting points at a fixed distance from a given point.

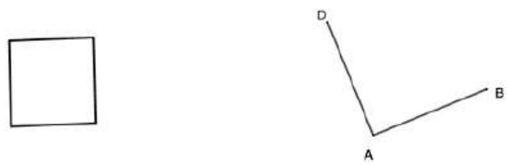

Figure 2 - Example: reproduce the square from two given consecutive sides

Thus, the reproduction of the square leads to the emergence of properties of the square, depending on the instruments available. Nevertheless, the mere aim of materially reproducing a model figure does not make it necessary to formulate the geometric relations that could underpin action strategies.

Formulation situations

Formulation situations are those that make the formulation of the implicit action model necessary and problematize it. Formulation is necessary when direct action on the *milieu* is prevented, for example because the model figure and the figure to be produced are distant in space or time. In the case of formulation situations for others, for example, a student has a model figure, and must produce a message that allows a classmate to reproduce the figure without having seen it. A key variable in these situations is the type of language used. It can be graphic (using diagrams) or verbal. If verbal, it can be oral or written. The verbal formulation itself covers different types of discourse. We can

[14] In the TDS, an *a priori* analysis is an epistemological reflection that does not have a predictive but a causal meaning; it involves describing various possible (and therefore potentially reproductible) phenomena in the context of a classroom situation (Margolinas & Bessot, 2021).

progressively impose constraints on the language to be mobilized, from a technical language that mentions the instruments to a geometric language that refers exclusively to the geometric objects and relations that are invoked.

In the example (Figure 2), in the case of verbal formulation situations for others, the first difficulty is to designate the elements of the figure: designation of points allows to designate segments, lines, etc. The second one is to designate the relationships between these elements. In the case students dispose of a ruler and a right-angle transfer and points are already designated, they must explain how to place their right-angle template and trace or to trace a line perpendicular to a side on a vertex. Technical language would be "place one side of your template along AD with the top on D and trace along the other side. Do the same along the side AB with the top on B. You get two lines that intersect in C". Geometric language would be "Trace the perpendicular to the AD side that passes through D and the perpendicular to the AB side through B. Their intersection is the point C." Using a ruler and a compass, the formulations could be: "Lay the compass on D and the mine on A and draw an arc. Do the same with the point on B. The arcs cut into C." or "Draw the circle with center D and radius AD, then the circle with center B and the same radius; these circles intersect at A and another point, that is C; trace the segments [CD] and [BC]. Thus, a formulation situation about this reproduction can confront students with the problem of naming and characterizing geometric objects and relationships.

The linguistic work involved in these situations, based on the reproduction, or instrumented construction of a figure, plays an essential role in the possible links between material geometry and theoretical geometry. On the one hand, there is a profound change in the object of the work, from the material figure to a text that allows the geometric definition of the represented object. In these situations, the need to construct a geometric language to designate the objects and relationships mobilized in the reproduction of a figure goes far beyond questions of lexicon. These situations confront students with the question of how to characterize geometric objects, that is, what information is necessary and sufficient to define a figure.

Validation situations and start proving

In action and formulation situations related to the reproduction of figures, the validation of drawing strategies or formulations intended to communicate these strategies is based on pragmatic evidence. The *validation situations* must make the students' geometric activity evolve towards the search for justifications and reasons for constructions and involve them in intellectual proofs.

In the example (Figure 2), questions can be asked like: "Does this message allow, for sure, to get a square and why?" This can be an opportunity to generate counterexamples to invalidate a message or to produce arguments to prove its validity, for instance: "it's a square because we have constructed a quadrilateral with three right angles and it has two consecutive sides of the same length"; "we have constructed a quadrilateral with four sides of the same length and one right angle". It can lead students towards a proof approach, based on the construction of the square using geometric instruments.

DISCUSSION: FROM LEARNING SITUATIONS TO TEACHING SITUATIONS

Our observations in class for many years are encouraging. However, it is difficult for primary school teachers to manage the complex set of didactic variables mentioned or to perceive the potential stakes of the Geometry of Tracing. The diffusion of this way of teaching geometry in regular classes requires new research with collaboration between teachers and researchers. In France, such research has now begun in some institutional places that make this work possible, bringing together primary or secondary school teachers, and researchers to collaborate and experiment in the classroom, even to produce resources (Guille-Biel Winder & Assude, 2023). These collaborative spaces seem to be particularly conducive to changing practices, as evidenced by a number of works that go in this direction (cf. ICMI Study 25 "Teachers of Mathematics Working and Learning in Collaborative Groups"). We emphasize the importance of developing now the research that allows to describe the professional teaching actions that can help teachers to build this new contract.

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APPROACH TO TEACHING GEOMETRY TO DYSPRAXIC STUDENTS

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The aim of this contribution is to present an overview of an approach to teaching geometry to dyspraxic fifth and sixth-grade students. These students face some important difficulties in using geometrical instruments (ruler, compass, set square) with required precision. We've developed a dyadic work system, based on the preserved skills of dyspraxic students, to allow them to study geometry without drawing geometric figures being an obstacle. We present this work system, its theoretical background, an implementation in a pencil-and-paper environment and the possibilities offered by a human interaction simulator to contribute to teaching geometry to dyspraxic students.

INTRODUCTION

In French primary school and at the beginning of secondary school (students aged 9-12 years), the teaching and learning of geometry is based primarily on situations requiring the use of instruments (ruler, compass and set square) in a pencil-and-paper environment. Tracing with instruments to reproduce, to represent or to construct geometric figures is an injunction of the school curriculum, and is supposed to contribute to the conceptualization of geometric concepts. However, this teaching method, which uses manipulatives, is not suitable for all students, and in particular for students with dyspraxia, as they cannot perform the skilled movements necessary for such academic tasks: their motor performances are slower, less accurate and more variable (Elbasan & Kayihan, 2012). These students invariably fail to complete tasks in geometry, whereas they do have the core abilities – reasoning, language and memory – that would otherwise enable them to conceptualize geometric notions.

Our research aims at proposing alternative ways of teaching geometry, accessible to all students, and in particular for those who have difficulties in manipulation in order to meet the objectives of the geometry curriculum. In the first part of this paper, we present a dyadic system for teaching geometry considering the handicap of dyspraxic students and their preserved capabilities. We next present an example of an implementation of this work system in a pencil-and-paper environment with two students (dyspraxic student and standard student). Finally, we present the possibilities offered by a human interaction simulator to contribute to teaching geometry to dyspraxic students.

A WORK SYSTEM FOR TEACHING GEOMETRY

The dyadic system (Petitfour, 2016) is based on the joint activity of two students, an *instructor* and a *constructor*, who interact to solve a geometric construction problem. This system is anchored in a conception of learning as a social phenomenon (Vygotsky, 1978): the formation of mathematical concepts is carried out through social interactions, in joint work around the resolution of a problem. The work takes place through free exchanges on possible construction techniques, the implementation of a chosen technique in the "instructor-constructor dyad" and discussions on the validity of what is produced. This dyadic work system is based on the language dimension of geometric activity, in order

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 245-252). ICMI.

to capitalize the preserved capabilities of dyspraxic students. It aims to replace instrumented action situations for dyspraxic student with situations that are as close to action as possible, but without any actual action, while producing similar effects in terms of geometric learning.

The construction proceeds as follows. The instructor expresses his/her *prior intention* (Searle, 1983) through language instructions to the constructor, accompanied by gestures. The constructor carries out the instrumented actions requested while the instructor observes the concrete implementation and obtains precise feedback. Let's specify the language, gestures and feedback specific to this work.

Geometric technical language

The instructor uses a technical language (Petitfour, 2016) to formulate his/her plan for the instrumented actions. In the field of geometry, the technical language relates to the use of instruments in connection with the geometric properties they convey. It is used to express the relationship between geometric lines and a given instrument in order to produce a new line. For a given instrumented action, the instructor first mentions the instrument to be used, then gives its position by describing the relations between the instrument and graphic objects already present and finally indicates the location of the line. For example, drawing a circle of center A and radius AB (geometric language) corresponds to the following instructions in technical language: take the compass, put the point of the compass on point A and the pencil lead on point B, draw the circle. This language must not be confused with "manipulative language" that expresses corporeal action (e.g., "Hold the compass by the top, not by the legs", "Press more on the point of the compass than on the pencil lead").

Mathematical gestures

Following McNeill (1992) in that gestures, together with language, help constitute thought, we propose to associate with oral language the use of mathematical gestures, i.e. any body movement, spontaneous or deliberate, carrying a meaning related to mathematics. In geometry, deictic gestures (Figure 1a) can, for example, link elements of the drawing (straight line, point) with parts of the instrument (side or vertex of the right angle of the set square); mimetic gestures (Figure 1b) can evoke the handling of the instrument (turning the compass); iconic gestures (Figure 1c) can represent a geometric relationship or a geometric object (triangle).

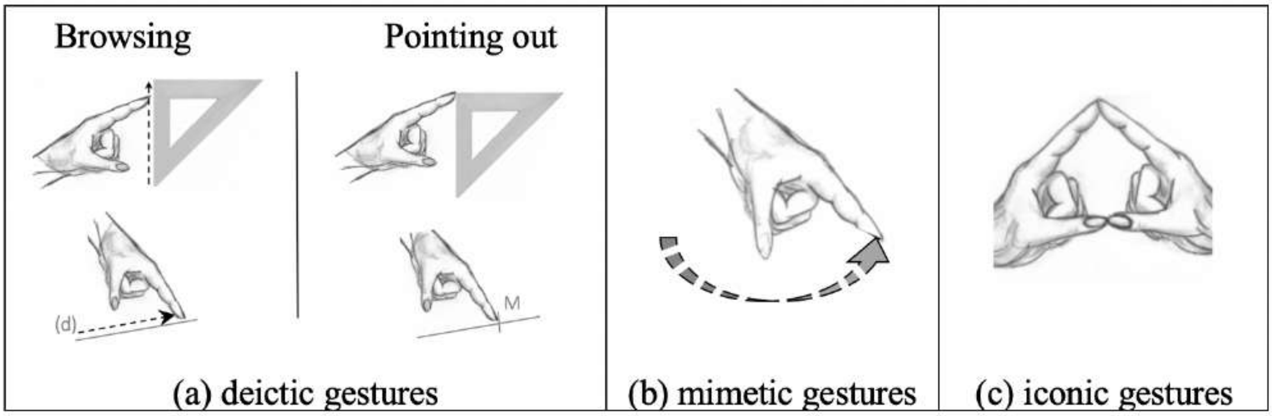


Figure 1: Example of mathematical gestures

These mathematical gestures can be produced by dyspraxic students: they require less precision than actions and they offer the advantage of expressing the planned drawing without leaving a graphic trace. Moreover, for the instructor, producing gestures help to preserve the role of bodily and kinesthetic experience in mathematical learning, as highlighted in various studies (e.g. Nemirovsky, 2003; Arzarello et al., 2009).

Feedback

The constructor follows the instructor's instructions: he/she takes the chosen instrument, positions it as requested and draws the required line. When he/she detects ambiguities in the instructor's instructions, he acts in the "least likely" way to help him/her remove the implicit meaning. In this way, the constructor needs to pay attention to the validity of the instructions for obtaining a correct figure, i.e. one constructed without the "guesswork" of using the instrument (such as positioning the ruler perceptually to draw a right angle).

By working in pairs, dyspraxic student (in the role of the instructor) can experience the instrumented actions he/she is planning by observing them, without worrying about their manipulative characteristics. He/she also benefits from feedback that is consistent with his/her instructions, on which he/she can base his/her thinking.

In the following section, we illustrate this dyad work system by analyzing an implementation in a pencil-and-paper environment with two sixth-grade students (dyspraxic student and standard student)

EXAMPLE OF AN IMPLEMENTATION

Marion is an 11-year-old student diagnosed with visuo-spatial dyspraxia. Her psychomotor assessment demonstrated an immature spatial organization and difficulties with digital untying and oculomotor co-ordination. She has always had difficulty writing and drawing geometric figures with instruments at school. Despite her many attempts, her attention and her efforts, she was never able to achieve the required precision. This is illustrated by the comments made by her mathematics teacher in the following extract from an assessment (Figure 2). The use of ruler and set square act as a barrier between what she knows, what she wants to draw and the act of drawing. She indeed knows how to draw the symmetry of a point or a straight line with respect to a straight line, but when she draws with the instruments, she places the set square imprecisely, the instrument moves when she draws, etc.

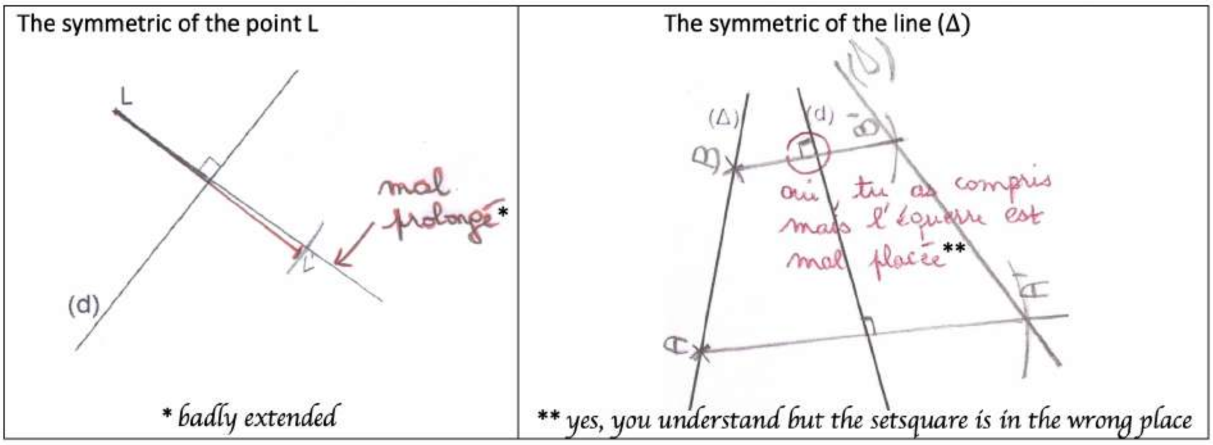


Figure 2: Marion's work

We report on the implementation of the dyadic work system that we tried out a month and a half later with Marion and Bonnie, a student in her class.

Given Marion's visuo-spatial difficulties, the instructions were given orally so as not to add a reading task to the proposed geometric task (Figure 3). In addition, the two straight lines were differentiated by color (the axis of symmetry was in red) and the starting figure was presented using gestures (browsing the straight lines, pointing to the points) to accompany the speech.

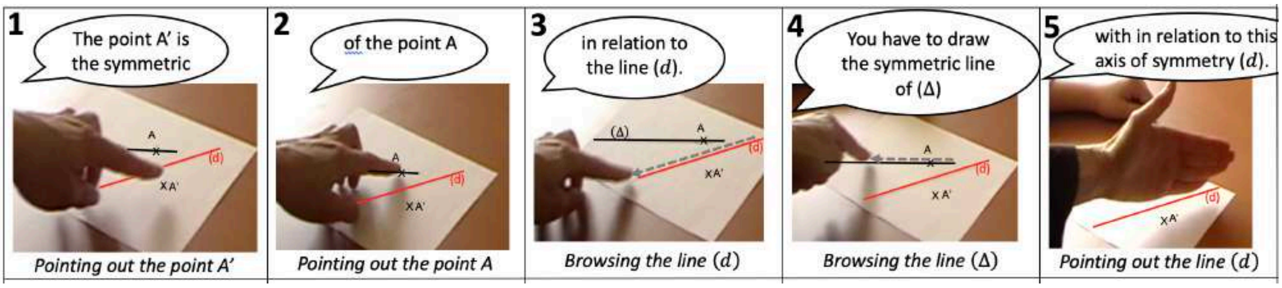
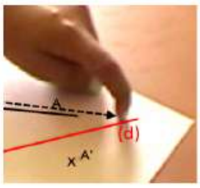


Figure 3: Instructions given to Bonnie and Marion

Bonnie and Marion both immediately announced that they knew how to do it, as soon as the instructions had been given. Bonnie gave a drawing technique (Tb) that involved extending the line (Δ) to its point of intersection with the axis of symmetry and then drawing the line through this point and point A'. Marion reacted by challenging this technique: "But actually, Bonnie, you can draw your line wherever you like, you can put it like this, like this, like this, so that it touches the point", pointing with her hand to different directions of the line that pass through the point of intersection that Bonnie was proposing to construct. Marion then gives her drawing technique (Tm), which involves drawing the symmetrical line of a second point on the line (Δ) in order to draw the line that passes through this symmetrical point and the point A'.

Both drawing techniques are shown in Figure 4 with the speech and gestures produced by Bonnie and Marion during their interactions.

Technique proposed by Bonnie (Tb)	
You take a ruler, you extend the line to the axis of symmetry	
Then you take the ruler, you draw a line passing by A' and the point of intersection	

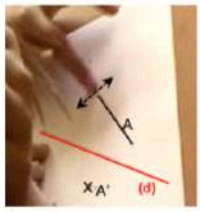
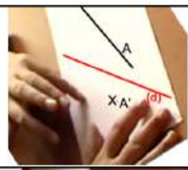
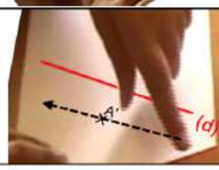
Technique proposed by Marion (Tm)	
I say we should put a second point at the bottom of the line, even if it's infinite, on the represented line	
and you do the symmetrical	
then you connect the two dots	

Figure 4: Two different construction techniques

Neither Bonnie nor Marion mentions point A', but they both point to it and run their fingers over it as they trace the symmetrical line they are considering. In this way, the two students can reflect together on the geometric problem they have been given, without any language barriers limiting their reasoning: it's difficult to focus on using geometric language properly while at the same time exploring and explaining possible solutions to a problem. The obstacles associated with the layout that Marion would encounter if the initial discussions to solve the problem were based on instrumented layouts were also ruled out, given her disability.

Both techniques (Tm and Tb) will be put to the test. The following four vignettes (Figure 5) trace the beginning of the interactions between Marion, the instructor, and Bonnie, the constructor.

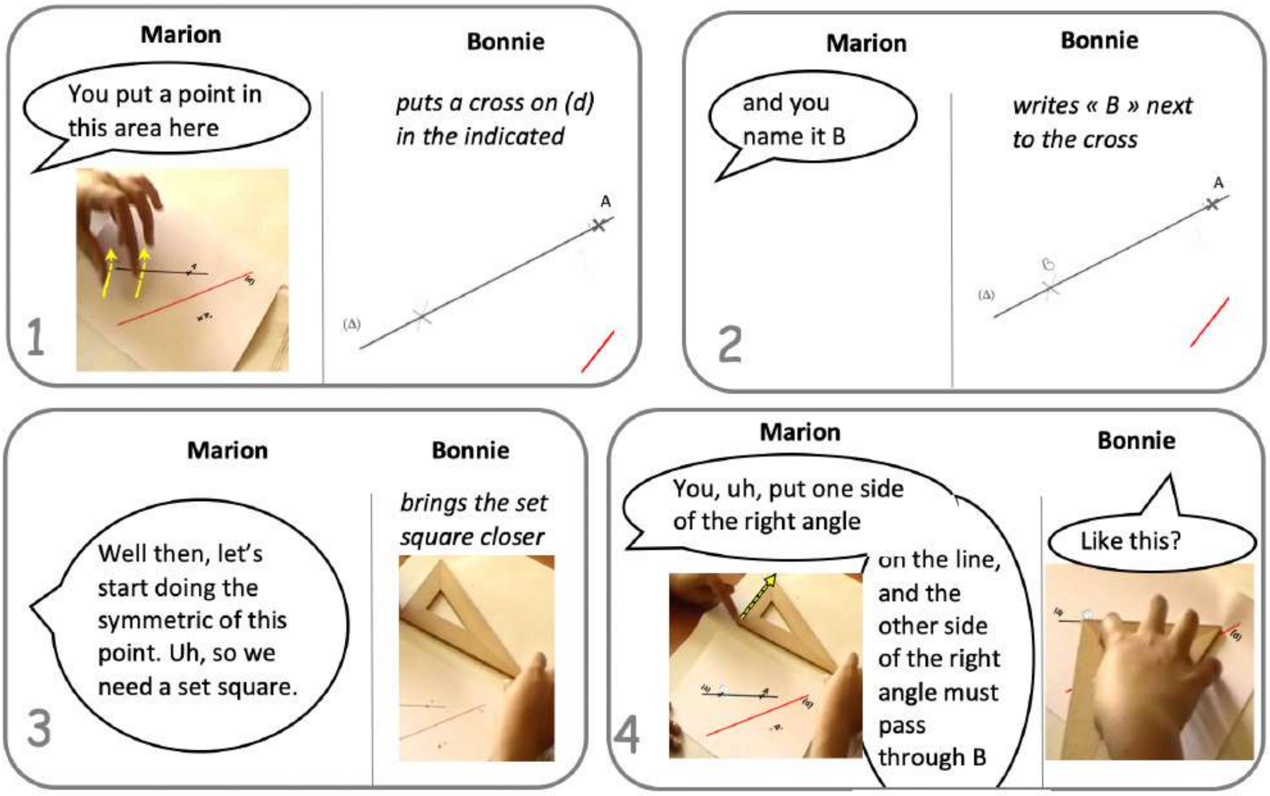


Figure 5: Carrying out the construction

Marion gestures to the part of the line (Δ) on which she wants Bonnie to place a point (Vignette 1). As well as the necessary indication of the line to which the requested point must belong, she is probably making sure that the point chosen is a long way from point A to allow precise drawing. In formulating her technique, she had in fact specified that the second point should be "at the bottom of the line" (Figure 4, Tm).

Marion then names the introduced point "B" (Vignette 2), and announces her intention: to make the symmetrical point of point B (Vignette 3). This information can help Bonnie to anticipate the drawings, if necessary correcting the position of point B on the line (Δ) so that its symmetrical point does not leave the support sheet. She can also help her to give Marion feedback on her instructions ("do the least likely thing"). After asking for the set square, Marion specifies its position: "one side of the right angle on the line and the other side of the right angle must go through B" (Vignette 4). The instruction was incomplete: Marion hadn't specified which straight line she was talking about (the axis of symmetry (d)) or pointed to it with her finger. Bonnie then suggests a position that corresponds to the instruction given, by choosing the line (Δ) and asks Marion if this is appropriate. Marion can then add the missing detail if this was not the position she had in mind.

The presentation of this short extract from the interactions between Bonnie and Marion suggests a certain potential of the dyadic working system: the two students can get involved in solving an instrumented construction problem, applying their knowledge and developing it. For the dyspraxic student, the obstacle of handling instruments is removed, giving way to geometric work.

Following this initial experiment outside the classroom, we continued our research in collaboration with teachers by implementing the dyadic system in the context of the class, adapting it to its constraints, so that it produced the expected effects in terms of geometric learning for all the students. We observed that it was not easy for some of students in the role of instructor to use the technical language and that the feedback from some of students in the role of constructor was not always appropriate. This led us to imagine a dyadic work in the digital environment, using an avatar, so that the students can learn the technical language, the rules of the dyad and the correct use of the instruments. We give an overview in the next section.

GIVING INSTRUCTIONS TO AN AVATAR

In collaboration with Fabien Emprin, we used the Virtual Training Suite[15] software as a human interaction simulator in an exploratory study (Emprin & Petitfour, 2021). With this software, reality is rendered by the setting of a classroom in which the avatar is installed as if next to the student instructor (Figure 6, left), and by videos of the hands of a person carrying out the required actions (Figure 6, right).

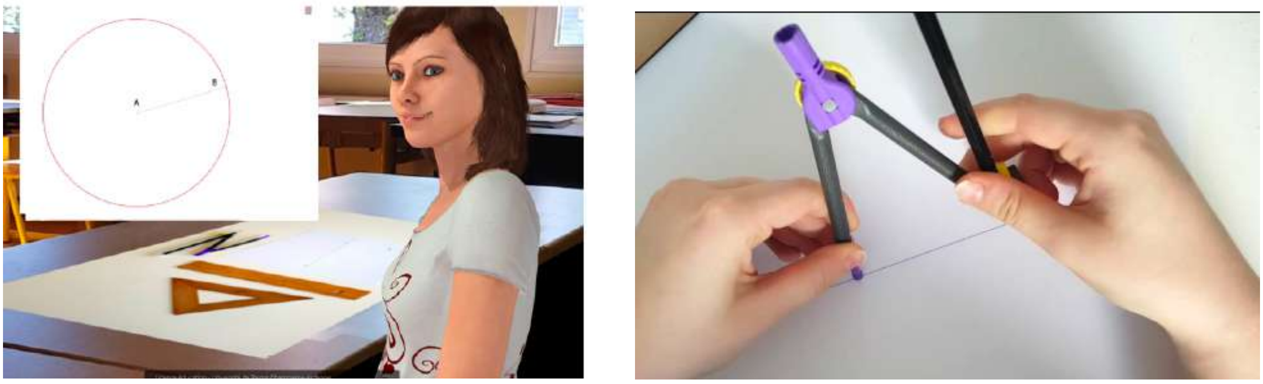


Figure 6: The avatar in the human interaction simulator

The simulator proposes five elementary instrumented actions (Figure 7) for which the instructor must get the avatar to produce the red line on the starting drawing. Each of the five drawings corresponds to an instrumented action, that is, an action that leads to the production of a line with an instrument (non-graduated ruler, set square or compass). A combination of several different types of actions allows to construct a complex figure.

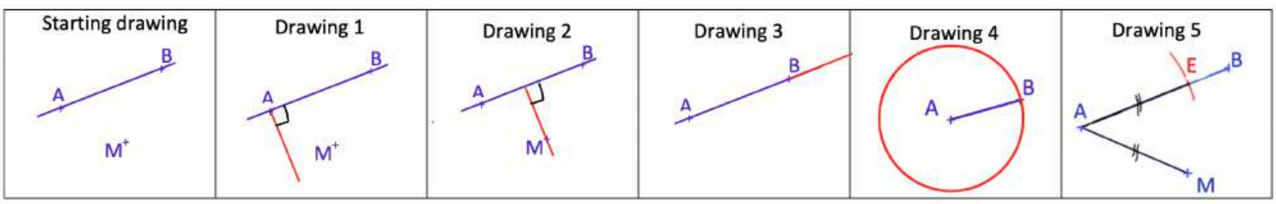
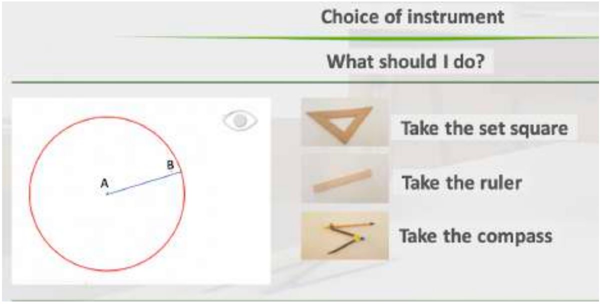


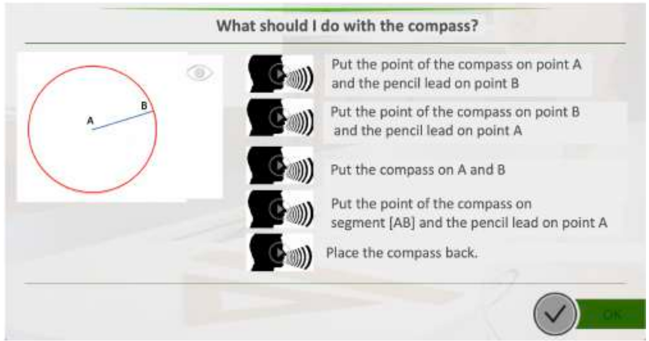
Figure 7: Elementary instrumented actions

[15] https://seriousfactory.com/virtual-training-suite/

Once a drawing selected, the student has to choose which instrument to use among ruler, set square and compass, in order to obtain the red line from the starting figure (Figure 8, step 1). He/she can then see the hand of the avatar taking the requested instrument in video. Four or five positioning formulation of the chosen instrument in relation to the geometric objects present are then proposed (Figure 8, step 2). One is valid, and the others are incomplete or not precise enough.



Step 1



Step 2

Figure 8: Example of choices proposed by the simulator

The student selects the instruction that seems to be appropriate, and then watches a video of the avatar's hands performing the requested positioning, while observing the "least likely" rule. The student can go back on his/her decision or confirm the avatar to carry out the drawing. In the latter case (step 3), he/she sees a video of the avatar's hands making a drawing from the position of the chosen instrument. The avatar compares the drawing produced with the expected drawing. If it is not correct, the student makes another proposal, if it is correct, the avatar formulates the drawing produced in geometric language (for example for the drawing 4 of Figure 7, the avatar says "Well done, you've correctly drawn the circle with center A and radius AB").

Computer simulation removes certain aspects of human interaction, which can have positive effects. Indeed, the fact that the computer shows no impatience or judgement can help to instill confidence in the student, particularly if he/she has a disability. He/she can think at his/her own pace, and benefit from interactions with the avatar that are less emotionally charged than they might be with a peer. Moreover, because the computer does what it's told without over-interpreting, the student can better understand the need to be explicit in his/her instructions. There are a few drawbacks to using the simulator, however. We had to abandon the development of the gestural dimension for the instructor to accompany the language: gestures cannot be considered by the software used, only language instructions can. What's more, the student's freedom of expression is restricted, since he/she has to select from a finite list.

Our first experiment was carried out as follows. The teacher began with focusing the whole class on the functions of the set square and the compass, then to the technical and geometric terms to be used, associating them with gestural exploration (see Figure 1a). The students then each worked freely with the software (student–instructor and avatar–constructor dyad), while the teacher observed their work. At the end, the teacher consulted a summary of the activities for each student memorized by the software: the number of (in)correct choices of formulation and instrument, the number of times the student (not) correctly anticipated that the positioning of the instrument was (in)valid; the number of times the student correctly identified that he/she had reached (or not) the goal. Next, a discussion led by the teacher highlighted the rules of the dyad work system (role of the avatar as constructor and language to be used by the instructor). The students then practiced the five elementary instrumented actions (Figure 7) in the pencil-and-paper environment, alternating between the roles of instructor and constructor. They then worked on other geometric construction, taking the avatar's responses, and some actions, as a model for their own behavior. An exploratory study of the contribution of the human interaction simulator thus designed to teach geometry to dyspraxic students led us to establish a proof of concept (Emprin & Petitfour, 2021, 2023). Our initial experimental results indicate that alternating student-avatar and student-student dyads is of interest for teaching geometry to dyspraxic students, but also for any student. These results still need to be confirmed by further experiments.

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LANGUAGE AND MATHEMATICS: A CASE OF GEOMETRIC SHAPE IDENTIFICATION

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This study was motivated by earlier work exploring the language and mathematics hypothesis. Extending the earlier studies that focused on the relation between spoken number words and speakers' understanding of 2-digit numbers, the current study investigated potential influences of geometric shape names on American and Japanese preschoolers' shape understanding. Japanese geometric shape names include the number of angles in the shapes – a defining characteristic of a shape. We wondered, given the transparent shape names, if Japanese preschoolers are able to more quickly recognize complex geometric shapes than their US counterparts. We briefly taught the participants each shape name and its corresponding number of angles. This should benefit Japanese preschoolers who speak transparent shape names. As expected, we found that the country was the strongest predictor of children's shape knowledge. What is more interesting is that Japanese children appeared able to apply knowledge of simple angles as a defining characteristic of shapes to complex shapes. The implications of the current findings are discussed.

BACKGROUND

The relation between mathematics language and mathematics achievement has been of interest to mathematic education researchers around the world. One area that has received much attention was an investigation into the influence of spoken number words on children's grasp of the base-10 numeration system. The initial work that sparked much interest was a comparison of Japanese and American children's understandings of two-digit numbers (Miura, 1987). In short, Miura found that Japanese-speaking first graders thought of two-digit numbers as consisting of tens and ones whereas English-speaking counterparts did so as collections of ones.

These findings were explained in terms of how Arabic numbers are spoken in English and Japanese. The Japanese spoken number words, beyond ten ("ju"), follow the rule of the traditional base-10 numeration system (Table 1). That is, a number word for any given two-digit number can be generated from a set of base-10 rules and a base sequence of spoken names. For example, 11, 12, and 13 are spoken as "ju-ichi" (i.e., ten-one), "ju-ni" (i.e., ten-two), and "ju-san" (i.e., ten-three). Compare these with how numbers are spoken in English: 11 (eleven) and 12 (twelve) are new words to young English speakers. The number, 13 (thirteen), is complex: instead of ten-three, the sound that hints 3 is pronounced first, followed by the base word of ten.

Miura's (1987) initial findings were later confirmed in a series of studies by Miura and her colleagues (e.g., Miura, Okamoto, Kim, Steere, Fayol, 1993) as well as those testing the language-mathematics hypothesis in other languages (e.g., Dowker, Bala, & Lloyd, 2008; Helmreich, Zuber, Pixner, Kaufman, Nuerk, & Moeller 2011). The consensus appears to be one that language does influence its speakers' mathematical thinking – at least in the area of two-digit numbers.

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 253-260). ICMI.

Language	Spoken Number Words
English 1-10	one two three four five six even eight nine ten
Japanese 1-10	ichi ni san shi go roku shichi hachi ku ju
English 11-19	eleven twelve thirteen fourteen fifteen sixteen seventeen eighteen nineteen
Japanese 11-19	ju-ichi ju-ni ju-san ju-shi ju-go ju-roku ju-shichi ju-hachi ju-ku

Table 1: Comparison of English and Japanese spoken words

The current study is an attempt to extend this language-mathematics hypothesis by examining children's early conception of geometric shape. Do shape names influence children's recognition of geometric shapes? This is the overarching question explored in this study. We want to emphasize that this is our first attempt at this question and to the best of our knowledge, no other studies exist at the time of writing this proposal.

Japanese shape names directly correspond to the exact number of angles each shape contains, with the exception of circle and rectangle. This naming system applies to all other two-dimensional (2D) shapes. Thus when one becomes aware of this feature, it is easy to name any 2D shape by counting the number of angles. As the literature shows (see the next section), young children are developing shape names and knowledge during preschool years. Beyond circles and squares, however, their understanding has yet to develop prior to entering kindergarten.

The question of interest is: would Japanese children speaking shape names that contain the number of angles – even if they are not familiar with complex shapes beyond rectangle – benefit from a brief instruction on the role of angles in defining shapes? Of course, angles are not the only defining feature of geometric shape. However, compared to English speaking children who need to memorize complex shape names that do not directly communicate shapes' defining characteristics, it seems likely that Japanese children would be the beneficiary of such language characteristics. Thus the specific question asked in this study is: With a brief exposure to the shape defining feature of angles, would Japanese children be able to apply this knowledge of simple shapes to complex shapes more so than American children would? In answering this question, we included age, gender, counting skills, and spatial thinking to determine the factor(s) that would most likely to explain the outcome.

Language	Shape names		
English	triangle	square rectangle	rectangle
Japanese	three-angle shape	canonical four-angle shape	long-side shape
English	pentagon	hexagon	octagon
Japanese	five-angle shape	six-angle shape	eight-angle shape

Table 2: Geometric shape names by languages used in this study

EARLY KNOWLEDGE OF GEOMETRIC SHAPES

Knowledge of geometric shapes takes root early in life. From infancy, children are introduced to basic geometric concepts through manipulative toys, books, and even forms of educational and interactive media (Hannibal, 1999; Resnick et al., 2016). Both parents and preschool educators also play a role in terms of exposing children to types of shapes or comparing shapes in different categories, though the extent or the language that they may use in talking about shapes is not clear (Verdine et al., 2016) and perhaps not enough (Rudd et al., 2008). Environmental input aside, other evidence suggests that shape discrimination might even be innate to some extent considering newborns' capability to discriminate between classes of shapes, and forms of shapes within the same class (Quinn et al., 2010). Collectively, these factors might help explain why young children are equipped with some degree of geometric knowledge by the time they enter preschool, including the ability to label (Heibeck & Markman, 1987), to distinguish, and to recognize simple properties of familiar shapes (Clements et al., 1999). However, they tend to vary widely with respect to their understanding of shape concepts upon entering kindergarten well into elementary school (Satlow & Newcombe, 1998). This variation may in some way account for the underperformance of US students' knowledge of basic geometric concepts compared to students from other nations, such as China (e.g., Starkey et al., 1999; Zhou et al., 2005) and Japan (e.g., Gonzales et al., 2008; Stigler et al. 1990).

The developmental progression of shape knowledge neither reveal a consistent pattern relative to the identification of basic shapes nor the ability to distinguish between canonical versus noncanonical representations. According to Keil (1989), developing knowledge of an individual concept is a function of its complexity, or more specifically, its defining properties or characteristics. Accordingly, when concepts differ along one dimension, they tend to show the same developmental progression. In contrast, when concepts differ along multiple dimensions, development is more likely varied. Geometric shapes are good exemplars of concepts that can be considered simple or complex in terms of their defining properties, and therefore, the developing knowledge of such concepts should follow suit.

With the exception of circles, defining properties of shapes include closure, straight sides, and a particular number of sides and angles. Other non-defining properties can vary adding to the perceptual complexity of a shape. For example, the defining properties of a rectangle refer to the number of lines and parallelism; orientation and length of sides are non-defining properties that differ across representations of rectangles. In a similar vein, a triangle's orientation as well as the degree of its angles may also differ, as is the case with isosceles and scalene triangles. Consistent with Keil's (1989) line of reasoning, young children tend to demonstrate knowledge of circles, and even squares with relative success, as early as age 2 (Zambrzycka et al., 2017).This suggests that from an early age, children are quite capable of identifying simpler shapes based on defining properties that translate to "stronger and fewer prototypes" (Clements et al. 1999) with a lower number of non-defining attributes (Aslan & Arnas, 2007).

Findings pertaining to rectangles and triangles suggest otherwise (e.g., Hannibal, 1999; Satlow & Newcombe, 1998; Zambrzycka et al., 2017), particularly when non-defining properties (e.g., orientation or angularity) seem to interfere with an accurate identification of a shape. Children are more apt to experience failure when they do not pay attention to the defining properties of a shape that determine shape classification. Classification tends to be based on select visual information, or falling back on more familiar, spontaneous concepts (Hannibal, 1999). Perhaps perfect representations of shapes often depicted in books and puzzles may skew their interpretation of noncanonical representations (e.g., a scalene triangle is not a triangle because it is too "crooked"). With age, visual information continues to influence classification decisions but to a lesser degree for some children compared to others. This developmental shift partly explains why older children's reliance on defining properties versus visual information enables more correct classifications (Aslan & Arnas, 2007). To reiterate, the current study examined potential influences of shape names in English vs. Japanese on preschoolers' developing understanding of geometric shapes.

BACKGROUND

Participants

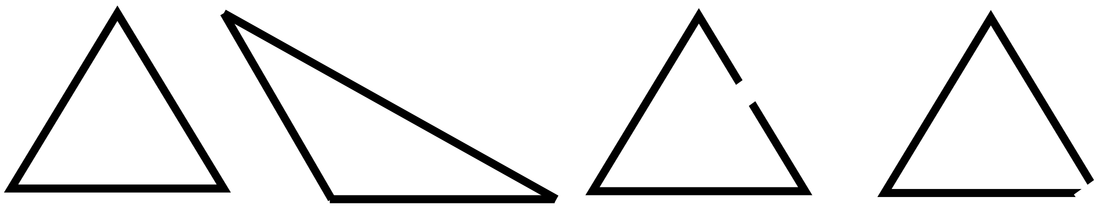
The participants were 46 Japanese children (24 girls, 22 boys, M_{age} = 59.46months, *SD* = 10.48 months, age range: 42-75 months) and 38 US children (15 girls, 23 boys, M_{age} = 50.18 months, *SD* = 12.37 months, age range: 27-74 months). The Japanese children were recruited from one nursery school located in Yokohama, Kanagawa of Japan (the prefecture adjacent to Tokyo) and the US children came from three different preschools located in a metropolitan city in California, United States.

Tasks and procedures

Children were seen individually in each nation by native speakers. They were first asked to count to 10 cubes. The highest number reached without any assistance was recorded. They were then given a brief lesson on geometric shapes. In the first part of the lesson, children saw three line drawings of an angle (45-degree acute angle, 90-degree right angle, and 135-degree obtuse angle) and were told, "This is an angle," as the interviewer pointed to the angle. In the second part of the lesson, they saw line drawings of triangle, rectangle, pentagon, hexagon, heptagon, and octagon and were told, for example, "This is a triangle. It has three angles." Following the brief lesson, children completed two tasks: spatial puzzle and shape identification tasks. For the spatial puzzle task, children solved three puzzles consisting of four, five, and six pieces of familiar objects (i.e., a cat, a dog, and a house). For example, they saw an outline of a cat and given pieces to fill in it (Figure 1). Children received 1 point for each correct solution for a maximum total of 3.

Figure 1: Outline of a cat used in the puzzle task

For the Shape identification task, children saw line drawings of six shapes (i.e., triangle, rectangle, pentagon, hexagon, heptagon, and octagon) one shape type at a time. For each shape type (e.g., triangle), they saw four different drawings of triangles (two valid triangles and one non-enclosed and one missing-angel shapes that look like triangles - see Figure 2). Children were asked if each of the four shapes was a triangle and why or why not they thought so. Children received 1 point for responding "yes" to each of the valid forms and "no" to each of the non-valid forms for a maximum possible of 24 points.



canonical triangles non-enclosed missing-angle

Figure 2: Shape identification task example: triangle

RESULTS

The primary interest of this study was to see if children speaking English or Japanese respond differently to a brief lesson that emphasized one defining feature of geometric shapes. Because Japanese shape names include the number of angles, a brief exposure to pay attention to the number angles might benefit Japanese children more so than their US counterparts who do not have language support.

Descriptive statistics are presented in Table 3. A quick observation reveals that the two groups' mean counting scores were relatively similar. However, there seemed to be differences between the groups in the spatial puzzle and shape identification performance.

We then carried out an analysis of variance. The dependent variable was the shape identification task and the primary independent variable was country (language groups). We also included as gender to see if any gender differences could be identified. Furthermore, we included age, counting, and spatial puzzle performance as covariates.

As expected, country was the strongest predictor of the shape identification tasks, F (1, 77) = 34.13, p = .000. In addition, age was a significant variable, F (1, 77) = 4.74, p = .033. No other variables, including gender, were significant.

	Japan (n = 46)				US (n = 38)			
Variable (Possible Range)	M	SD	Min	Max	M	SD	Min	Max
Counting Task (0-10)	9.09	2.84	0	10	8.97	2.06	0	10
Spatial Puzzle Task (0-3)	2.43	.83	0	3	1.56	1.01	0	3
Total Shape Score (0-24)	18.30	4.12	4	24	9.34	6.30	1	24

Table 3: Descriptive statistics of variables by country

Source	SS	df	MS	F	$Sig.$
Corrected model	1854.90	6	309.15	11.62	.000
Intercept	114.95	1	114.95	4.32	.041
Country	908.14	1	908.14	34.13	.000
Gender	10.50	1	10.50	.395	.532
Age (mon)	126.12	1	126.12	4.73	.033

Counting10	.09	1	.09	.032	.953
Puzzle	2.50	1	2.50	.94	.760
Gender*Country	11.35	1	11.35	.43	.519
Error	2048.85	77	26.61		
Total	20961.00	84			
Corrected total	3903.75	83			

Table 4: Analysis of variance for shape knowledge by country and gender with age, counting to 10, and spatial puzzle as covariate

If the language-mathematics hypothesis is at work, we should see that Japanese children are able to accurately identify two canonical shapes as valid and one missing-angle shape to be non-valid. We carried out a MANOVA to see where children show strengths and weaknesses. For this analysis, we summed across all six canonical shapes, all six noncanonical shapes, all six missing-angle shapes, and all six non-enclosed shapes – all which were the dependent variables. The independent variable was country with age as a covariate (we did not include other variable as no other variables were significant in the previous analysis).

The results indicated that country was significant F (4, 78) = 14.18, p = .000: so was age F (4, 78) = 3.83, p = .007. Univariate analyses for country revealed all but noncanonical shapes turned out to be significant: canonical, F (1, 81) = 11.84, p = .001; noncanonical, F (1, 81) = .74, p = .391; missing angle, F (1, 81) = 58.20, p = .000; and non-enclosed, F (1, 81) = 47.24, p = .000. Similar trends were found for the age variable.

	Japan (n = 46)		US (n = 38)	
	M	*SD*	*M*	*SD*
Canonical	5.54	0.94	4.11	1.78
Noncanonical	2.30	1.98	2.00	1.68
Missing angle	5.32	1.49	1.66	2.21
Non-enclosed	5.13	1.73	1.58	1.68

Note: Possible range for scores is 0-6.

Table 5: Comparison between Japan and US

DISCUSSION AND IMPLICATIONS

This study was motivated by earlier studies examining the language and mathematics hypothesis in the area of two-digit numbers. Earlier studies found transparency of Japanese number words that correspond to the rules of base-10 influenced first-graders' developing concepts of 2-digit numbers. The current study extended this language and mathematics hypothesis by examining a different area of mathematics, namely geometric shape understanding. Japanese geometric shape names incorporate defining characteristics of shapes – angles. Thus it would make sense to predict that Japanese preschoolers, once taught to realize how shape names are constructed, should be able to apply this knowledge even after a brief instruction on it in comparison to English speaking counterparts.

As expected, we found that the country (language group) was the strongest predictor of children's shape identification. Age was also a significant variable. The current results are in-line with the language and mathematics hypothesis. The only puzzling finding is the performance of the Japanese children on noncanonical shapes. If the language-mathematics hypothesis is at work, they should be able to identify noncanonical shapes as valid (though they were correct more than half the times).

One possibility is the effect of age. Our analyses revealed that age was a significant covariate. Our participants, however, varied in age. It would be ideal to use methods such as stratified sampling to ensure that data can be analyzed by age groups.

Given this and other limitations, what do the current findings suggest? Theoretically speaking, the language and mathematics hypothesis appears to extend to young children's shape knowledge development. This contributes to the debate that specific aspects of language influence the corresponding cognitive mind.

In terms of applications, preschoolers should be encouraged to explore shapes that go beyond triangles and squares, and discuss defining features of geometric shapes. In English-speaking nations, it might be useful to introduce a triangle as a three-angle shape, etc. at early stages of learning. Once this feature is understood (along with other features), English terms might be introduced. As we delve further into our data, we hope to have clearer directions for future research and teaching.

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GEOMETRY EDUCATION IN IRANIAN SCHOOL MATHEMATICS: CURRENT SITUATION AND FUTURE CHALLENGES

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The main purpose of this paper is to discuss the role of geometry education in Iranian national curriculum document, Iranian school mathematics curriculum standards and Iranian Teachers education programs. This paper provides a good sense about what happens in geometry education within the Iranian educational system with a population of 80 million which has more than 14 million students at different grades at school. In the current Iranian mathematics curriculum, geometry has a prominent presence and geometry lessons are taught in almost all grades. Students learned about geometrical shapes and used geometrical concepts to foster reasoning and proof capability. But in the near future, geometry education would change somehow. New ideas like computational thinking and programming were introduced to school mathematics curricula in many countries around the world and according to a previous ICMI study (24) curriculum changes started. So, one of the important challenges for the future of geometry education will be how we can integrate geometrical concepts with computational thinking to raise the next generation.

INTRODUCTION

The issues related to geometry in the school mathematics curriculum have been of interest for a long time, so that two topic study groups have been dedicated to geometry in the international congresses of mathematics education. According to Howson and Wilson (1986), geometry has been of particular interest to mathematicians and educators. Reviewing the history of geometry education in school mathematics curriculum shows that this field has witnessed many developments; For example, Sharigin and Protasov (2004) point out that some people tend to remove geometry from the school mathematics curriculum, and some also tend to reduce the amount of geometry in the school mathematics curriculum. In fact, they think that we need other knowledge to educate active citizens in the 21st century. This is while there are various reasons for including geometry in the school mathematics curriculum in the research literature in the field of mathematics education; For example, Sharigin and Protasov (2004) advocate teaching geometry in 21st century schools by mentioning three reasons. First, Geometry is a phenomenon of human culture. Second, Geometry prepares students' minds for higher education and develops the sense of aesthetics in students. Third, Geometry shows the history of human thought well (pp. 167-176). In this line, Shahshahani (1375) also lists three reasons for teaching geometry at the secondary school level. The first reason: geometry is the language of all sciences. Because geometry is historically the science of space and shapes, and all natural phenomena occur in space; The second reason: Geometry is the first theoretical science, and in it, a series of results are derived from other results based on reasoning and thinking, and its history

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 261-268). ICMI.

goes back to before Christ. The third reason: geometry is a basis for strengthening imagination and creativity. According to Zanganeh and Gooya (2001), one of the main characteristics of geometry is creating the ability to visualize, and perhaps this is the same characteristic of geometry that warrants its existence in the school mathematics curriculum, for the same reason. Almost no other course can replace geometry.

Rafiepour and Gooya (2014) believe that geometry should be included in the school mathematics curriculum because geometry is one of the subjects that has a high potential to be taught in the school mathematics curriculum and if it is taught correctly, it can develop creativity and thinking of students. In this line, Jones (2002) believes that geometry should be included in the mathematics curriculum because the study of geometry helps students develop the skills of visualization, critical thinking, intuition, problem-solving, conjecture, deductive reasoning, logical reasoning, and proof. Furthermore, although geometric representations can be used to help students understand in other areas of mathematics, like subtraction and multiplication in arithmetic, relationships between function graphs and graphical representations of data are used in statistics. spatial reasoning in other subjects; Like science, geography, art, design, and technology, it is important, and having the knowledge of geometry seems necessary because of its use in solving mathematical problems and problems of daily life. The National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the importance of geometry in school mathematics and acknowledges that teaching geometry is an opportunity to develop students' reasoning and logical skills.

If we look at geometric proofs as a means of discovery, geometric proofs can be a means to strengthen students' imagination and creativity. An example of this discovery, which is the result of combining geometry education with digital technologies, can be found in Rahmani and Rafipour's paper (2017). Regarding the importance of teaching and learning geometry in the school mathematics curriculum, it should be mentioned that the main focus of the ninth ICMI study was related to teaching geometry for the 21st century. Regarding the importance and necessity of geometry in Iran's school mathematics curriculum, it should be mentioned that "geometry and measurement" is included as one of the domains of mathematics learning in the Iranian national curriculum document. Also, traces of geometry can be seen in relation to the real world, reasoning, and modeling (from different parts of the national curriculum document). The main focus of this paper is to show the role of geometry in the Iranian school mathematics curriculum. In this regard, the Iranian standards for geometry will be present and the teacher education program at pre-service will be reviewed.

IRANIAN CURRICULUM STANDARDS IN GEOMETRY EDUCATION

So far, several reasons have been mentioned for the importance and necessity of teaching geometry in the school mathematics curriculum. Based on theories of teaching and learning geometry; Including Van Hiele (1999) theory of geometric thinking, it is possible to teach geometry from preschool to the end of high school. Following are the general standards of teaching and learning geometry at the Iranian school mathematics curriculum.

Teaching geometry with consideration of the Van Hiele Model of Geometric Thinking: Van Hiele (1999) study showed that students' learning of geometry has levels. Paying attention to these levels of geometric thinking in the process of teaching and learning geometry could increase the quality of students' learning. Van Hiele theory began in the 1950s with a family team (wife and husband) in the

Netherlands, Dina and Pierre van Hiele. Research on the levels of Geometric Thinking has continued to develop over the years and now there are three levels: Visual, Descriptive, and Informal Deduction. There are several studies about the role of Van Hiele theory on geometry education in the Iranian context that show the capability of Van Hiele theory in enhancing the process of teaching and learning geometry. For example, Liaghatdar, Soleymani, and Sadrarhami (2012) investigated the effect of using Van Hiele theory in teaching high school geometry on students' achievement and published it in a refereed journal in Persian language (the official language in Iran).

- Modeling and application in the context of geometry: students should be able to use geometric concepts and relationships in solving everyday problems in real-world situations, such as measuring the length, area, and volume of shapes and volumes in the real world and be able to solve historical modeling problems (especially problems related to the history of Iranian-Islamic mathematics) - which are related to geometry. Also, have the ability to understand geometric drawings in the context of real applications in tiling and architecture of historical and national places. The history of Iranian-Islamic mathematics is a treasure of practical problems and creative modeling, especially in the branch of geometry and trigonometry, which can be used in modeling problems. The benefit of such use is that many problems in modeling the history of mathematics have benefited from relatively simple and at the same time creative mathematics, and in addition, it creates a sense of national self-confidence in students. See Figures 1 and 2 for examples of Iranian-Islamic cultural heritage.

Figure 1: Yazd Mosque, Yazd, Iran

Figure 2: Ali-Agha Mosque, Isfahan, Iran

- Connection and integration of geometry with other branches of mathematics: students should be able to understand the connection of geometric shapes and relationships (circle, right triangle) with trigonometry. They also would be able to communicate between geometry and arithmetic; For example, they can understand length, perimeter, area, and volume, coordinate geometry, and geometric shapes in two- and three-dimensional coordinate systems. In the field of combining geometry with algebra, it is necessary for students to be able to use equations and linear algebra in solving geometric problems and solving equations of conic sections.
- Transformations in Geometry: students should be able to do all kinds of transformations; Identify and draw including transmission, rotation, reflection, and homogeneity. To understand the combination and relationship between types of transformations and to be able to express coexistence, similarity, and symmetry with the help of geometric transformations. Also, understand the connection and analogy between transformational geometry and Euclidean geometry (the connection between the proofs of the similarity of triangles and congruence, the connection and analogy between the collocation of triangles and translation, rotation, and reflection) and be able to, with the help of the principles of the subject of geometry transformations, prove related theorems. The geometry of transformations is one of the content standards of geometry in the document of the National Council of Mathematics Teachers (NCTM, 2000). Hollebrands (2003) states three reasons for the serious presence of transformative geometry in the high school curriculum: transformative geometry provides opportunities to reflect on mathematical concepts; It provides functions such as range and range on the page. Also,

transformational geometry makes students realize that mathematics is a connected scientific field. Finally, transformational geometry provides opportunities for reasoning at higher thinking levels with the help of different representations.

- Geometrical manipulation: students should be able to make replicas of geometric shapes (cubes, pyramids, cylinders, cones, etc.). They can also use dynamic geometry software, such as GeoGebra (to make geometrical components and shapes and relationships tangible, induction, and conjecture), and use their facilities in geometric drawings and finding the geometric location of points. Drawing, physical construction of geometric objects and the use of dynamic environments for geometry, especially the use of dynamic geometry software, in addition to helping to improve the understanding of shapes at elementary levels, can help to understand the concepts of coexistence, symmetry and similarity, intuition. The skill of conjecture and hypothesizing will help and be a bridge between inductive reasoning and deductive reasoning. This importance is taken into consideration in the standard of geometric manipulation.
- The standard related to shapes (two-dimensional and three-dimensional shapes): The standard related to shapes (two-dimensional and three-dimensional) and the analysis of their relationships is one of the four general standards stated for geometry in the National Council of Mathematics Teachers (NCTM, 2000). In the common core state standards, shapes, especially triangles and circles, are part of the set of standards; They are like coexistence and similarity of forms (2010). "Connection and integration of geometry with other branches of mathematics" standard; It includes the subject of trigonometry and analytical and coordinate geometry. In the state standards of the common core, part of the content of this standard, i.e. coordinate and analytical geometry, is given under the title "Description of geometric relationships using equations" and trigonometry in relation to right triangles is discussed separately. In two-dimensional shapes, triangles, parallelograms, polygons, and circles will be discussed. In the two-dimensional section, a three-dimensional coordinate system, line, and plane in three-dimensional space, Rectangle cube, cube, cone, pyramid, cylinder, sphere, and spatial visualization will be introduced.

THE ROLE OF GEOMETRY IN IRANIAN TEACHERS EDUCATION PROGRAM

In the latest version of the mathematics teacher education program, there are 4 different courses that are related to geometry. The titles of these courses are Basics of geometry 1, Basics of geometry 2, Introduction to proof, and Mathematics and art. Table 1 shows these courses and their credit.

#	Name of Course	Credit
1	Basics of geometry 1	2
2	Basics of geometry 2	2
3	Introduction to proof	3
4	Mathematics and art	1

Table 1: Title and credit of geometry-related courses in the latest version of the mathematics teacher education program (in-service)

In total 8 credit points from 150 credit points were related to geometry at the in-service mathematics teacher education program. In the past in-service mathematics teacher education program, there were three compulsory courses related to geometry with the titles Foundation of mathematics, Basics of geometry, and Introduction to school geometry. There are two extra arbitrary (selective) courses related to geometry that students could choose with titles Differential geometry and General topology (see Table 2).

#	Name of Course	Credit	Compulsory or Selective
1	foundation of mathematics	3	Compulsory
2	Basics of geometry	3	Compulsory
3	An introduction to school geometry	2	Compulsory
4	differential geometry	3	Selective
5	General topology	3	Selective

Table 2: Title and credit of geometry-related courses in oldest version of the mathematics teacher education program (In-service)

In these courses, prospective teachers of mathematics gain a vision of geometry and geometrical proofs. So, it could be reasonable to expect to implement geometry standards at the highest level that could be imagined.

FINAL REMARK

In this paper, the place of geometry education in the Iranian national curriculum, Iranian school mathematics curriculum standards, and Iranian teacher education program are briefly introduced. Furthermore, there are some other concepts and ideas like computational thinking and programming which started to be added at the school mathematics curriculum (e.g. Kadijevich, Stephens, Rafiepour, 2023). This movement started in many countries; in Iran, it also started to add computational thinking to the school curriculum gradually (Rafiepour, & Farsani, 2021). So, one of the future challenges in this area would be the integration of geometry concepts with computational thinking ideas and in some cases probably geometry has to be replaced by some ideas from computational thinking.

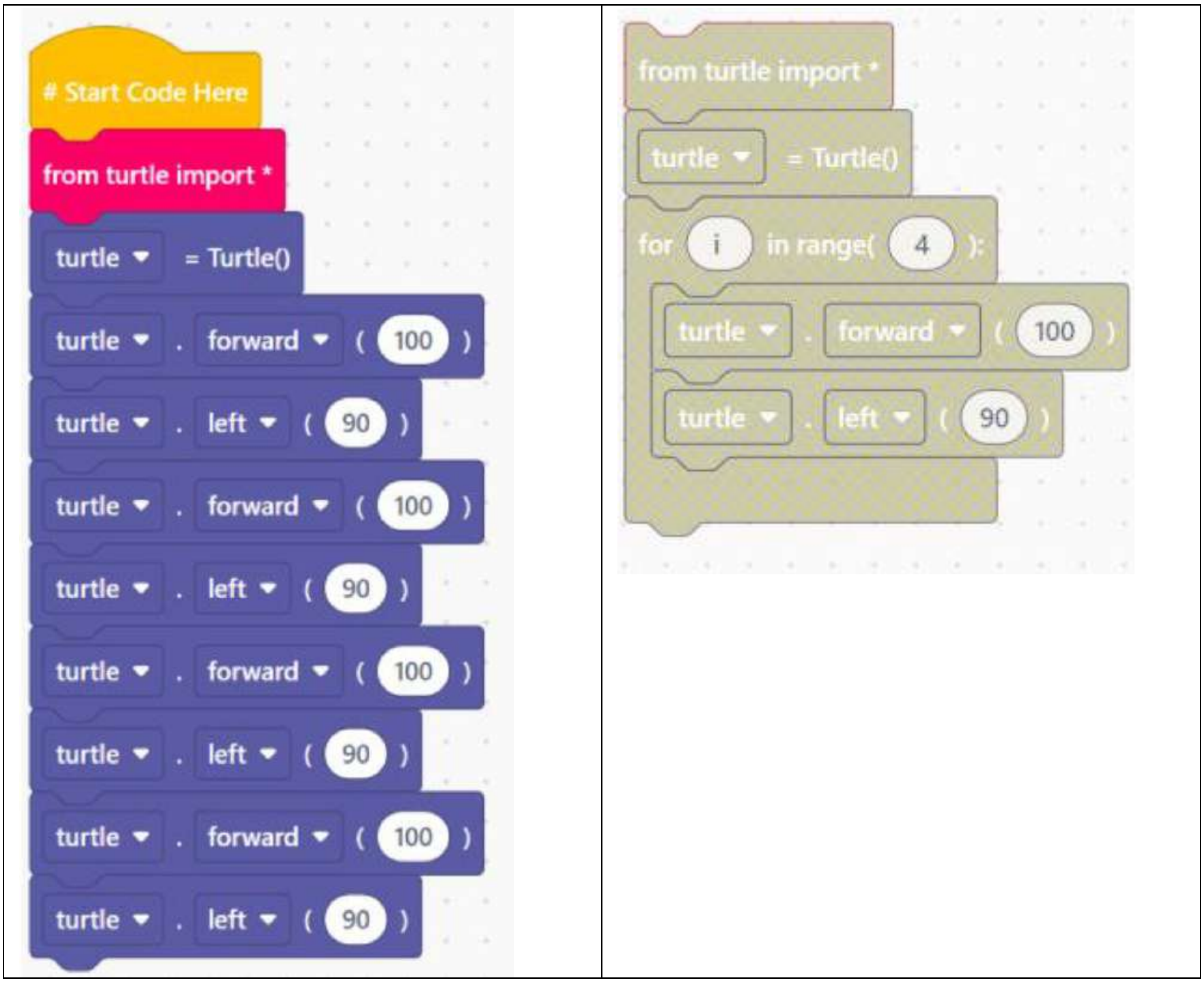

Figure 3: Two different blocks of codes for producing a square at Scratch

In one of the recent systematic reviews, four researchers from Hong Kong investigated the integration of computational thinking in K-12 mathematics education (Ye, Liang, Ng, and Chai, 2023). They selected 24 articles that were published after 2018 to provide illustrations of computational thinking-based mathematics instruction. They found that geometrized programming supported productive learning in CT and mathematics. But we have to become careful about this integration. In many cases, the writer of this paper is aware that mathematical ideas became a suitable context for discussing about computational thinking (programming) challenges. For example, in Figure 3, a programming challenge is introduced regarding drawing a square using Scratch. In the meanwhile, from the mathematics side of STEM, we want to give more weight to mathematical challenges in the context that new technological tool provides for us as mathematics education.

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DESIGNING MEANINGFUL TASKS TO PROMOTE ARGUMENTATION SKILLS IN DGE – A CONCEPT FOR A PROFESSIONAL DEVELOPMENT PROGRAM

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In this article we describe our approach to a professional development (PD) program for teachers that focusses on ways to enhance the argumentation and proving skills of students. While research in technology-based geometry education has shown the positive effects of theory-based tasks in interactive of dynamic geometry environments (DGE), teachers struggle to implement such tasks in their own teaching appropriately. As a consequence, the PD program is using a twofold approach combining tasks that can be readily used in the classroom and the theory that has been used to build these tasks, such that teachers can use their newly acquired knowledge immediately and transfer it to other subject areas as well.

INTRODUCTION

The introduction of interactive of dynamic geometry environments (DGEs) in the last decade of the last century has opened up a wealth of new tasks and activities that have the potential to improve mathematics instruction in the long term, and there has been lots of research demonstrating its effectiveness in small settings. Still, even teachers with a positive attitude to the use of technology are not always able to raise that potential, some are even are reluctant to use technology at all. (Thurm et al., 2017), so large-scale effects are missing despite the general availability of technology in schools. A key to remedy this situation can be professional development (PD), however, the effectiveness of PD is not guaranteed.

Starting from principles of good teaching in mathematics and theoretical results on proof and argumentation using DGE, we describe the design of a PD course that is being implemented within the framework of the German QuaMath program for developing quality for mathematics classrooms and mathematics professional development. The course focuses on improving argumentation skills of students using DGE and, while using a specific flagship example, is not tied to a particular topic. Through a scaffolding approach, teachers are enabled to design tasks that promote quality in mathematics teaching. We claim that our design is capable to bridge the gap between theory of DGE-based geometry education and the everyday practical jobs of teachers, in particular designing learning activities.

THEORETICAL BASIS FOR THE TEACHER’S PD PROGRAM

Principles for high-quality teaching

The ultimate goal of our PD program is to enable teachers to plan and execute mathematics lessons on a high standard. Since there is no universally accepted definition of good teaching, we first have to operationalize this goal. We follow Prediger et al. (2022), who identify five principles for high-quality teaching, which have been derived by combining normative, empirical, epistemological, and pragmatic perspectives on instructional quality. These principles—*conceptual focus, cognitive demand, student focus and adaptivity, longitudinal coherence,* and *enhanced communication*—are a common guideline for teaching used in the larger context of the German nationwide teacher PD initiative QuaMath, where they are considered in the context of typical situational demands for teachers (Prediger, 2019). PD in QuaMath should enable teachers to set goals and plan teaching progressions, to select and adapt activities and media, to diagnose students' learning states and processes, to support students in learning, and to moderate conversations.

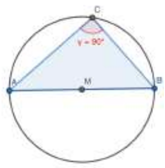
Argumentation and proof in geometry education

Argumentation and proof characterize mathematics as a discipline (Heintz, 2000) and also play an important role in mathematics education. In literature, the term *mathematical argumentation* is defined and differentiated from the terms *proving* and *reasoning* in various ways (see for example Brunner, 2014; Reid & Knipping, 2010). In this article, we align with an understanding of these terms explicated by Reiss & Ufer (2009). According to their terminology, mathematical argumentation generally aims at the formulation, investigation and validation of hypotheses and open questions. It is a multifaceted activity that encompasses both exploratory activities, such as investigating and discovering mathematical relationships and formulating corresponding conjectures, or testing the plausibility of given conjectures, as well as substantiating a given and assumed plausible conjecture. The latter aspect can be summarized as *reasoning* and in particular includes *proving* which is defined quite consistently in literature. Proving refers to the validation of a claim through its purely deductive derivation from known axiomatic definitions as well as from accepted derived statements. These specific characteristics distinguish proving from other forms of mathematical reasoning like, for instance, justifying one's own solution strategies for a mathematical task. In mathematical argumentation, non-deductive inferences such as induction or abduction are also possible (Brunner, 2014; Reiss & Ufer, 2009). Although the conjecture generation phase has an exploratory character and does not yet aim at formulating a deductive chain of arguments, there is a close cognitive link between the conjecturing and the proving phase, as Boero et al. (1996) also describe in connection with the concept of *cognitive unity*.

Geometry constitutes a part of the spiral curriculum from elementary to upper secondary school in proving (Bescherer & Hoffkamp, 2022; Reiss & Ufer, 2009). Due to its axiomatic foundation since Euclid, geometry is considered particularly suitable for learning argumentation, reasoning and proving. This is also because of the different registers of representation in geometry, that allow for various approaches and levels of argumentation for both conjecture generation and reasoning (Wittmann, 2018). The advent of interactive and dynamic geometry environments (DGE) in the 1980s/90s expanded these possibilities by allowing interaction with dynamic representations, and researchers have since been investigating the role that DGE can play in supporting argumentation processes in mathematics education (e.g., Healy, 2000; Jones et al., 2000; Laborde, 2005).

DGE, the dragging mode and the role of invariants

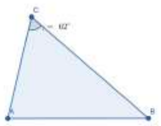
Working with DGE to solve geometric problems offers a special affordance, the drag mode, which allows to vary the position, size and shape of objects in DGE (seemingly) continously. The fact that dragging changes the drawing on the screen, but not the constructed figure (Laborde, 1993) makes the drag mode a powerful tool to check the correctness of geometric constructions, and to initiate mathematical argumentation processes. Figures in DGE are given by a system of relations, that are either directly constructed or implications from constructed relations according to the Euclidean theory (Kortenkamp, 1999). Those relations can be explored as motion invariant properties of the figure during dragging. Moving parts of the figure and identifying invariants form the core of generating conjectures about mathematical relations with DGE (Baccaglini-Frank, 2012; Baccaglini-Frank & Mariotti, 2010; Bescherer & Hoffkamp, 2022; Healy, 2000; Hölzl, 1996; Laborde, 2005).



The following task is a very typical example for a conjecture generating activity with DGE in a german textbook: *"Use a DGE to draw a circle with center M and diameter AB. Place a point C on the circle and connect it to A and B. Measure γ and drag C along the circle. What can you discover? Formulate a conjecture."*

The first invariant in this first example is the position of point *C* on the circle. During dragging, a second invariant can be observed: γ is a right angle. According to Healy (2000), both properties of the figure are *robust invariants* because they are invariant under any kind of dragging (Laborde, 2005). The position of *C* is called a *direct* robust invariant, because *C* is directly constructed to be located on the circle, whereas the right angle at point *C* is a consequence of the construction within the Euclidean theory (Thales' Theorem), an *indirect* robust invariant (Baccaglini-Frank, 2012).

The following second example demonstrates the counterpart to "robust invariants":



"Use a DGE to draw a triangle ABC. Measure the angle γ at point C. Drag Point C so that γ is a right angle. Use the trace tool to describe the positions of C. What can you discover? Formulate a conjecture."

The right angle at point *C* is no permanent property of the figure. It must be induced and maintained by dragging *C* in a special way and is called a *soft invariant*. During this activity, students can observe that the *path* of the dragged point *C* has the shape of a circle with diameter *AB*. The *path* is an ideal, mental construct, which would be traced by C if the soft invariant could be maintained continuously (Baccaglini-Frank & Mariotti, 2009, p. 235). The property "*C* lies on the circle" is a second soft invariant (Healy, 2000). Since in this task, the right angle γ is directly and intentionally established by dragging *C*, it is a *direct* soft invariant, whereas the circle shape of the path of *C* is a consequence of maintaining the size of γ (according to the reversal of Thale's Theorem), an *indirect* soft invariant. Baccaglini-Frank and her colleagues call the kind of dragging required in this task with the aim of maintaining the right-angle property *maintaining dragging* (Baccaglini-Frank, 2011, 2012; Baccaglini-Frank & Mariotti, 2010).

Recognizing invariants in DGE and formulating conjectures about the underlying mathematical relations, moving from the phenomenological experiental world of DGE to the formal axiomatic and static world of Euclidean geometry, is challenging for students (Baccaglini-Frank, 2010). In the case of the examples above, the observation of the invariants can lead to the following conjectures: "If *C* lies on the circle with diameter *AB*, the angle at point *C* of the triangle *ABC* is a right angle" (Ex. 1) and "If the angle at point *C* of the triangle *ABC* is a right angle, *C* lies on a circle with diameter AB" (Ex. 2). The *premises* of these conjectures correspond to the observed *direct* invariants and the *conclusions* to the *indirect* invariants. Concerning maintaining dragging, Baccaglini-Frank and her colleagues suggest that this interpretation of motion dependence between the two invariants as logical dependence between two mathematical statements acts as a key element for conjecture generation processes with DGE and provides a bridge for the 'experimental-theoretical gap' (Lopez-Real & Leung, 2006) between the two mentioned worlds. Example 2 further shows how using the trace tool can help to find invariants (Baccaglini-Frank, 2012; Baccaglini-Frank & Mariotti, 2010).

Moving from Dragging Exploration to Documentation

The discovery and formulation of mathematical relationships can also be supported by further features of DGE. Let us revisit Ex. 2: instead of continuously drawing the trace of point *C* during maintaining dragging, we introduce a *stamping tool* such that the location of *C* can be marked for only a finite number of sample triangles. Compared to maintaining dragging with an activated trace, there is an advantage in reduced demands on hand-eye coordination. Instead of permanently maintaining the right angle at *C* during dragging, one can first search for a triangle that satisfies this desired condition. Stamping then transitions from dynamically varying C on the screen to fixing a specific position of C on the drawing surface, moving from an experimental playground (*Spielraum)* to a document (*Dokument)* that conserves the experimental findings (Kortenkamp & Wollring, 2017). We will use the additional benefits of stamping for conjecture generation in DGE, particularly regarding mathematical equivalence statements, in our PD.

Opportunities and Challenges of Using DGE

In the previous section, we already took a *theoretical perspective* on the opportunities offered by DGE through the construction and dynamic variation of geometric figures for making explicit the relations between the constructed objects, for free experimentation and the discovery of mathematical relationships, as well as for the formulation or refutation of conjectures (Bescherer & Hoffkamp, 2022). From an *empirical perspective* as well, the potential of DGE for mathematical argumentation is supported by numerous studies. For example, it has been shown that DGE have a positive impact on the sub-competency of conjecture generation in open problems (see, e.g. Baccaglini-Frank, 2010, 2011; Baccaglini-Frank & Mariotti, 2010). Building on (Arzarello et al., 2002), who analyzed and classified various dragging modalities in students' solutions to open problems, Baccaglini-Frank et al. put a focus on maintaining dragging and describe in detail how DGE can enhance cognitive processes during conjecture generation within the context of maintaining dragging. There is even a hint that maintaining dragging can lead to a *cognitive unity* (Boero et al., 1996) and thus can promote the construction of deductive proofs of the generated conjectures.

However, the use of DGE for argumentation also comes with pedagogical challenges. The transition from the *process*-oriented, empirical exploration in DGE and conjecture generation to the logical-deductive proof of these conjectures as a *product* of the exploration process is described in the literature as one of the major difficulties associated with argumentation in DGE (Bescherer & Hoffkamp, 2022; Healy, 2000). In addressing these challenges, teachers play a crucial role. They must not only provide students the opportunities for independent exploration and conjecture generation in DGE but also strengthen their students' needs to prove their conjectures, as has been extensively and controversially discussed in the literature. In particular, there is the concern that the persuasive power of dynamic figures in DGE might diminish the need for proof (Olivero, 2003). They also have to assist the learners in developing supporting arguments and arranging them into a deductive proof, without providing too much guidance and reducing the cognitive demand of the learning activity. The selection and adaptation of meaningful tasks—as a situational demand of the teacher—is crucial for a rich learning activities in argumentation (Bescherer & Hoffkamp, 2022).

A condition for addressing these challenges is a sound professional knowledge of teachers regarding argumentation in DGE and, especially, the creation of effective DGE-supported tasks that stimulate argumentative processes. Our experience with teachers in Germany indicates that so far, only few teachers have incorporated rich DGE-supported activities for argumentation into their teaching. While there is still a lack of research with respect to open exploration in proving in school, the existing literature describes that there is currently less exploration than desirable in the classroom (Jahnke et al., 2023). Therefor, we will now show strategies to transform traditional robust construction tasks into meaningful, cognitively demanding, and discovery-oriented tasks for the process of conjecture generation.

MEANINGFUL TASKS FOR ARGUMENTATION IN DGS IN TEACHERS' PD

Starting from a task related to a robust construction (Healy, 2000) of the mathematical relationship, meaningful tasks promoting argumentation skills can be generated by systematically relaxing and/or changing conditions. It becomes evident that the complexity of the tasks varies depending on how many conditions of the theorem's premise are passed to the DGS and mentioned in the task, providing different levels of space for students' own discoveries. We illustrate this approach to task construction using the already mentioned converse of Thales' theorem as an example: If the angle at point *C* of the triangle *ABC* is a right angle, *C* lies on a circle with diameter *AB*.

(1) Robust Construction: If each condition of the theorem's premise is robustly constructed in a DGE, a triangle with an invariant 90°-angle at Point *C* over segment *AB* (direct invariant) is obtained. The following task prompts students to discover the position of C on the Thales circle (robust indirect invariant) through dragging: "*The triangle ABC has a right angle at Point C. Drag C to change the shape of the triangle. Which path does Point C move on? Generate Conjectures.*" A task like this, focusing on generating conjectures about relationships between robust invariants, relies on students' ability to differentiate between direct and indirect invariants (Baccaglini-Frank, 2012). Students can also perform the construction themselves, enhancing not only tool competencies but also an awareness of all required conditions in the theorem while getting direct feedback on the correctness. Tasks involving robust constructions are also suitable for demonstrating mathematical relationships by the teacher (Laborde, 2005).

(2a) Relaxing/changing conditions: As in Ex. 2 above, some conditions of the theorem's premise are omitted from the construction. Here, the condition $\gamma = 90°$ is mentioned but not constructed robustly. The position of point *C* on the circle is not obvious due to this soft construction, making it more challenging for students to discover the geometric path of Point C through maintaining dragging. This variation is more open compared to (1).

(2b) Relaxing/changing conditions: The condition $\gamma = 90°$ can be further modified to create room for additional interesting geometric discoveries. Instead of right angles, any angle γ can be investigated. The instruction "*Investigate other angle sizes. What do you observe?*" can follow task 2a and leads to the converse of the Inscribed Angle Theorem. Students can discover that for various angles γ, point *C* lies on an circular arc. The Thales' relationship is a just special case of this.

(2c) Relaxing/changing conditions: Another task variation is: "*Draw a triangle ABC. Use the angle tool to measure all interior angles of the triangle. Move Point C and investigate special types of triangles (regarding angles). Describe the positions of Point C for each triangle type. Use the trace tool for Point C to assist. Verify your conjectures through construction.*" Through the request "*investigate special types of triangles*," the condition regarding the interior angle of triangle *ABC* is not only omitted in the DGS construction as in (2a), but also left out from the task description. This variation allows for considering all interior angles and is not restricted to right triangles. For instance, investigating isosceles triangles reveals that Point *C* moves along the perpendicular bisector of segment AB. For equilateral triangles ABC, the position of Point C is uniquely determined. Thus, this task is the one most open to discovery. Solving this task requires students to independently determine whether and which interior angles should be kept invariant. In other words, the soft invariants directly induced by maintaining dragging must now be determined freely by the students, demanding higher problem-solving skills than before.

The process of task construction demonstrated here can be applied to any geometric relationship and can be combined with the *stamping* approach described earlier: Select one or more conditions from the premise of the theorem to be discovered and relax these conditions in the DGS construction and in the task statement. This creates room for additional (related) interesting geometric discoveries that enhance the understanding of underlying concepts and increase students' activity. Baccaglini-Frank (2012) also argues that tasks involving conjecture generation using soft invariants offer potential for richer classroom discussions. In contrast to robust constructions, students must first identify the conditions under which a figure exhibits specific properties or—speaking about motions of the dragged point—the path on which point *C* must be dragged on.

The strategy for task construction can also be applied to other concepts, for example the angle bisector theorem: For angles up to 180°, if a point *P* has the same distance from both sides of the angle, then *P* lies on the angle bisector. For the robust variant, the observation and conjecturing task would be *"Given a point A and two rays, g and h, starting at A. Point P is constructed in such a way that the distances from P to g and h are equal. Use the trace tool to describe the positions of point P.",* while relaxed/changed variants using the soft construction of *P* could start with *"Given a point A, two rays, g and h, starting at A, and a point P."* and then (a) *"Move P in a way that the distances to the two sides of the angle are equal. Use the trace tool to describe the positions of P."* or (b) *"Move P in a way that the distances to the two sides of the angle are in a ratio of 1:3. Use the trace tool to describe the positions of P."* or (c) *"Measure the distances d_g and d_h from P to g and h. Investigate different ratios of d_g and d_h by moving point P. On which line does point C lie in each case?"*

Another example of transferring this task construction technique is the investigation of the area of a triangle ABC: Here, the role of $\gamma = 90°$ is taken by $area(ABC) = 50$ (or any other value), and the locus of C is a straight line parallel to AB instead of a circle, and the tasks are reformulated accordingly.

The implementation in the PD program is based on a sandwich model of in-person PD and self conducted teaching experiments. Equipped with the technique of opening up proving and argumentation tasks the teachers are enabled to use this technique for their own, current teaching, even if the content is different. In fact, in the get-together phase after the teaching experiments, they are able to report on different content that could be converted to theory-based lesson activities, and compare and reflect their approaches. As the next step, there is still the challenge of transitioning from *conjecturing* to *proving*. However, as the teaching content has been designed to follow the principles of *cognitive demand* and *longitudinal coherence* (Prediger et al., 2022), teachers can build on these. In our guiding Thales example, students will be able to argue using isosceles triangle and their base angles after working with the soft constructions, and use this knowledge to build a proof using an auxiliary line connecting the midpoint of *AB* to *C*.

CONCLUSION AND OUTLOOK

In this paper we describe a technique for opening up proving tasks by converting robust constructions into soft constructions. This technique is suitable for inclusion in practice-oriented PD, as it enables teachers to address their situational demands, in particular selection and adaptation of tasks and activities. We will use the results of the implementation in the next years within the QuaMath program for further refinement of this approach.

As a counterpart of the relaxation approach described, it is also possible to use DGE to add additional constraints, for example through pseudo-objects (Baccaglini-Frank et al., 2018). Again, it will not suffice to provide a single example of this proof technique, but we need to provide a general schema to enable teachers to create substantial tasks and activities on their own.

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ANALOGIES: A WAY TO PROMOTE THE LEARNING OF PROOF IN 3D GEOMETRY USING DYNAMIC GEOMETRY ENVIRONMENTS

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Research on the influence of dynamic geometry environments on the teaching and learning of 3D geometry is embryonic. Some proposals that use these dynamic environments have involved analogy as a methodological approach to studying geometric objects in space through comparison with objects on the plane so that some properties of plane geometry are taken as the basis of the exploration and formulation of properties in space. Adhering to this line of work and as part of ongoing doctoral research, we present in this document some ideas about the role that analogy plays in students' mathematical activity when they solve construction-and-proof problems with the help of a 2D and 3D dynamic geometry environment. We emphasize the relationships between analogy, elaboration of proofs, and the use of a digital environment when students' productions are analyzed.

INTRODUCTION

The impact and benefits of dynamic geometry environments (hereafter, DGE) in the development of different processes of mathematical activity, including learning to prove, are significant (Sinclair & Robutti, 2013). However, the focus and development of the research has been mainly on 2D configurations, with the attention paid to 3D geometry or the use of 3D-DGE not being similar (Gutiérrez & Jaime, 2015). And, in the schools, 3D geometry is scarcely studied in depth. Some arguments that support this decision allude to the complexity of the study of 3D objects when plane representations of them are involved, since it is difficult to correctly visualize or represent the properties of these objects in these representations (Parzysz, 1988). However, mathematically gifted students can benefit of the challenges provided by 3D geometry problems and, in particular, proof problems. Then, in our research, we analyze the learning by mathematically gifted students because it is an aspect on which there is scarce literature (Jaime & Gutiérrez, 2017).

Some proposals for the study of 3D geometry have involved 3D-DGE and analogies between 2D and 3D objects as a method to recognize similarities and differences between analogous objects in both domains (Echeverry et al., 2018; Ferrarello et al., 2020). We are determined to provide results along this same path. We are conducting a doctoral research to analyze the learning of proof by mathematically gifted students in 3D geometry with the support of 2D and 3D GeoGebra. This research focuses on the establishment and use of analogies between 2D and 3D objects when students solve construction-and-proof problems.

This document aims to show how the use of analogy, as a driver of students' mathematical activity when entering the world of 3D geometry, with the support of a 3D-DGE, promotes the learning of mathematical proof. We analyze the sequence of problems we posed to a sample of mathematically gifted students and the results of its implementation. We focus on the relationships built between analogy, the use of 3D-DGE, and the elaboration of proofs.

THEORETICAL BACKGROUND

Analogy: A way to extend ideas

An *analogy* is a relationship of similarity between two domains through specific objects and relationships. A domain is a representation of certain aspects of a situation, model, problem, conceptual structure, etc. (Schlimm, 2008). By establishing a set of relationships in one domain and taking them to another domain, in which these relationships are valid, analogy allows the formulation of hypotheses and the simplification of complex mental operations in the second domain, when these are performed in the first domain that is already known (Fishbein, 2002).

Making analogies is also a product of human activity, with several benefits (Richland & Simms, 2015). This process demands recognizing corresponding conceptual structures between different domains and their similarities or differences, thus advancing from a comparison based on superficial aspects or characteristics to one based on relationships between those domains. It means that this process has implications for the learning of mathematics (English, 1997) since new objects and relationships in an unknown domain can be discovered by extending ideas from a known domain.

Learning of proof and support of dynamic geometry environments

To analyze the learning of proof, we consider different strategies of the students when they validate mathematical statements. We consider that students, as learners of this practice, use with understanding theoretical elements, modes of reasoning, and forms of communication that are valid within the frame of this activity. We find support for this need in the approaches of Stylianides et al. (2016), making explicit the different nature of the ways of validating. In this sense, we understand the proof as an empirical or deductive mathematical argument, a sequence of connected assertions for or against a mathematical statement.

In our study, we emphasize the learning of proof through construction-and-proof problems. These problems ask i) to create on the DGE a geometric figure having some properties required by the problem that must be preserved under dragging, and ii) to prove that the procedure used to create the figure is correct by explaining and validating the way of construction (Mariotti, 2019). The statement to be provided is that the sequence of actions of the construction produced a figure that fits the conditions of the problem.

The tools provided by a DGE are related to theoretical elements of Euclidean geometry. When students use these tools to construct geometric objects, personal meanings are produced, thanks to the dependency relationships they discover and verify through dragging. Using DGE tools to solve different problems, personal meanings can progressively become mathematical meanings incorporating theoretical elements. Bartolini-Bussi and Mariotti (2008) call this relationship the semiotic potential of an artifact. Therefore, solving construction-and-proof problems allows students to take advantage of the possibilities of DGE, the logical system that underlies it, and evoke theoretical meanings of the tools they have used in the solutions (Mariotti, 2019).

METHODOLOGICAL CONSIDERATIONS

The content of this paper is part of a doctoral research adopting case study methodology. The research design considered a hypothetical learning trajectory (Simon & Tzur, 2004), whose objective was the learning of proof by mathematically gifted students. To achieve this objective, we developed a sequence of 18 construction-and-proof problems that involved the equidistance relationship and the analogy between 2D and 3D domains. Equidistance offers an opportunity to observe solution strategies guided by theoretical and perceptual aspects, as well as similarities and differences between the geometric objects of the compared domains. To solve the problems, students only used GeoGebra. This software was used because it provided simultaneity between the 2D and 3D configurations, an aspect that we consider useful to support the establishment and use of analogies.

The sequence brought students closer to the perpendicular bisector and circle and their 3D analogs (sphere and bisector plane) to construct these objects, define each object as a locus, and provide valuable properties to solve subsequent problems. Some problems required the construction, first in 2D and then in 3D, of geometric objects that satisfied equidistance properties (e.g., construct an equilateral triangle given its side). Other problems required constructing a 2D object and an analogous 3D object (e.g., finding the center of a circle in 2D and then the center of a sphere in 3D).

Four Spanish mathematically gifted students participated, aged between 11 and 14 years, belonging to different school grades and with different academic experiences. We recognize the students as mathematically gifted because they had participated in workshops on general and mathematical giftedness. The students had used GeoGebra 2D previously but not to carry out robust constructions in a context like the ones described. His knowledge of GeoGebra 3D and spatial geometry was scarce. The characteristics of the students were suitable to the process of solution of each problem to provide them with useful conceptual and instrumental elements to solve subsequent problems and for them to mobilize their knowledge of 2D geometry to the 3D domain, where it would be amplified and deepened by converting it into support for the mathematical activity deployed there.

For each problem, the students had to solve it and then discuss the solution and justify its correctness to the teacher (the first author of this paper), who led the conversation. The teacher's participation was relevant; his questioning of the students' productions promoted the ideas and arguments to advance from empirical and perceptual approaches to deductive and theoretical ones. The sessions were held by video conference and recorded in audio and video after informed approval by the students and their parents. As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

We present three moments in the sequence and what we expected the students to do in each one regarding the analogies developed, the use of GeoGebra tools, and the elaboration of proofs. Student names have been changed by pseudonyms. We contrast these considerations with the students' productions when implementing the sequence. Supported by this information, we discussed the articulation between the three elements that intervened in the sequence design.

THREE MOMENTS OF THE SEQUENCE

First moment: discovery of new geometric objects

The first problems were solved in 2D and 3D GeoGebra. These problems aimed to introduce the circle, perpendicular bisector, and their 3D analogs (sphere and bisector plane) as loci. Drag and trace functions were relevant. This characterization would be the basis of the analogies established between the circle, sphere, perpendicular bisector, and bisector plane. In instrumental terms, the recognition of tools and schemes that would allow each object to be constructed robustly was anticipated. As in this phase, geometric objects and properties were discovered, deductive proofs were not expected.

Instead, the use of empirical and perceptual evidence was expected, as well as the description of each object or set of points in terms of its appearance.

The students quickly incorporated the circle and perpendicular bisector into their activity since they knew them in advance. His school experiences also allowed the sphere to be introduced in 3D as a substitute for the circle and support equidistance concerning a point. This did not happen with the bisector plane. The students were unaware of this object, which provoked different exploration strategies when solving the problem that involved it.

Mario relied on the trace and drag function to move points of the perpendicular bisector out of the XY plane and drag them on it, controlling this action with the dotted line that drag function provides in GeoGebra 3D (Figure 1a). He used the labels of the distances of the displaced point concerning the two fixed points to validate his strategy inductively and said: *Point C continues going through the perpendicular bisector... by a perpendicular, a vertical line, then it will always be at the same distance*. Hector realized the sufficiency of changing the height of the points of the perpendicular bisector (z component of its coordinates) to preserve equidistance in 3D, so he saw in vertical dragging a way to obtain points that solved the problem, as he explained: *As high as point D is, up or down the perpendicular bisector, there will always be the same distance*. To prove equidistance, he referred to the line determined by the displaced point and the midpoint of fixed points as a perpendicular bisector, without being aware that this lacked sufficient theoretical support (Figure 1b). Both students characterized the set of points obtained as a perpendicular plane to the XY plane.

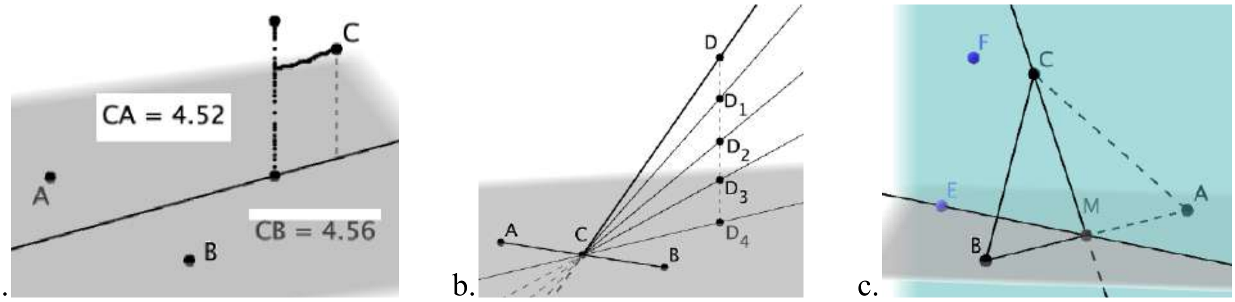


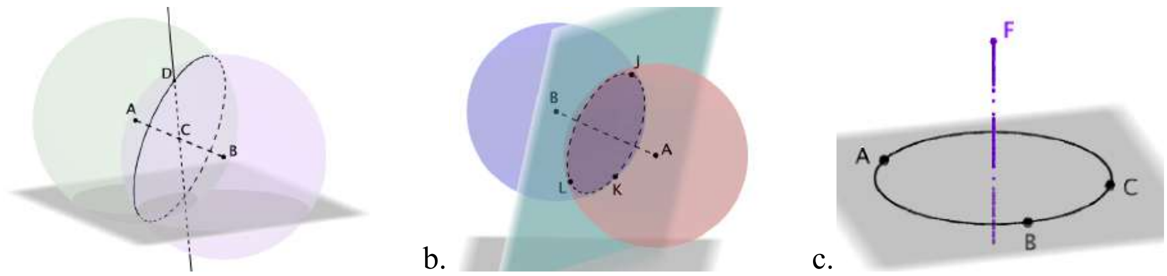
Figure 1. Discovering the bisector plane

Rafa anticipated a plane as a locus and the solution to the problem (Figure 1c). Like Hector, Rafa obtained this plane by dragging the points of the perpendicular bisector vertically, and he justified his actions: *We have moved point C on the Z axis, A and B are at the same height, so it does not vary (the distances between A, C and B, C remain the same)*. However, Rafa also explained that it was possible if the configuration taken as a base (perpendicular bisector and points that determine it) were contained in a horizontal plane. Like Hector, he proved the validity of his construction by referring to the line determined by each dragged point and the midpoint of the fixed points as a perpendicular bisector. Juan developed a different strategy. From the beginning, he recognized the existence of infinite perpendicular bisectors of a segment in 3D, so he constructed one of those lines and rotated it to determine all the solutions to the problem: *There are infinitely many perpendiculars to segment AB. I could draw any one and move point C on it*. The proof of his construction took advantage of the fact that the line constructed was the perpendicular bisector of the segment. The set of bisectors was characterized as a plane.

Second moment: substitution of 2D objects with 3D objects

The second block of problems was intended to use the geometric objects introduced. Students had to build objects with specific characteristics, first on the plane and then in space. Replacing the plane with space was intended to exchange the circle and perpendicular bisector for the sphere and bisector plane when necessary. Working in 3D would take advantage of strategies for constructing objects and proving properties worked out on the plane.

The substitution of 2D objects for 3D objects was seen, for example, in the strategies of some students to construct points in space that are equidistant from two fixed points. Hector and Rafa, aware of the lack of a tool that would allow constructing the perpendicular bisector or the bisector plane, adapted the procedure to construct the perpendicular bisector on paper with a ruler and compass, replacing the circles with spheres and using one of the points of their intersection and the midpoint of the given points to determine a perpendicular bisector (Figure 2a), or three points of the intersection to determine the bisector plane (Figure 2b). For them, replacing circles with spheres, as long as they preserve the equidistance to a fixed point, supports the validity of their construction. Hector argued: *I have done the same as in 2D, instead of circles I used spheres... I have created the perpendicular bisector*.



a. b. c.

Figure 2. Construction of perpendicular bisector and bisector plane in 3D

When the problem conditions did not suggest the replacement of 2D objects with 3D objects, the students used the configuration represented in 2D as the basis of their exploration in 3D. An example of this was obtaining the set of points in 3D that are equidistant from the points of a circle. Previously, the center of the circle had been constructed in 2D. Mario combined vertical dragging and the trace of the center of the circle to determine a perpendicular line to the XY plane as a solution to the problem (Figure 2c): *It is a circle where the center is at a different height*. The other students anticipated this locus, ignoring the vertical dragging and constructing the perpendicular line. Mario proved his construction based on the conservation of equidistance under vertical dragging. Hector combined empirical evidence and theoretical elements to prove it. Juan and Rafa defined the perpendicularity relationship between a line and a plane, which allowed them to prove the equidistance.

Third moment: sophisticate use of objects and properties

The final problems requested constructions only in 3D, leaving behind the simultaneous work with the plane and the use of analogies. At this sequence stage, robust constructions of geometric objects and fluidity in using their properties to solve problems were expected. The proofs developed should be deductive or have features very close to these.

The students' actions when solving the problems reflected a purposeful use of 3D objects and robust constructions of them. This was the case of Rafa, who used 3D objects in the construction process and proved it based on their properties, leaving behind the reference to vertical drag or the analogy between these objects and their corresponding ones on the plane. Juan and Hector, in addition to Rafa's action, explained some constructions in similar ways, both based on an analogy between the plane and space. An example of this is what was made when they constructed the center of a given sphere. They constructed chords of the sphere and their bisector planes (Figure 3a), stating that the point of intersection of the planes was the center of the sphere by simultaneously equidistant from the ends of the chords. Juan explained that the choice of the bisector planes was based on an experience in the plane, in which they used a perpendicular bisector of chords in the circle to construct its center (Figure 3b): *In 2D, we used the intersection of the perpendicular bisectors of points in the circle. In this case* (3D)*, instead of perpendicular bisectors, we used the bisector plane*.

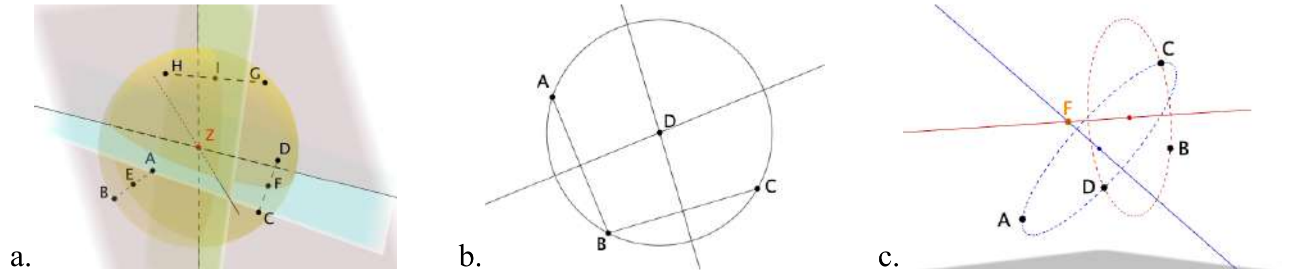


Figure 3. Construction of perpendicular bisector and bisector plane in 3D

Mario's actions do not allow us to generalize the ideas presented above. Although Mario knew procedures for constructing geometric objects that solved problems and he was sure about the result obtained, he could not develop a deductive proof of its construction. As the vertical dragging of points marked his experience moving objects from plane to space, he inductively established these results as properties that, although true, did not have theoretical support. He determined the center of the sphere with the help of bisector planes, as did Juan and Hector. When he solved another problem that asked to construct an equidistant point from four non-coplanar points, he used lines containing points that were equidistant from the points of the circles determined by three of the given points and their intersection (Figure 3c). In both cases, the robustness of his constructions was accompanied by gaps in the elaborate proofs.

DISCUSSION AND CONCLUSIONS

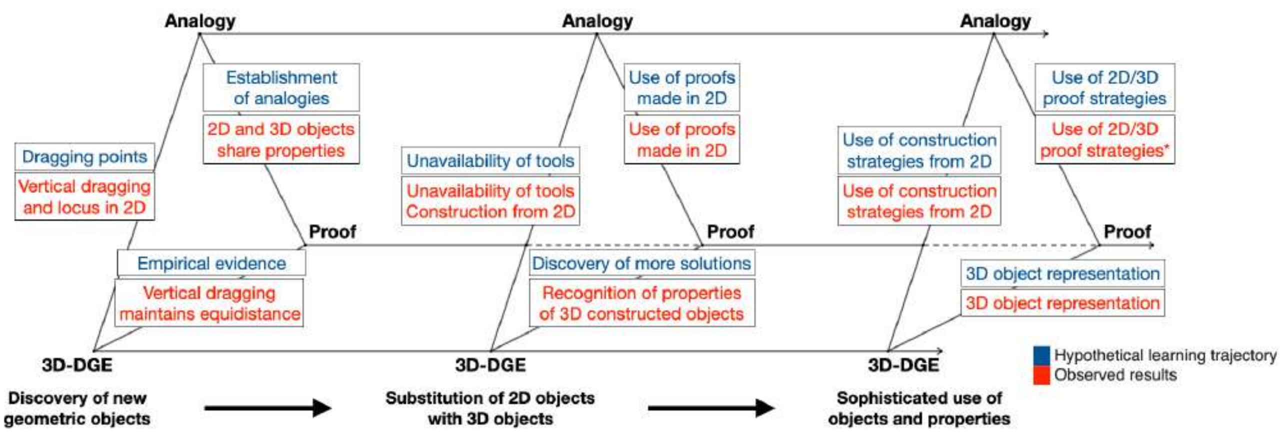


Figure 4. Moments of construction of perpendicular bisector and bisector plane in 3D

Figure 4 shows a diagram with the trajectory described previously. Each moment is represented through a triad in which the analogy, the use of the DGE, and the elaboration of proofs are articulated. The articulation derived from the hypothetical learning trajectory is presented in blue. In red is the articulation of the elements at each moment, according to the results obtained.

In the first moment, the combined use of vertical dragging and a locus in the plane became the basis for exploring and discovering analogous objects in space. Although we expected that the dragging of points would be an expected action, we did not expect the ways some students used it to guarantee the conservation of equidistance. This relationship led them to formulate proofs for equidistance in space, characterizing the discovered object in 3D through the 2D object taken as a base.

The relationship between vertical dragging and equidistance is a personal meaning promoted by using DGE functions. This meaning evolved in the second moment of the sequence when the characteristics of one of the problems led the students to replace dragging points with the construction of the perpendicular line to the plane. In this way, vertical dragging became a vehicle for establishing the relationship between perpendicularity and equidistance in 3D (mathematical meaning). Determining the meaning of the perpendicularity between a line and a plane allowed some students to deductively prove compliance of equidistance and not offer empirical proof as an explanation.

The absence of some construction tools favored the analogy between the circle and the sphere, an aspect considered when the sequence was designed. However, the procedures of some students to construct the perpendicular bisector and the bisector plane in 3D were not part of our assumption. These students simplified 3D mental operations to build these objects by recovering and adapting to the 3D-DGE construction protocols with a ruler and compass. Proving these constructions led to evoking the corresponding proofs in 2D, a valid strategy for them since the circle and sphere shared the property of equidistance.

The third moment shows a more mature state of transition in which the analogy begins to disappear. The use of geometric objects is observed, endowed with useful properties for constructing and elaborating deductive proofs. The analogy, if used, allows ideas to be extended from one domain to another by recovering construction strategies in the plane that are then adapted to the characteristics of the 3D configuration, thereby advancing the resolution of the problem in the latter domain.

Mario's actions offer a different picture. The use of vertical dragging in the exploration and characterization of sets of points had a greater presence when compared with the actions of his colleagues. Mario transformed vertical dragging into a tool for construction and proof when he worked in 3D, but the inductive nature with which this tool was developed did not allow the level of his proofs to be similar to that of his colleagues. Although the analogy allowed him to solve problems in 3D, his journey to get to this point was slower. This was evident in the episodes in which other students anticipated locus while he resorted to vertical dragging in search of solutions or used the discovered geometric objects without being aware of why they provided a solution to the problem.

Construction-and-proof problems provide a scenario in which the need to construct objects with specific properties leads to the need to articulate the available tools and knowledge. In the context of 3D geometry, there is also a search and connection with what is known about 2D geometry when the tools provided by 3D-DGE do not allow replicating the actions executed in the plane. Analogy can support these processes but requires the design of appropriate problems that promote it. As we have presented, using a 3D-DGE in a scenario like this produces personal meanings that progressively, and at unequal rates among students, will become mathematical meanings or approximate them.

Acknowledgements

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CRITERIA USED BY PROSPECTIVE ELEMENTARY SCHOOL TEACHERS WHEN APPROACHING A 3D FIGURE CLASSIFICATION TASK

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Classification is a fundamental geometric activity since it mobilizes a series of knowledge and competencies, which allow the construction of criteria to distinguish and associate different structures following elements in common. The objective of this communication is to characterize the professional knowledge used by students in the second year of the teaching degree for Primary Education, through the analysis of the criteria they use when classifying 3D figures. The study considers a descriptive approach, under a qualitative methodology. A professional task is designed and implemented with 148 students; structured in four sections in which construction, composition and decomposition, visualization, identification of attributes of 3D figures, and their classification are promoted. For the analysis of the productions of the future teachers, tools of the model of Didactic-mathematical Knowledge and competencies of the mathematics teacher are used. It is observed that most of the future teachers recognize mainly perceptual elements of the figures, which makes them construct dichotomous classifications and only a few interpret the classification using global characteristics.

INTRODUCTION

Research on geometry, its teaching, and learning has been a focus of attention in Mathematics Education for decades (Sinclair et al., 2016). Several studies highlight the importance of having tools that allow teachers and students to explain the world around them and the significant contribution of geometry in this regard. The Principles and Standards for School Mathematics (NCTM, 2000) identify geometry as one of the fundamental contents of school mathematics. In addition, it highlights that, for mathematics from preschool through eighth grade, the focus is on students' abilities to identify geometric ideas and objects in their world; and to describe, model, draw, compare, and classify shapes according to their properties, analyze compositions and decompositions of 2D and 3D figures, and relate geometric ideas to dimensions and measurement systems. All these elements are of particular interest to the scientific community in mathematics education, both among mathematics educators (Lannin, 2003) and learning researchers in general (Roth, 2005).

Although there is a need for research on geometry and particularly on skills such as classification, currently, research on the classification of figures has been developed mainly within the framework of Van Hiele's model and focusing mainly on 2D geometry with the study of some types of polygons (Guillén & Figueras, 2005). Gutiérrez and Jaime (1998) point out that most geometric activity is centered on the study of figures (both 2D and 3D) with prototypical examples and in the usual position, and that this explains some of the difficulties for students to advance from levels 1 or 2 to Van Hiele's level 3. Some of the research on classification focuses on presenting figures that students

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or future teachers must recognize and analyzes how students identify properties underlying these figures and whether this helps them to classify them (Guillén, 2000; Patkin, 2015).

Many investigations serve as a reference for the discussion that we wish to raise here; In this sense, some examples of research on the classification of figures are found in Fujita (2012), who concludes that, when students decide if an object belongs to a certain category or not, they may give biased answers because in their decision they privilege prototypical or common forms. For his part, Guillén (2000) reports research with future mathematics teachers in which they sought to address problems of identification and enumeration of families of solids with one or more classification criteria, as well as stating and verifying relations.

However, the research landscape does not differ much from that of the school. There is a clear consensus that geometry is a relevant topic in the school curriculum; However, there is not enough work in the classroom. Many of the teachers who address geometric concepts have a which favors the memorization of the names of the figures and some of their characteristics (Copley, 2000). Furthermore, generally, the representations used to talk about/show such figures focus on a prototypical vision of them. It is worth mentioning that the geometric work developed in this stage (6-12 years) focuses mainly on the study of flat figures and not so much on three-dimensional ones. On the other hand, authors such as Sinclair and Bruce (2015) point out that little research has focused on children's geometric thinking regarding this type of figures.

We consider that studies on the knowledge of future teachers become relevant since they will allow characterizing the ways of thinking, and the inputs available and outlining strategies for improvement and development of professional competencies to improve the teaching of geometry. Recent research has pointed out that the geometric knowledge with which future teachers arrive at the university is not enough to promote the development of geometric thinking in their future students (Burgos et al., 2018; Font et al., 2018; Vargas et al., 2023), so identifying the knowledge available at the time of initial training is key to adapt and improve the training proposals to promote geometric reasoning in future teachers.

Authors such as Ball et al. (2008), Carrillo et al. (2017), and Pino-Fan and Godino (2015) have identified and defined components to characterize types of mathematics teacher knowledge. One such model is the model of Mathematics Didactic-Mathematics Knowledge and Competencies of the mathematics teacher - DMKC, defined by the Ontosemiotic Approach (OSA), (Godino, 2009; Pino-Fan and Godino, 2015). In this model, it is stated that to achieve an ideal teaching, the mathematics teacher must possess different types of knowledge. On the one hand, he/she must know the school mathematics of the educational level at which he/she teaches. In addition, he/she must know elements of subsequent levels, which is called "mathematical content knowledge *per se"*.

This knowledge is divided into two types: *common mathematical knowledge* and *extended mathematical knowledge*. The former refers to the knowledge about the mathematical object that is necessary to solve problems and/or activities related to a specific (mathematical) topic at a given educational level. It is generally associated with the level at which it is taught. The second refers to the fact that the teacher, in addition to knowing how to deal with problems/activities on a given topic, must possess more advanced knowledge, which is part of higher levels.

This paper aims to show an approach to the geometric knowledge evidenced by prospective elementary school teachers when they propose criteria for the classification of 3D figures. For this purpose, some DMKC tools will be used. More broadly, we are interested in analyzing the types of knowledge of prospective elementary school teachers and in contributing to their understanding and continuous improvement.

METHODOLOGY

This communication presents some results of research with an exploratory qualitative approach (Hernández et al., 2010) and follows the case study methodology (Yin, 2014). The case is that of the practices of the human group that configures a class of didactics of geometry, belonging to the degree of Primary Education, of a Spanish university.

A professional task (PT) is designed and implemented with a total of 148 students of the Primary Education Degree of a Spanish university. The PT is carried out in groups (4-5 people), resulting in 33 work teams. The data are the written protocols constructed by each of the groups.

For the development of the PT, future teachers are asked to elaborate a manipulative material: four half cubes, for which they are provided with templates (one with a white background and the other with a grid). The PT consists of four activities, the second of these is addressed in this communication. In this activity, the future teachers are asked, in the first part, to build solids using the four half cubes, to assign a name to each one, and to detail its characteristics. In the second part, they are asked to propose a classification of the solids obtained by them and to state the classification criteria considered.

To analyze the types of knowledge of future teachers, we will base ourselves on the analysis of the answers to three questions of the activity developed. Table 1 shows the questions asked, their intentionality, and the type of knowledge (according to the DMKC) assessed with each one.

No.	Question	Type of Knowledge	Intentionality
1	Build all the polyhedra that can be formed with the four pieces (half cubes). Make a table where you present the graphic representation, the name, and its characteristics.	Common	Obtain figures through composition. Identify a variety of figures. To know the specific language to describe figures.
2	Define criteria and classify the figures you have obtained. Explain what you have considered to define the criteria.	Extended	Generate categories from characteristics of different figures.
3	Why do you consider that when we study various types of figures a fundamental process to work on is Classification?	Extended and Meta-Didactic	Promote an attitude of inquiry. Recognize the intentions and motivations of a classification task.

Table 1. PT questions, type of knowledge, and intentionality.

In Figure 1, some examples of solids constructed by some groups are shown together with the names given to these figures by the future teachers.

Irregular pentagonal prism (G - 22)

The double ramp (G - 18)

Irregular polyhedron "Shower" (G - 03)

Figure 1. Examples of figures constructed by students with the respective names.

For the analysis, we proceeded to group similar responses (regarding the type of response, materials used, and initial classification criteria). From this grouping, information is obtained that allows us to describe difficulties, errors, and justifications present in the common knowledge of the content possessed by prospective teachers concerning 3D figures and their classification. Finally, systematize the answers given on classification, we show the process of recognition of emerging categories associated with the production of the future teachers (Table 2). To systematize the look at mathematical objects, we look for the type of characteristics used from the proposed classification. For this, we use the idea of the configuration of objects and processes of OSA, looking at the definitions, arguments, and propositions used.

Productions of future teachers	Emerging category
They establish a table grouping according to the number of vertices. [They identify that the constructed figures have 8 vertices. Most have between 6 and 12 vertices. None have 11...]. (Groups 22, 25)	C1: Perceptual appearance
"The figures cube, octahedron, and triangular prism can be classified within the same group since they are regular polyhedra, while the figures 4,5, and 6 are irregular, and therefore do not have all the same faces. Thus, we have two groups, regular and irregular". (Group16)	C2: Attempted classification by grouping known 2D features (regular, irregular) but with errors.
"We have taken into account the type of polyhedron according to its bases, the regularity, the number of faces (bases and sides), and the shape of its bases" (Group 12).	C3: Correct recognition of 2D features applied to 3D.
"We have considered whether it is concave or convex, the number of vertices and edges, and symmetry." (Group 3)	C4: Global observation characteristics of 3D figures.

Table 2. Emerging categories and classification criteria

For extended and meta-didactic knowledge, the arguments given by prospective teachers to assess the classification process are analyzed. The responses are categorized as shown in Table 3.

Example of the future teachers' argument	Emerging category
"...allows children to group objects according to their similarities and differences, based on different criteria: shape, color, size .... These relationships serve as the basis for the construction of logical-mathematical thinking. Piaget considers that these logical relationships are the basis for classification, seriation, the notion of number, and graphic representation" (Group 23).	C5: Allows establishing logical relationships and discovering new concepts.
"The fact of classifying allows us to compare and see differences from other groupings. It also allows us to look for regularities between shapes and their properties, compare similarities and differences using the appropriate vocabulary, understand relationships between different three-dimensional figures, using properties that define them, and look for regularities and changes that occur in a collection or sequence" (Group 17).	C6: Allows to compare, to recognize common characteristics, through similarities and differences.
"It is important because it allows to order or organize the figures in groups following common criteria, it facilitates the development of logic" (Group 22).	C7: Allows an organization

Table 3. Categories of arguments are given to the value of the classification.

RESULTS

Initially, when referring to the types of solids constructed by the future teachers, it is possible to detect characteristic elements of their knowledge and way of proceeding; particularly, most of the figures constructed are of the type known as cubes and rectangular prisms; however, when naming them, the geometric name is not used, but, on the contrary, an everyday name such as "box" is assigned. It is striking that most of the figures constructed are not polyhedrons, but configurations that recall some everyday element such as a *shower* or a *candy*, due to a lack of knowledge of the criteria for the construction of polyhedrons.

When analyzing the criteria established by the future teachers to classify the solids obtained through their different configurations, we find that most of the groups point to elements of a visual and descriptive nature to classify the solids. They mention, for example, the number of faces, vertices, sides, and bases of the polyhedra they were able to construct. In these cases, they classify considering these attributes. Although other groups allude to global characteristics proposing a classification of the figures as regular and irregular polyhedra, they do it wrongly, since none of the figures constructed is a regular polyhedron. Other groups in making the classification speak of figures that are also not possible to obtain with half cubes such as pyramids. Few groups speak of prisms as convex polyhedra, which would be a suitable answer. Table 4 shows the percentages for the emerging categories (C1 - C4), according to the elements that the future teachers considered for the classifications.

Category	C1	C2	C3	C4	None
Number of groups	18	10	1	3	1
Percentage	54.54 %	30.3%	3.03 %	9.09 %	3.03%

Table 4. Elements alluded to for classification by prospective teachers.

Most of the classification criteria proposed by the future teachers are based on a basic description of the object, i.e. they do not address any more advanced theoretical element, but remain in a process of visual verification; only one group manages to include elements of the origin of the *half cube* as a section of the cube (figure 3D) for the construction of other polyhedra.

Finally, regarding the value of the classification process. The importance given to the classification process by prospective teachers focuses on aspects such as the establishment of relationships; the identification of properties; the recognition of common and uncommon characteristics or the idea of classification as a grouping. All 33 groups (100%) focus on the importance of classification as it allows distinguishing attributes based on comparison (C6). Additionally, 17 of them allude to classification as separation, proposing simple dichotomous groupings, but without an adequate appropriation of the characteristics of the figures (C5) and only 4 of the 33 groups understand classification as an element that allows organization (C7). Most of the groups identify a great value of classification the recognition of common characteristics, but at the moment of classifying, they do not explicitly state the criteria, but emphasize the name given to a group of figures as evidence of the classification.

CONCLUSIONS

Concerning *common knowledge*, it is found as in other studies (Bernabeu et al., 2017; Gonzato et al., 2013) that perceptual appearance dominates over certain conceptual aspects. Future teachers construct classification criteria based on visual aspects, similar to those used by children at early ages. An application of the mathematical knowledge assumed from the known classifications of figures in the plane to the case of prisms is not observed.

We agree with Walcott et al. (2009) who state that the classification process depends on the ability to identify similarities and differences between figures and explain why a certain figure is an example of that class. However, when analyzing the classification criteria used by future teachers (referring to their *extended knowledge*), it became evident that they usually show the examples and characteristics of the bodies achieved, but not so much the common and distinguishing criteria, although they are valued as important to achieve classification. Most of the works analyzed did not report the class to which the bodies belonged or, in most cases, did not indicate the common criterion used, which leads to the conclusion that this is not a decisive element for future teachers, even though it is known that in geometry it is.

Similarly, we found that future teachers have difficulty recognizing how certain visualization elements are associated with representations that allow them to recognize properties. For example, we expected that there would be classification proposals where the number of concavities or symmetries would be considered.

We can say that we interpret our results in the sense of showing the difficulty in relating the comprehension of geometric figures, with the coordination of two semiotic systems of representation. The discursive one (oral or written) and the non-discursive one (drawings, pictures of the figures) (Duval, 2017). Indeed, despite having manipulative elements, students are not able to formulate mathematically well-formulated explanations.

The analysis carried out has identified a perceptual domain in the characterization of 3D figures and a low level of structuring based on properties. Although the arguments given by the future teachers about the value of classification are at an initial level, it is an indication of how they could in the future deal with this process in the mathematics class in primary education. Resolving the question about the importance of classification led future teachers to question the models (of classification) that they had learned in their school years, but since they did not have a broad appropriation of the attributes of solids, they were not able to recognize generalizable properties that would allow them to make meaningful and/or richer classifications.

The use of manipulative material as a mediator of observations and responses allowed opportunities for reflection on the congruence of figures that can be perceived in different positions. Likewise, it is important to mention that the use of the material alone is not enough to make adequate *common mathematical knowledge* emerge, because many groups are not able to systematize the process of constructing figures and are satisfied with having found a few.

Acknowledgements

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ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
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THE USE OF STRAIGHTEDGE, COMPASS, AND FIXED SHAPES TO ENHANCE THE CONTENT KNOWLEDGE OF MATHEMATICS TEACHERS IN THEIR PROFESSIONAL TRAINING IN GEOMETRY EDUCATION

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The paper discusses a special approach to professional learning with a focus on building geometrical reasoning and argumentation through the extensive use of straightedge, compass and fixed shapes together with some special constructions. The novelty of this method is in combining the elements of content knowledge with specially designed rich geometry tasks, where the use of instruments takes the dominant role until the end of the solution for each task.

INTRODUCTION AND THEORETICAL BACKGROUND

Despite the wide range of professional learning models related to geometry education that are available in literature (Jacobs et al, 2020; Smith et al, 2005) and known from practical implications (Seago et al, 2010; White et al, 2012), an appeal for new resources and educational tools that can be developed and used for professional training and development in this area remains high. The proposed model of teachers' professional development aims to answer these questions and outline further perspectives bringing attention to a special method of teacher training. The method is based on the use of special constructions from different topics of school geometry affiliated with meaningful tasks that require the active use of drawing instruments while making reasoning and argumentation. It is considered from the perspective of activity theory, where mathematical ideas are regarded as historical artifacts of human culture (Vygotsky, 1987). Artifacts may also include drawing instruments such as straightedge, compass or fixed shape(s) that may be used individually or collaboratively within a group to solve mathematical problems. The novelty of this method is in combining the elements of content knowledge with specially designed rich geometry tasks, where the use of instruments takes the dominant role until the end of the solution for each task. The active use of straightedge and compass for proof and problem-solving activities in geometry was never under a question mark in geometry education. However, its importance diminished within the last several decades. As a result of that, school graduates in Australia, even those who complete 'Specialist Mathematics' and 'Mathematical Methods' – the two most challenging mathematical subjects in Grades 11 and 12, can hardly use a straightedge and compass beyond drawing an initial figure for a geometry task. In other words, these instruments are mainly considered as drawing instruments that do not have any relation to reasoning and argumentation in geometry education. A very similar situation takes place with teachers. The drawing instruments provide unlimited opportunities in school geometry which are not explored often, unfortunately. They help to see appropriate steps towards solutions in a more logical way and, more importantly, help to connect different parts of geometric knowledge together and look at the same piece of content from different perspectives. There is a consensus among mathematics teachers and educators that geometrical facts and statements

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 293-300). ICMI.

that require additional constructions to be added to the initial figure for their proofs are regarded as more challenging for students (Bartolini Bussi et al, 2009; Vondrova and Divisova, 2013; Gridos et al, 2022). As a consequence, such situations are more difficult for teachers to manage in the classroom and achieve desirable learning outcomes for students. This method of training aims to address these issues so teachers can acquire practical skills in using different drawing instruments while making reasoning which is expected to have positive effect on their content knowledge and enrich their professional experience. Specifically, the paper addresses the following research question: How can a program of professional development for mathematics teachers in geometry education integrate geometrical constructions, meaningful tasks and the use of drawing instruments to enhance content knowledge of mathematics teachers and their skills in geometrical reasoning?

THE METHOD OF PROFESSIONAL TRAINING AND DEVELOPMENT

A geometrical construction representing some special property or fact of Euclidean geometry is assigned with a task to complete using a straightedge, or straightedge and compass, or fixed shape(s). A straightedge can be used to construct a line or a line segment through two points. A straightedge cannot be used for measuring any distance. It is also called a ruler (without scaling). A compass can construct a circle or some arc of a circle. However, its main functionality is in measuring distances which cannot be carried out with a straightedge. Fixed shapes represent special geometrical shapes that can be used for drawing and remain unchanged. If a fixed shape contains a straight part of its shape, it can be used for constructing line segments. For example, a fixed shape of a right-angled triangle I discuss later can be used for constructing right angles or line segments up to the length of its hypotenuse. For practical usage, fixed shapes can be made of cardboard or veneer. These instruments play a central role in this method of professional training. A card with a special construction is given to the participants of the training as well as a card with the task to work on. In some cases, a proof (or justification) of the given construction is provided together with its statement, in other cases participants are encouraged to prove some constructions on their own before moving on to work on the tasks. Some cards may contain a set of constructions rather than just a single construction as it happens for some topics, where a number of different constructions can be often considered and used together. For example, many properties of circle geometry are worth to be considered together. The solution to complete the task requires participants to use some of the aforementioned instruments. It also requires to look for the ways of how a special construction assigned to the task can be used as part of a solution for that task. While working on each task, teachers refresh their knowledge of some constructions and make acquaintance with others. Whatever their case is, they need to demonstrate the active use of the prescribed instrument(s) to complete the task successfully. This gives participants the opportunity to be involved in constructing solutions to the given tasks at the time when they are involved in studying some geometrical content and applying it through their practical work. Modules containing constructions and their corresponding tasks within a certain topic are called activities. In the next section I discuss several examples of activities set in an increased order of difficulty. Within each activity I provide a brief analysis of what teachers were asked to do, why that activity was important, how the use of instruments was organized to stimulate teachers' thinking and enhance their content knowledge of the topic. Also, I identify directions for potential questions for teachers to think and implement the tasks in learning environments and present some participants' work and reflections on the selected tasks. The data was collected from task-based interviews with small groups of participants and observations made in the training sessions. The teachers' tasks were analyzed at three levels: competence to solve or make progress in the proposed task, effective (or ineffective) use of the required instruments and skills to see connections between geometrical constructions assigned to the tasks and the tasks.

THINKING GEOMETRICALLY WITH STRAIGHTEDGE, COMPASS, AND FIXED SHAPES

Activity 1: A right angle inscribed in a circle

The first activity is based on a stand-alone, simple property of a right angle inscribed in a circle. An instrument for the first activity is a fixed shape which represents a right-angled triangle. One can use this instrument for constructing right angles or line segments of size up to the length of its hypotenuse.

Construction

A right angle inscribed in a circle is subtended by the diameter of the circle (Figure 1).

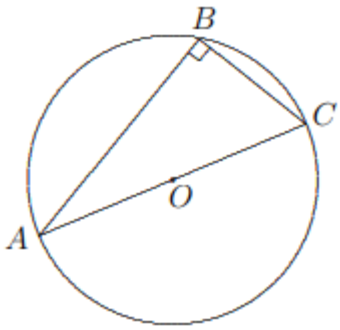

Figure 1: Right angle inscribed in a circle

Task

A circle on the left side is drawn on paper and a fixed shape of the right-angled triangle on the right side is given as an instrument (Figure 2). The location of the centre of the circle is not known for participants. Using the fixed shape, construct a point which is the centre of the circle.

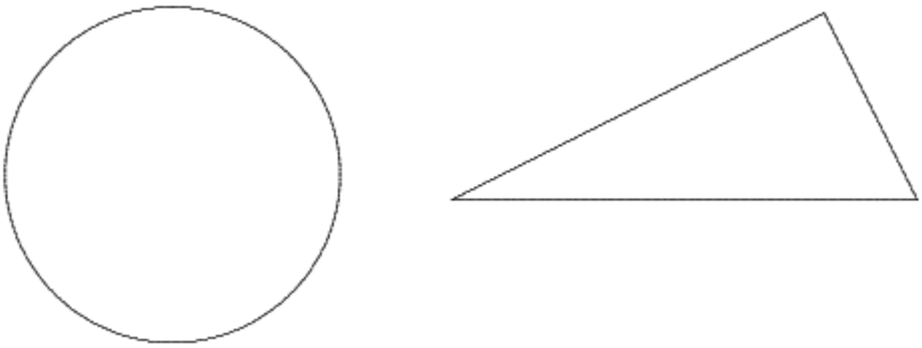

Figure 2: Task for Activity 1

Analysis

The main idea of this task is to stimulate teachers' awareness that each part of the meaningful task in geometry can be actively used in reasoning and argumentation. A geometrical object can be modified to serve the goals of the current task. The straightforward use of the construction above does not look to be workable here since the hypotenuse of the fixed shape even visually is longer than the diameter of the circle. However, the modified version of the fixed shape where the right angle only is important should work. Indeed, put the right angle vertex of the fixed shape on the circle. Then, the construction assigned with the task can be in use. Indeed, let the sides of the fixed shape intersect the circle at points *A* and *B* (Figure 3).

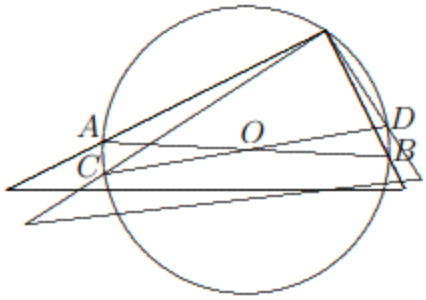

Figure 3: Solution

According to the construction, *AB* is the diameter. If the right angle vertex of the fixed shape remains at the same point on the circle and the location of the fixed shape is changed by rotation, the sides of the fixed shape intersect the circle at other points, say *C* and *D*. Note that *CD* is also a diameter, distinct from *AB*. Hence, using the fixed shape, two diameters *AB* and *CD* can be constructed. Both contain the centre of the circle, so the intersection *O* of *AB* and *CD* is the centre of the circle.

This activity helps teachers to advance their knowledge of the simple property of the right angle to a higher level of understanding where they can use that property in the new teaching environment through meaningful tasks that support reasoning and proof building from basic initial ideas. For example, joining points *B* and *O* (Figure 1) contributes to potential questions for teachers to think and implement the above task to the concept of the central angle ($\angle BOC$) and its relationship to $\angle BAC$, and to the concept of a tangent line at point *B*. One more potential question to problematize this task relates to constructing the midpoint of *AC* and *BD* (Figure 3) using a straightedge and compass. The transcript below shows reflections of two participants of the training about the task above.

Interviewer: Did you find a fixed shape helpful in this task?

Participant A: Yeah, it was. I see the benefits of the fixed shape of a right-angled triangle here. But, after this task, a straightedge and compass seem more meaningful to me than they were before and I can change this task in my class and go in the opposite direction. First, I can ask students to construct a circle using a compass. So, we know the centre. Then, construct a line through the centre using a straightedge. And by connecting each point of intersection the line and the circle have with a point on the circle, we get a right angle constructed.

Participant D: I think so. The solution is clear to me after discussion. A bit tricky though. I am trying to find another way to use the fixed shape here. What if we ask students to construct a circle that this fixed right-angled triangle is inscribed in?

Activity 2: Corollary of Pythagoras Theorem

The second activity is based on the famous Pythagoras theorem. Instruments for the second activity are a straightedge and compass. This activity helps participants to look at Pythagoras theorem from a different point of view and develop more understanding of how useful Pythagoras theorem can be in situations where this famous property does not seem at first glance to be part of the solution.

Construction

In an acute-angled triangle *ABC* with altitude *AD* (Figure 4) the following property holds

$$AD^2 = AB^2 - BD^2 = AC^2 - CD^2.$$

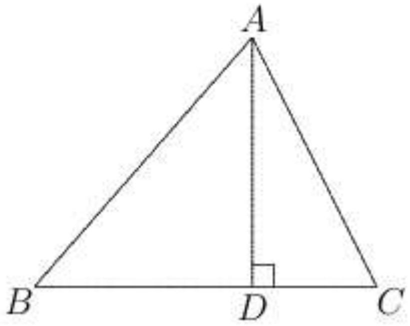

Figure 4: Corollary of Pythagoras theorem

Task

An acute-angled triangle *ABC* with altitude *AD* is given (Figure 4 again). Using a straightedge and compass, divide the triangle *ABC* into two triangles that have the equal sums of squares of their sides.

Analysis

The construction is not hidden in the figure for the task. In fact, both, construction and task, refer to the same Figure 4. However, to be connected with the task, this construction requires some preparatory work to be done before it can be used. It can be rewritten as $AB^2 + CD^2 = AC^2 + BD^2$. However, *AB* and *CD* as well as *AC* and *BD* cannot form triangles. Thus, if *CD* can be replaced with a congruent line segment emanating from *B*, say *BE*=*CD*, then the required property of the equal sums of squares could come to the scene (Figure 5).

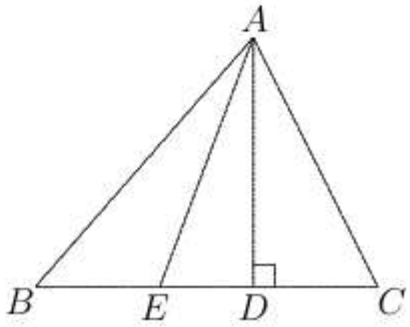

Figure 5: Solution

Distances can be measured with a compass, so one can construct such a point *E* on the side *BC* so that *BE*=*CD* which implies *BD*=*CE*. Thus, using a straightedge to draw *AE*, the two triangles, *ABE* and *ACE*, with the required property can be constructed. This activity helps teachers to gain confidence in using a compass for measuring equal distances when required, and constructing new points that have some special properties. It has connections with Activity 4. This task can be made more challenging for participants, if the altitude *AD* is not given in the task. The following transcript demonstrates participants' views in relation to Pythagoras theorem and the use of drawing instruments in the task above.

Participant B: It makes sense to me. I mean that the use of compass and straightedge in tasks like this makes Pythagoras theorem alive and usable in classroom activities.

Participant J: Looks to me that Pythagoras theorem could be somehow used in the task where we constructed the centre of the circle [the task from Activity 1]. Need to think more.

Activity 3: Symmetry of a rectangle

The third activity is based on the central symmetry of a rectangle. An instrument for the third activity is a straightedge only. This activity helps participants develop more understanding of how some sophisticated constructions can be carried out with a straightedge only.

Construction

Let *ABCD* be a rectangle and *O* be a point where the diagonals *AC* and *BD* intersect. Then, *O* is the centre of symmetry of *ABCD* and any straight line through *O* divides the rectangle *ABCD* into two congruent parts.

Task

Let *ABCDEF* be a hexagon with five angles of size 90° and one of 270° (Figure 6).

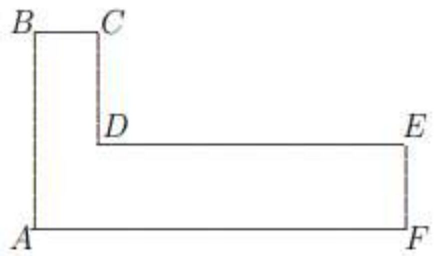

Figure 6: Task for Activity 3

Divide the hexagon *ABCDEF* into two parts of equal area using a straightedge only.

Analysis

To see the benefits and connections of the construction above with the task, participants are expected to construct a larger rectangle *ABKF* (Figure 7).

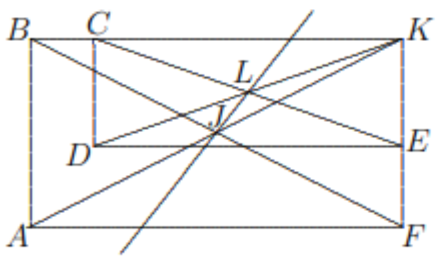

Figure 7: Solution

Then, the diagonals of the rectangles *ABKF* and *CKED* can be constructed, where *J* and *L* are the points of their intersection, respectively. According to the construction above, *J* and *L* are the centres of symmetry of the rectangles *ABKF* and *CKED* respectively, so a straight line through *J* and *L* divides the hexagon *ABCDEF* into two parts of equal area. The next transcript summarizes the difficulties experienced by many participants.

Participant M: I see why I was stuck with this task… All three points that can be constructed with a straightedge are hidden here, and the conclusion is still not obvious after that... A very convincing example to demonstrate how powerful a straightedge can be in good hands.

Activity 4: A cyclic quadrilateral

The fourth activity includes a special construction related to cyclic quadrilaterals. Instruments for this activity are a straightedge and compass. This activity helps participants develop more understanding for the case where additional constructions need to be completed.

Construction

A cyclic quadrilateral *ABDC* is such that *AB>AC* and *D* is the midpoint of the arc *BC* not containing *A*. Then, there exists a point *K* on the side *AB* such that the quadrilateral *ABDC* can be divided into three isosceles triangles, *ACK*, *DKC* and *DBK* (Figure 8).

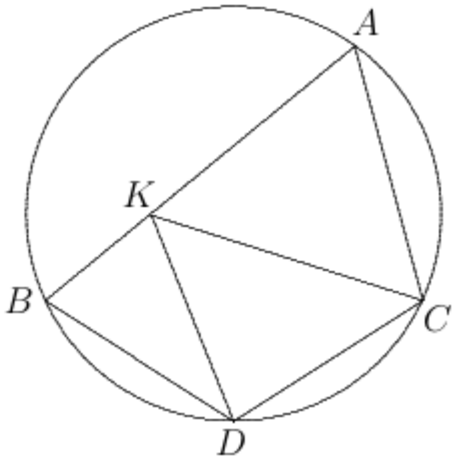

Figure 8: A property of a cyclic quadrilateral

Similarly to Activity 3, participants need to prove this property first, and then work on the task that follows. At first, using a compass, point *K* is constructed on *AB* such that *AK=AC*, so participants need to show that the triangles *DKC* and *DBK* are isosceles. Using a straightedge to connect *A* and *D*, and considering congruent triangles *AKD* and *ACD* (SAS) one can conclude that *CD=DK* which implies that triangles *DKC* and *DBK* are isosceles.

Task

Given triangle ABC, where *AB>AC*, and its circumcircle. Using a straightedge and compass, construct the midpoint of the arc *BC* not containing *A*. Both instruments can be used just once.

Analysis

The main challenge for participants remains the same as in Activity 3 – to find out where the construction above is hidden in the task. Extending *DK* beyond *K*, let *DK* intersect the circumcircle of *ABC* at point *L*. Taking into account that angles subtended by the same arc are equal, and vertically opposite angles are equal, $\angle ALK = \angle ALD = \angle ABD = \angle KBD = \angle BKD = \angle AKL$ which implies *AL=AK=AC*. This paves the way to complete the task using a compass to construct a circle of radius *AC* with the centre at *A*, and then a straightedge for a line through the points *K* and *L* (Figure 9).

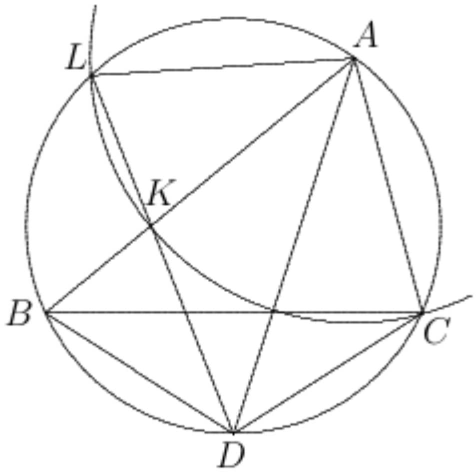

Figure 9: Solution

The following short transcript summarizes the most common responses received from participants.

Interviewer: What useful conclusions did you draw from this task?

Participant C: I think I got it. Isosceles triangles represent a good example of how some constructions with a compass can be interpreted. Using one vertex of a triangle as the centre of a circle and the equal sides as radius, we can construct the isosceles triangle that we need.

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TOPIC C

Resources for teaching and learning geometry

DESTRUCTURING / RESTRUCTURING OF A VIRTUAL GEOMETRIC ARTIFACT: THE CASE OF THE NUMBER LINE

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The use of artifacts in mathematics teaching has become a widely used practice with the introduction of laboratory activities. In this regard, even the advent of information technologies has given impetus to both research and new educational aspects. In particular, the use of artifacts virtually manipulated with the DGS (Dynamic Geometry Software) opens up new scenarios available to both teachers and students. In this work we will present an alternative teaching proposal of manipulating a virtual artifact: its destructuring (reduction into its fundamental elements) and its restructuring (reconstruction of the same artifact) to give students the possibility of also grasping the intrinsic information. We will analyze this didactic proposal of manipulative typology of an artifact in the case of the Number Line.

ARTIFACT AND ITS MANIPULATION

Mathematical concepts fundamentally face two difficulties in the communication phase. The first is the need for a formal language supported by a natural one that conveys it. From this it can be deduced that mathematics also needs words. The second is that its foundations do not have a counterpart in reality but only a representation of them, if one wants to adhere to Platonic vision (D'Amore, 2003). This situation, which involves the Author from a professional point of view as an Italian secondary school teacher, has prompted throughout history the 'materialization' of the concepts themselves into objects that contained this concept or were effective mediators of it for the students' use. According to what has been said, this materialization is achieved with objects called artifacts.

Artifacts can be of the most diverse shapes, built with the most varied materials and digital. Some are in common use (abacus, ruler, compass, etc.) others are used specifically for educational purposes as we will see in the case of the Number Line (Frykholm, 2010).

The useful theoretical reference regarding the didactic use of artifacts is in the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). Here the artifact differs from the tool where the first is a product of Man while the second results from the union of an artifact and a pattern of use (Mariotti & Maffia, 2018). The aforementioned theory analyzes the use of the artifact by the students and the process they follow in solving an assigned task. As far as we are concerned, artifacts can be considered both concrete and digital. In the use of the artifact, its semiotic potential is also highlighted, i.e. its close link between the meaning that is evoked with its use and the mathematical meaning itself evoked. In this context, the teacher has the role of activity planning and guide. The result is that the manipulation of an artifact puts almost the entire body and mind in motion and in relation in the student, involving manual skills, vision, the coordination between the two, cognition and metacognition.

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THE GEOMETRICAL NUMBER LINE AND THE NUMBER THEORY

The Number Line is an ideal didactic artifact practically unchanged from the 17th century since it was first used by J. Wallis (1616-1703) in *Treatise of algebra* (1685). In his drawing the negative part was indicated with the dotted line (Figure 1). For this reason it can be defined as a fossil artifact. However, it does not seem that Wallis had designed the segment with the intention of placing the known numbers to order them but only as an example of a 'road' traveled in one direction and/or the other, i.e. more as a reference system for motion on which, then, he operated with the relative integers.

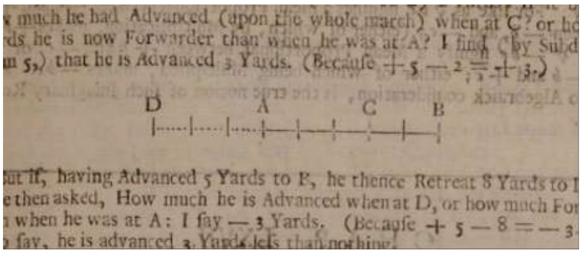

Figure 1: The first Number Line published by the English mathematician J. Wallis in 1685

The Number Line must be considered fundamentally abstract since its definition requires geometric points and ideally it has an infinite extension, which cannot occur in material translation. Therefore the artifact that physically finds itself in the hands of the students is a materialisation of those ideal geometric concepts.

The choice to study the Number Line derives from the Author's professional experience regarding the problems of secondary school students about numbers. It was observed that:

- numbers are all considered 'the same', highlighting the predilection for the 'integer' and/or 'decimal' forms as a consequence of the use of the calculator.
- difficulty in managing numbers written in 'mixed' form in which numbers, signs and symbols must be considered at the same time: $-7 \quad 2^n \quad \frac{4}{5} \quad 2\pi \quad \sqrt{2}$.
- the non-differentiation of numbers leads to a non-deep understanding of their use in certain fields (e.g. Science → measurement).
- the incomplete knowledge of the typology of numbers results in the inaccurate application and understanding of approximation techniques.
- difficulty in ordering the numbers due to their different writing.

Therefore, the geometrical visualization of numbers can be, together with their different way of drawing them, an aid to move with greater awareness in the fundamental notions of mathematics.

Before drawing them, let's see which numerical sets we can place on the Number Line. From number theory we have the following set relationship:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \equiv \{\mathbb{I}_A \cup \mathbb{I}_{NA} \wedge (\mathbb{I}_A \cap \mathbb{I}_{NA} \equiv \emptyset)\} \subset \mathbb{C} \equiv \{\mathfrak{Re}(\mathbb{C}) + \mathfrak{Im}(\mathbb{C})\}$$

where $\mathbb{I}_A$ are the algebraic irrationals (numbers that cannot be expressed as rationals but solution of polynomial equations with rational coefficients) and $\mathbb{I}_{NA}$ non-algebraic or transcendent irrationals (e.g. e, π). It should be noted that $\mathfrak{Im}(\mathbb{C})$ cannot be reported on the Number Line since the Argand-Gauss plane is required. Therefore, only the complexes with zero imaginary part, i.e. the real ones, will be considered.

The drawing of the Number Line will follow the method described in Euclid's *Elements* (3rd century BC) using only straightedge and compass (Hartshorne, 2000). Ruler and compass for the Greeks were simply tools for tracing straight segments and not for measuring lengths, for tracing circles with a given center and passing through a given point or having a given segment as a radius. These were ideal tools for realizing properties of incidence between points, lines and circles. Their exclusive use started a fruitful analysis of those problems that could not be solved with this method such as the trisection of the angle, the squaring of the circle or the duplication of the cube.

DESTRUCTURING / RESTRUCTURING OF AN ARTIFACT

We have seen how the artifact must be given to the students with an assignment. The achievement of the task is documented in various ways: written, oral, multimedia. All this allows the teacher to convey the external information *x* of the artifact. It must, however, be taken into consideration that the restructuring of the artifact itself was possible with another knowledge which in the moment of its final realization became intrinsic and not immediately capturable in the teaching activity (Rizzo, 2000). In order to exploit the full potential of the artifact it is necessary to also be able to transmit its intrinsic information *y*.

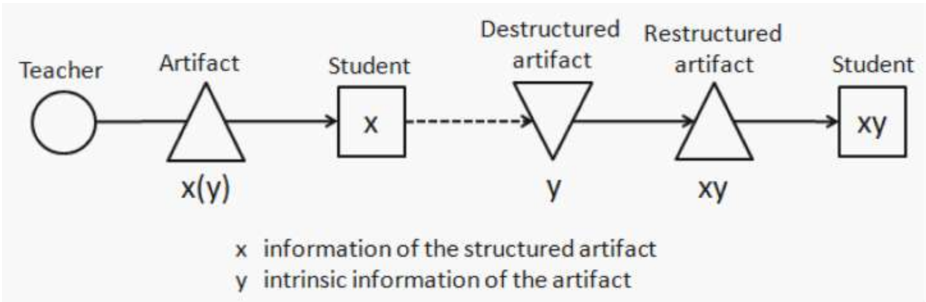

Figure 2: Logical diagram of the destructuring / restructuring of an artifact

The activity of deconstructing and restructuring the artifact begins in the classic way with the teacher delivering the artifact to the student (Figure 2). The student's activity operates on the artifact to conclude with the acquisition of information *x*. At this point, part of the artifact's metacognitive potential is lost. To make the intrinsic information of the artifact usable, the student must deconstruct the artifact itself (not a similar one!; see dotted arrow in Figure 2) to dissecting it into its fundamental parts. The destructuring phase is followed by the restructuring phase during which the student reconstructs the artifact by acquiring the two types of information *x* and *y*.

With this methodological approach, the Number Line is a useful geometric artifact that lends itself very well to the aforementioned logical scheme if transformed into a virtual artifact, i.e. if it is built and manipulated in a DGS environment like GeoGebra©, the environment used for our purposes.

The problem with the geometric Number Line in school textbooks is that this, apart from some variations in the drawing such as the presence of double arrows at the 'extremes' which create disorientation, is date already drawn with the numbers positioned. The delivery of such an artifact already drawn also involves the total disregard of the fact that, as will be highlighted, one does not discriminate between constructible numbers ($\mathbb{K}$ set) and those that cannot be built with straightedge and compass. The placement of the various dashes with the numerical equivalents is not defined, remaining an unexplained fact and to be accepted uncritically. Ultimately, the use of the Number Line in school textbooks is weakened. In fact, it is used in primary and the first year of secondary school, losing track of it in the following years when new numbers are introduced. All this does not allow for an in-depth critical analysis and discovery of the students guided by the teacher. To all this we must add that important historical knowledge is lost, i.e. the idea of a static mathematics is consolidated. On the contrary, the Number Line brings together millennia of theory much of which, like the identification of a geometric point with a pair of ordered real numbers by R. Descartes (1596-1650) and the rigorous definition of real numbers by R. Dedekind (1831-1916) and G. Cantor (1845-1918), relatively close to us. Obviously, in the case of the secondary school, the connection with the philosophical and epistemological themes is lost.

The geometric Number Line has the characteristic of one-dimensional space to satisfy the total ordering of the various numerical sets. For the Euclidean construction of the Number Line, the *Elements* make available, in addition to the two aforementioned tools, the following fundamental geometric entities (Sbaragli, 2005): a) the point (Deff. I,I; I,III; Post. I,I); b) the 'straight line', the segment in modern parlance (Deff. I,II; I,IV; I,VI; Post. I,II); c) the circumference (Def. I,XV; Post. I,III); d) the 'surface', the plane in modern parlance (Def. I,VII).

Below is the schematic description of the construction of the Number Line for numerical points that can be constructed starting from the set ℕ. The restructuring phase of the Number Line as virtual geometric artifact will also be an opportunity to think about numbers' features and their history.

Construction of ℕ

To construct ℕ it is necessary to choose a point to identify it with 0. Since we want to construct a straight line through points it is necessary to choose one in a one-dimensional space (geodetic line). But you also have to consider the straightedge and compass construction types that require at least a two-dimensional space or plane Π (Barbieri, Maschietto, 2012).

Having defined the geometric locus where to locate the point 0, we must ask ourselves whether by choosing it at random we are sure of finding that point. This requires that the plane consists of points that are compact and this property, in general, is not explicit in the *Elements* but it seems that it can be deduced from the Post. I,III. However, this certainty comes with the introduction of the Cartesian plane and the axiomatic analysis carried out by Hilbert at the end of the 19th century.

The placement of the 1 is very important as it identifies the unit referred to in Def. VII, I and for the Def. VII,III this unity, both arithmetic and geometric, will be the measure of all our construction (Figure 3). The choice of the point corresponding to 1 is completely arbitrary among those which are distant from 0 of the distance taken as unit. To do this, we use the compass to draw a circumference. At this point the basic structure of the Number Line for ℕ has been built following the Post. I,I. The segment $\overline{01}$ has deliberately not been drawn because we want to highlight the existing numerical void between two consecutive naturals since N is non-dense ($\nexists x \in \mathbb{N} \mid n \in \mathbb{N} \wedge n < x < n+1$).

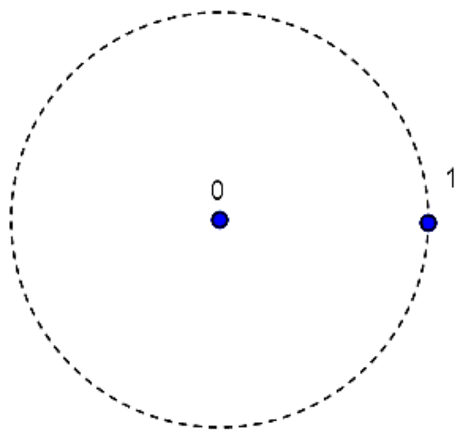

Figure 3

As for the construction by induction of the natural number consecutive to n, a geometric construction must be found to identify the point corresponding to 2 which is $\overline{01}$ from 1 and is on the straight line being constructed. Not wanting to enforce the Post. I,II the assurance that the construction of successive naturals is possible without the use of the straightedge is based on the Mascheroni-Mohr theorem (1672/1797). To satisfy the unit 'distance' from 1, draw the circumference with center at 1 and radius $\overline{01}$ by identifying the two points A and B on the plane (Figure 4). The constructability problem is to identify the point 2 on the circle with center 1 aligned to 0 and 1. This point can be identified as the opposite to 0 of the diameter $\overline{02}$ with respect to 1, viewed as midpoint. Therefore one must construct the vertex opposite 0 of a polygon inscribed in the circle with center 1. The candidate polygons are all those with an even number of sides (square, hexagon, etc.) and with a vertex at 0. Among these polygons it is preferable to choose the hexagon since its side is congruent with the radius of the circumscribing circumference. Ultimately, therefore, point 2 is constructed by pointing the compass at A or at B with amplitude $\overline{A1}$ or $\overline{B1}$ to identify, respectively, either point C or point D intersection with the circumference with center 1. Again, centering in C or D with the same amplitude identifies the point 2 aligned with 0 and 1. The construction of all subsequent natural numbers is carried out with the same procedure in sequence.

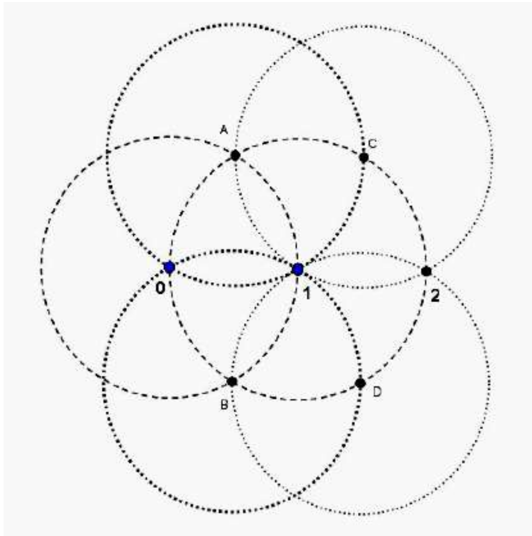

Figure 4

Construction of $\mathbb{Z}$

The relative integers $\mathbb{Z}$ will be constructed with the same procedure described for $\mathbb{N}$ but along the negative direction (on the left by Western convention).

Construction of $\mathbb{Q}$

In this case the set is dense i.e. $\frac{p}{q}, \frac{s}{t} \in \mathbb{Q}$, $\frac{p}{q} < \frac{s}{t}, \exists \frac{k}{m} \in \mathbb{Q} \mid \frac{p}{q} < \frac{k}{m} < \frac{s}{t}$ but not compact, i.e. $\frac{p}{q}, \frac{s}{t} \in \mathbb{Q}$, $\frac{p}{q} < \frac{s}{t}, \exists x \notin \mathbb{Q} \mid \frac{p}{q} < x < \frac{s}{t}$. If the subdivision of the interval between n and $n+1$ is equidistant, the number of rational points with denominator m to be identified will be $m-1$, unless there are any overlaps with previous points as the equivalent rationals (fractions reduced to lowest terms or not).

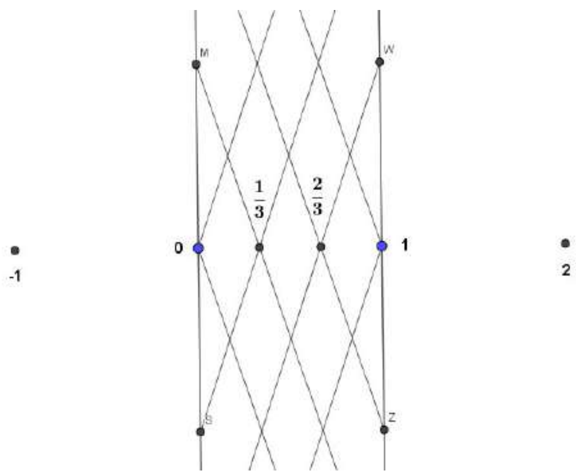

Figure 5

Due to the lack of compactness, we will continue not to take in account the segment that joins two consecutive naturals and we will use Thales' theorem, i.e. intercept theorem, with the following construction.

For the points corresponding to the two consecutive natural numbers, the straight lines $\overline{SM}$ and $\overline{WZ}$ are drawn with reference to Prop. I,11 and with the protocol of Prop. I,1 (Figure 5). For the identification of the *m-1* rationals along the aforesaid passing lines, in our case between 0 and 1, the $k_1, \ldots, k_m$ points equidistant from each other are identified, starting from 0 and 1, along the four half-lines. Subsequently the point 0 joins the k_m on the straight line passing through 1. The same is done by joining the point 1 with the k_m on the straight line passing through 0. The drawing continues by joining the remaining points of one perpendicular with those of the other perpendicular which they find in the half-plane opposite to the one containing them with a segment parallel to the outgoing segment, respectively from 0 or 1 and joining the opposite k_m. The intersection of the segments joining the points on the perpendiculars will identify the rationals sought.

Construction of ℝ

Among the reals, the simplest numbers to construct there are the radicals $\sqrt{n}$ with $n \in \mathbb{N}$. The design of these rationals is possible by applying the Pythagorean theorem (Prop. I,47) considering the formula

$$\sqrt{n} = \sqrt{\left(\sqrt{n-1}\,\right)^2 + 1^2}$$

that is, constructing a right triangle with one side congruent to unity and the other congruent with the immediately preceding natural one: the number sought will be that of the hypotenuse. Also in this case, not wanting to materialize the Number Line, we must consider that the construction with the Pythagorean theorem will have to be carried out in the two semi-planes identified by the unit segments constructed on the perpendicular passing through the point corresponding to $\sqrt{n-1}$.

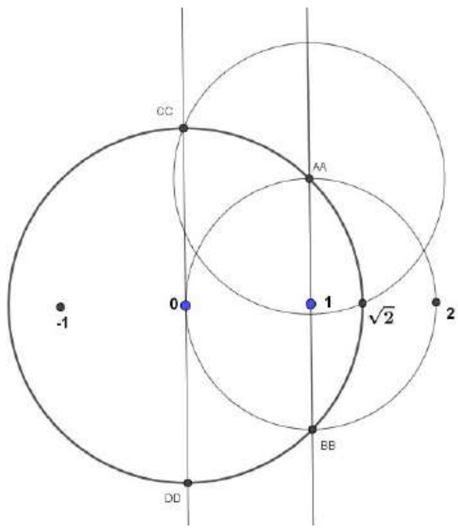

Figure 6

The drawing of the perpendicular passing through an algebric irrational not coinciding with a natural one poses the problem, if one uses the procedure seen for the construction of the rationals, of identifying two points along the diameter of the circumference with center $\sqrt{n-1}$. In reality, the problem is not in identifying both: it is possible to identify the distance with the immediately lower natural k as radius but it becomes impossible to identify, along the diameter of the circumference, the antipodal point at the distance $\sqrt{n-1} - k$, if not coinciding with a natural. To overcome this, one can make use of Prop. I,31 which allows the drawing of a parallel for a point external to a given straight line, i.e. the perpendicular drawn for the point identifying the natural immediately lower than the irrational considered. For example, $\sqrt{2}$ is identified by drawing the perpendicular for point 1 and on it the unit distance $\overline{10}$ (point AA) is reported (Figure 6). Then the circumference with center 0 and radius AA is drawn which is equal to the length $\sqrt{2}$.

This circumference crosses the numerical void between 1 and 2, so we need to draw an arc that identifies the point $\sqrt{2}$ aligned with the naturals. Considering that the triangle 01AA is isosceles and right-angled with a central angle equal to $\frac{\pi}{4}$, we have that AA is the vertex of the octagon inscribed in the circumference with center 0. Therefore, the point $\sqrt{2}$ is identified by intersecting the circumference with center 0 and the circumference with center in AA and radius $\overline{AACC}$.

Up to now we have identified the following points (for rationals there are only fractions with a denominator of 3) of a Number Line by points which gives a better idea than what is deduced in school textbooks with the presence of 'empty spaces' of non-constructible numbers, i.e. the place of the transcendentals (Figure 7).

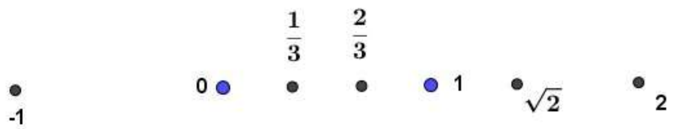

Figure 7: Some constructable points along the Number Line

To locate the geometric point corresponding to $\sqrt{3}$ you can repeat the procedure seen before but now the angle at 0 is 35.2° i.e. between 32.7° and 36° corresponding, respectively, to the angles at the center of an 11-regular agon and a 10-regular agon. This tells us that it is not possible to identify $\sqrt{3}$ starting from a vertex of the inscribed polygon as was done for $\sqrt{2}$. The construction with the Pythagorean theorem is similar to the one carried out for $\sqrt{2}$. In order to identify the point corresponding to $\sqrt{n+1}$ without prolonging the segment $\overline{01}$ we can apply both the drawing of a straight line parallel to a given point through an external point (Prop. I,31) and the 2nd Euclid's theorem (Prop. VI,8).

From what we have seen we are able to identify all the geometric points of the irrationals of even index. Those of odd index, such as $\sqrt[3]{2}$, cannot be drawn with straightedge and compass according to the theory of Galois groups. Even less constructively identifiable are the non-algebraic irrationals, i.e. the transcendentals such as e and π. At the end it is possible to draw the straight line according to the Post. I,II resulting in the only way to draw but not identify those numbers that form the continuum.

CONCLUSIONS

This proposal aims to highlight the intimate connection, both theoretical and didactic, between geometry and number theory mediated with DGS.

The aspect that is brought to light with the deconstructuring and restructuring of the geometric Number Line is that of a virtual artifact that is educationally useful and susceptible to an internal dynamic that allows us to grasp the fundamental elements of mathematics: numbers and geometrical points. In particular, it allows the study of constructible numbers both from a theoretical and historical point of view offering a geometric view to understand the different features of numbers.

About this didactic proposal, only the theoretical part concerning numerical sets was carried out in a secondary school in past years while the historical study and the constructability of numbers are activities that will be a subject of experimentation, always in secondary school, for the purposes of doctoral work. As regards the theoretical part carried out, obviously, in the secondary school the approach was more qualitative with, however, some operational considerations with the natural and the rational numbers up to the radical irrational ones in connection with the Pythagorean theorem and the relative design.

By highlighting the geometric aspect, the euclidean construction of the Number Line proves to be a useful tool for understanding the important and fruitful historical-scientific period known as the 'crisis of the foundations' which had in D. Hilbert (1862-1943) the protagonist of rigor of Euclidean geometry thus opening up other avenues of research (Hartshorne, 2000). For this reason, it is important to highlight the theoretical and historical aspects of geometry presented must also be seen as cultural enrichment of each teacher to modulate her/his didactic transposition.

Acknowledgements

This study is part of the doctoral research topic.

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DYNAMIC GEOMETRY IN PRIMARY SCHOOM

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In this paper we want to investigate Dynamic geometry Environment (DGE) in primary school. DGE is a way to develop powerful mathematics knowledge thanks to its ability to visually make explicit geometrical concepts. This theoretical possibility to better understanding geometry is not so easily to be shared by students in the classroom. The aim of this paper is to expound the epistemic value of DGE when reproducing figures in geometry in primary school, as a didactic milieu. Situations are designed by teachers and researchers in a cooperative engineering, based on habits and geometrical knowledge (didactic contract). Then they are implemented in the classroom. The shared understanding is a way to highlight geometrical properties. Doing so we want to show how the limits during teaching class when reproducing figures may be exceeded thanks to the DGE and a deep understanding by the collective of teachers and researchers.

INTRODUCTION

In this paper we want to investigate Dynamic Geometry Environment (DGE) in primary school. DGE is a way to develop powerful mathematics knowledge acquisition in a micro world (Leung, 2008). As we know, geometry is a kind of knowledge area based on two registers: the visualization of shapes in order to represent the space and the language for stating some properties (Duval, 2005, 2007). Furthermore the DGE's ability leads students to visually make explicit the implicit through the use of properties and the dragging shapes. The dragging mode is usually used to help students for proof and explanation (Hanna, 2000; Jones, 2000). But this research is up to students in high school (for example 9th or 10th grade). A few research concerns young students (4th or 5th grade, Assude 2005) or disabled students (Athias, 2021). We want to highlight the dragging mode to help these young students to develop their geometrical skills to better understand the properties in stake.

This work is grounded in a cooperative engineering (Sensevy & Bloor, 2020). Cooperative engineering refers to a methodological process in which a collective of teachers and researchers work together to design, implement, and re-implement a teaching sequence. In our case, it means to design a sequence about geometry by using a dynamic geometry software (DGS), GeoGebra (https://www.geogebra.org/classic?lang=fr), at primary school (4-5th grades). The cooperative engineering are built throughout different phases, which are not *necessarily* chronological. The first phase is a moment to work on the aim of the geometry teaching. It is necessary to focus on the links between geometric knowledge and the drawing. Each teacher member of the collective becomes familiar with the reproduction of figures with usual tools (ruler, compass, square set) and with the software in order to progress towards the conceptualization of geometric objects. The second phase focuses on the design of a sequence in geometry. Based on the knowledge acquired in the first phase, a teaching sequence is designed and implemented by the teachers, based on this shared understanding. Responsibility for implementation is shared by the whole team, teachers and researchers. The third phase involves actual implementation by one or more teachers. Most of the time, this implementation

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is followed by a session report, organized around the students' productions, screen copies of the blackboard or videos of the practice. Each video is commented on by the teacher, who becomes increasingly familiar with the sequence as the implementation progresses. The fourth stage involves an analysis of the implementation, by the group of teachers and researchers. This highlights the sequence's interests, limitations and prospects. The fifth phase involves iterative implementation, using the same approach as above. Finally, to conclude on these five phases, it's clear that the implementation and its analysis (phases 3 and 4) allow us to delve deeper into the geometrical knowledge involved (phase 1) and the improvement of the teaching sequence (phase 2). In addition, the new iteration (phase 5) enables the sequence to be reworked. In another way, the goals of designing teaching sequences and developing theories of teaching and learning are intertwined in a cooperative engineering. Moreover, cooperative engineering could be seen as fundamental research within an anthropological approach (Chevallard & Sensevy 2014). It aims to elaborate a theory of practice, and in the same time, it assumes an engineering function such as to build better educational designs (devices and practice). In this paper we develop this engineering function throughout a case study, which is remarkable: it represents what could be done in a sequence of geometry in primary school.

*I*n a first part we will present some theoretical concepts. In a second part we will show the possible limits when reproducing a figure with geometrical instruments (ruler, compass, square) in the paper and pencil environment during a specific lesson (at the end of the year), written by the team of the teachers and the researcher. In a third part we will try to expose how students use the dragging mode to represent a new shape in the DGE (at the end of the year), written by the team of the teachers and the researcher. In a fourth part we will engage a discussion before concluding.

THEORETICAL FRAMEWORK AND METHODOLOGY

Part 1: Figure versus drawing

I will explain the relationship between drawing, figures and theoretical geometry. Laborde and Capponi (1992) propose to reinterpret this distinction between drawing and figure by using the triplet referent, signifier, signified. As a material entity, the drawing (the visible trace on a material support) can be considered as a signifier of a theoretical referent (that object of theoretical geometry). The geometric figure thus consists of the pairing of a given referent with all its drawings. We can see it as a pair (referent/drawing, one of the drawings taken from the set of all possible drawings). In this approach, the relationship between a drawing and its referent is constructed by a subject, the signified of the geometric figure. A graphic representation gives access to the analogical dimensions of perception, while the text enables digital processing of the same theoretical object. It's precisely this point that interests us at primary school level. In the following I use the term of “figure” (Laborde & Capponi, 1992) to take into account that it’s not only a drawing (for example a circle), some geometrical properties are embodied thanks the use of tools.

In primary school doing geometry could be characterized by the use of instruments (such as ruler, compass and square set) to draw shapes in the paper and pencil environment (Mathe et al., 2022). Most of the geometry's knowledge in primary school is expressed in the same way as that of theoretical geometry, for example the circle. But a circle (in geometry) has to be characterized by a center and a radius (or a point of the circle). The geometric use of instruments (ruler, compass and set square) could be a way for conceptualizing the objects of theoretical geometry. Petitfour (2015) is interested in the potential of a geometric technical language "relating to the manipulation of instruments in connection with geometric properties, without mentioning these properties”. We consider this technical language in conjunction with the DGE. Selecting the "circle" button is like "taking the compass". If you can set the compass point in the paper and pencil environment, without having to explain it, you can't set a point anywhere on the dynamic figure. This is because the point moves with the mouse, as it is a "free" point.

I wonder how the change of instrument could encourage students to act on geometric properties.

Part 2: Joint action theory in didactic

I want to investigate what it happens in the classroom when the students use these different instruments (ruler, compass, set square and DGS). I will analyze it throughout both concepts of didactic contract and the didactic milieu. The didactic contract (Brousseau, 1997; Sensevy, 2014) can only be understood in relation to the didactic milieu. That is to say, when students work on a problem (any situation that confronts the student with a difficulty action, here reproducing a figure), they will look for ways to solve the problem within their existing knowledge and habits (the use of the geometrical instruments). The didactic milieu could be seen as the problem the students have to deal with. Recognizing this clarifies how the already-there capacities of the didactic contract only become meaningful in relation to the problem that the student must work on. These capacities are necessary but not sufficient to successfully undertake this work, otherwise the student would not learn anything new. There is thus a kind of initial imbalance between the contract (the already-there, which mainly results from the previous joint action) and the milieu (that which is to be known, and which is embedded in a specific symbolic structure). A new aspect of the didactical contract due to the DGE must be stressed (Laborde, 1998). Some pupils try systematically some menu items on every object: they hope to discover the solution or a hint. The effects of the dragging test could lead the students to have some clues to reproduce the figure.

The question which is addressed here is the following: how reproducing figures in the paper and pencil environment and the dynamic geometry environment could lead young students to a better understanding of geometric properties?

To try to answer this question we rely on various data. These data come from the implementations of a sequence in geometry based on the DGE and in particular on classroom videos and students’ drawings. Moreover I add a lot of exchanges during the cooperative engineering meetings. In this paper I have chosen to describe and analyze what happens in the classroom of one of the teachers who is part of the team of two teachers and one researcher. This class (4 and 5th grade in France), while being specific, is part of the team’s culture and the culture of dynamic geometry. The teacher and his students worked with the software early in the year. For example, they reproduce figures in the paper and pencil environment and in the dynamic environment. The students are on their own to reproduce the figure. The teacher is there but she doesn’t help. The specific session was video-recorded and transcribed. I selected two moments in particular since the geometric properties are tangible, but not achievable. The tasks are co-designed by the team as mentioned before. Their implementation led us to a better understanding of the link between both environments and their strengths and limits.

RESULTS

Result 1

The compass doesn't allow to take into account some geometric properties of a specific figure. The student can’t become aware of the properties.

I will describe and analyze the situation in the following. It was designed by the team of the teachers and the researcher. We decided to let students acting alone. The specific figure I want to present now is the circle into the square. The teacher S gives students this figure on the white board (cf fig. 1).

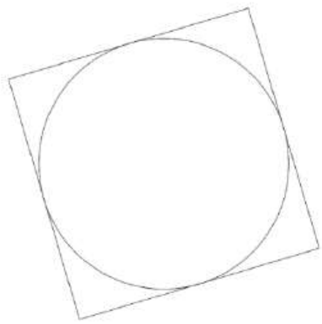


Figure 3: the circle into the square

The first task given by the teacher is to let students to describe the figure. The aim is to perceptually recognize a circle, a square and the link between the two geometric objects: the circle is internally tangent to the square. For elementary school pupils, the circle is inside the square, the circle “borders” the square. Only perceptual recognition of geometric objects is expected. For students from 4^{th} or 5^{th} grade, there's no problem.

The second task given by the teacher is to let students to reproduce the figure using geometrical instruments. They have to begin by drawing the square. I explain the choice we made of drawing the square first. Drawing the circle and then the square means using the tangents to the circle, drawing the radii first. Discussions within the engineering team have ruled out this possibility.

In this paper I am only interested in the construction of the circle. During the description phase, nothing was said about the characteristic elements of the circle. This is up to the students.

The teacher wanted the students use geometrical knowledge. One can notice that the center of the circle is not written. The students have to analyze a geometric figure with a view to reproducing it using instruments. This implies that students can make hypotheses (more or less explicit) about geometric properties and check them. The prior knowledge (the didactic contract) that pupils need to understand the problem and draw up a procedure is: (recognize a square and construct it with a ruler, compass and square) recognize and draw a circle with a compass, knowing its center and radius or a point. Pupils must reproduce the figure, without measuring.

Now I present a student in the classroom.

A student was drawing the figure. After drawing the square he took his compass. He hesitated where to put the tip of the compass. He decided to draw the diagonal of the square thanks to his ruler. He took the compass again, pushed one leg of the compasses into the paper with the spike, put the pencil on the paper approximately on the square and moved the pencil around while keeping the legs at the same angle. At the end he puts the model and his figure next to each other (c.f. Figure 2).

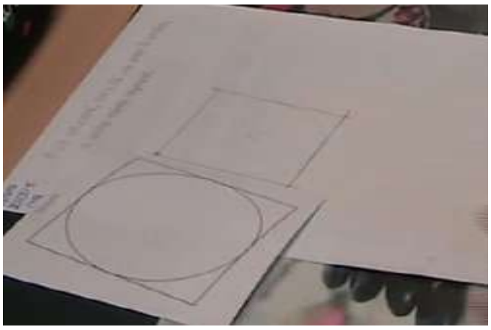


Figure 4: Both figures

The center of the circle is the point of intersection of the diagonals. At first the feedback of the milieu throughout the use of the compass in the paper and pencil environment is useful: the student is able to deal with it (you can see this extract here: https://mediacenter.univ-fcomte.fr/permalink/v1263dd7dfe545uq3gcy/iframe/). A kind of initial imbalance between the contract and the milieu is achieved. Then to finish the circle a point of the circle must be determined, here the midpoint of side of the square. The student's habits thus enable him to act in such a way as to solve the reproduction question. But they don't allow him to look any further into geometric relationships.

Result 2

In the dynamic environment, the dragging test highlights the "invisible" point (the midpoint of side) and allows the student to take into account the geometric properties of the figure.

I will describe and analyze the situation in the following. This time the teacher S asks the students to reproduce the same figure thanks to the DGS GeoGebra. As before, I describe a student reproducing the circle (c.f. Table 1).

The student is familiar with the software and the dragging test. He knows that he has to drag any point. The student is able to say that the reproduction is not right. Habits and geometric knowledge in the dynamic environment enable the student to begin appropriately, to understand the feedback of the milieu. After a few attempts the students is to visualize that point G must be declared as the midpoint of [AB]. As a lot of students, this student uses the well-known button "midpoint of a segment". But, as Laborde (1998) explained, some others tried some other buttons, for example, the unknown perpendicular bisector. After all, in the dynamic environment, the dragging test highlights the point, the midpoint of a segment, "invisible" in the paper and pencil environment.

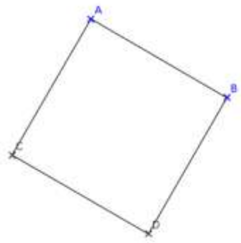
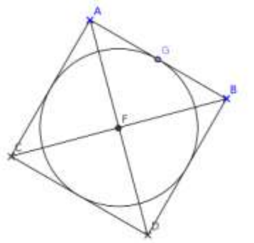
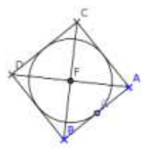
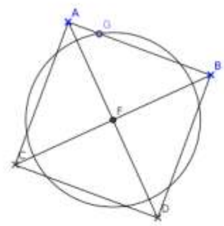
The square is already done	The circle is done	A and B are moved	The circle is outside the square.
			
We don't look before that moment: we want to inquiry how the student finds the characteristic elements of the circle, the center and a point of the circle.	The student drew the two diagonals of the square. He named F the point of intersection of the two diagonals. He chose the "circle" button. The circle moves with the mouse. A point G is created by a simple click on the segment [AB].	The student used the drag-test, by dragging the points A, B and C. The circle seemed to be fine.	The student move the point G. This time, the circle is out of the square.

Table 1: What the student does

Result 3

The teacher plays a decisive role in clarifying the geometric properties involved in a reproduction task.

I will describe and analyze the situation in the following. After that moment, the teacher summarizes the work done: "when you make a circle, you need to know its center and one of its points". Doing so she is able to point out knowledge issues to students. As we explained, the situation was written by the team of the teachers and the researcher. Even if the responsibility of the implementation of this situation is shared by the collective of teachers and researchers, the teacher has to challenge this situation in the classroom. She has to share her understanding of geometrical concepts. During a meeting, she explained to the collective that she had found it difficult to help students in the dynamic environment. She suggested changing the moment of description of the figure (description phase). The following year, when she proposed the figure (the circle in the square), she distributed a sheet to each student, asking them to use their instruments and code it. Only after this individual moment of analysis did, the students described the figure (c.f. Figure 3).

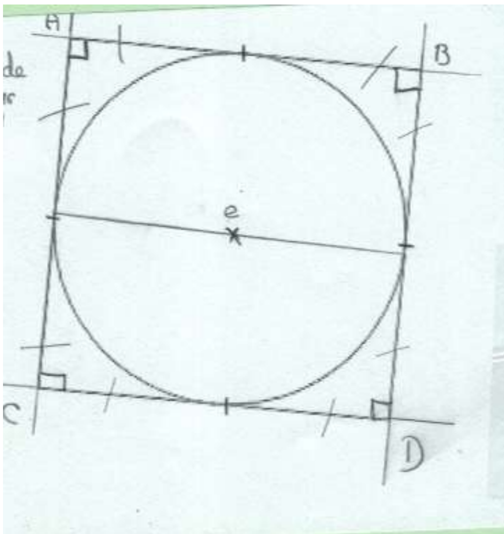


Figure 5: An example

I chose this example to show how the students become aware of geometrical properties. This student named the points (around 20 students over 23 in the classroom). This student showed that she take into account the midpoint of the segments (around 18 students over 23). He pointed the center of the circle (around 15 students over 23).

Based on this first analysis, the inquiry is easier. When students do not define the midpoint of the segment in the dynamic environment, the teacher can then refer them to the preliminary analysis of the figure. The distance between the already-there and the problem is adequate. It is thus a kind of balance between the contract and the milieu. What's more, these moments highlight the need for the two characteristic elements of the circle, in the paper-and-pencil environment and in the dynamic environment. As a result, recognition of the work done "when you make a circle, you need to know its center and one of its points" is more effective.

DISCUSSION AND CONCLUSION

Our research question is about reproducing figures in the paper and pencil environment, and in the dynamic geometry environment, to lead young students to a better understanding of geometric properties. This situation illustrates the need to declare geometrical properties in the dynamic environment, whereas the need for this property is invisible in the paper and pencil environment. However, at elementary school level, it is not possible to justify this property: we need to show the midpoint of the segment, but it's not possible to explain why. It is based on the visualization of shapes (Duval, 2005). Furthermore, the teacher summarized the geometric work done. She was able to understand the geometrical properties at stake. Substantial international research into the use of digital technologies in the teaching and learning of mathematics has yielded powerful results. Despite the massive investment of resources, impact on the reality of school practices has been limited (Lagrange & Kynigos, 2014). I am searching a way to reconsider the research results by working together with the teachers. The situation we want to provide in the engineering work leads us to a better understanding of the link between both environments by the teachers and the researchers. The visualization enabled by dynamic geometry increases the power of action for both students and teacher. Students who are able to use geometric concepts in their everyday lives: "I see the circle with center A that passes through B (Athias, 2019). Teachers and researchers are taking into account the importance of planning appropriate situations collectively. Based on these results, we have developed a program for pupils with long-term learning difficulties (Athias, 2021).

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THE NOTION OF ANGLE AND THE GGBOT AS A TOOL-TO-THINK-WITH... OR WITHOUT

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This study is about how geometrical reasoning can be supported through experiences with artifacts. Specifically, taking a semiotic perspective, we focus on the potential of primary school activities exploiting a drawing robot, the GGBot, a descendant of Papert's robotic turtle, in relation to the concept of angle. We analyze the interview of a 5th grader who participated in such activities to highlight his way of seeing the contour of a figure, that, we claim, showcases the extent to which GGBot's functionalities can become a "tool-to-think-with" (or, actually, without).

INTRODUCTION

Managing the theoretical nature of mathematical objects and the spatial nature of their (physical or mental) representations is a complex process with a long development (e.g., Fischbein, 1993). Geometrical reasoning is strongly connected with developing various ways of seeing and managing representations of geometrical objects, also making use of analytical reasoning (Duval, 1995). Here we will consider analytical reasoning as a specific kind of geometrical reasoning, that is related to recognizing subunits and composing and decomposing representations of figures (Duval, 1995).

It is widely recognized that the use of artifacts can provide experiences promoting multiple and dynamic representations of geometrical objects, that with appropriate tasks can be highly effective in fostering analytical and geometrical reasoning (e.g., Leung et al., 2023). This study provides an example of how during such experiences the artifact can support human cognition, becoming a tool-to-think-with, and it can proceed to become internalized, giving birth to a psychological tool (Vygotsky, 1978). More specifically, we focus on the geometrical notion of angle, because it is crucial in primary school geometry and it is one of the most challenging and misleading geometrical objects (e.g., Fischbein, 1993; Devichi & Munier, 2013), although it is experienced quite early in school (e.g., Bartolini Bussi & Baccaglini-Frank, 2015). Indeed, such notion is epistemologically challenging: for example, as described by Freudenthal (1973), when considering a pair of half lines with a common origin, in *Elementary Geometry* an angle is viewed as a static part of a plane comprised between such an unordered pair, while in *Goniometry* it is viewed as a dynamic turn between an ordered pair.

The protagonist of our study is Sam, a 5th grade student who learned about angles in grades 4 and 5 through a sequence of activities with GGBot, a descendant of Papert's drawing turtle (Baccaglini-Frank & Mariotti, 2022) that we will introduce in the following sections. Here we analyze part of a task-based interview about a polygon and its angles. The analyses focus on the signs Sam produced through geometrical and analytical reasoning. We will argue that such results showcase the extent to which GGBot's functionalities can become a "tool-to-think-with" (or, actually, without).

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THEORETICAL FRAMEWORK AND RESEARCH QUESTION

This study is grounded in *Theory of Semiotic Mediation* (Bartolini Bussi & Mariotti, 2008), a Vygotskian socio-constructivist theory, according to which students are guided by their teacher to construct mathematical knowledge by solving appropriately designed tasks with the use of appropriately chosen artifacts – that is, artifacts and mathematical tasks that are considered helpful for unfolding the target mathematical meanings. During a first phase of the *didactic cycle* students use the *artifact* (e.g., a physical compass, the dragging tool in a Dynamic Geometry Environment) to solve a task and develop *utilization schemes*. The artifact together with such schemes constitutes an *instrument* in the sense of Rabardel (1995), through which the students produce *artifact signs*, closely related to the task and to the artifact. We can now look at two developmental processes that the activity of working with an artifact can foster.

One process concerns the unfolding of the artifact's *semiotic potential*, the artifact's double relationship with personal meanings (related to artifact signs) and with mathematical meanings (related to *mathematical signs*), that the teacher aims to construct through the orchestration of mathematical discussions during the didactic cycles.

A second process concerns the student's *internalization* (e.g., Bartolini Bussi & Mariotti, 2008), the "transformation" of what has been a mainly external process into an internal process. At this phase the student will produce artifact signs and possibly make references to the artifact, even in absence of the artifact. Moreover, this process can go a step further: the instrument can be used as a *psychological tool* (Kozulin, 1998) to reason about and solve a problem. At this phase it is possible to observe the emergence of artifact signs, and possibly mathematical signs, without any observable physical action or presence of the artifact. In other words, it will have become a tool-to-think-with(out).

The semiotic potential of the GGBot with respect to the notion of angle

How does this apply to our study? Our artifact is the GGBot (short for "GREATGeometryBot"), an object moving on wheels (see Figure 1), similar to Papert's turtle, that can be programmed using a graphical block-based coding language similar to Scratch (it is realized in SNAP!). The two back wheels and a metal sphere in the front allow it to move on the plane (Figure 1).

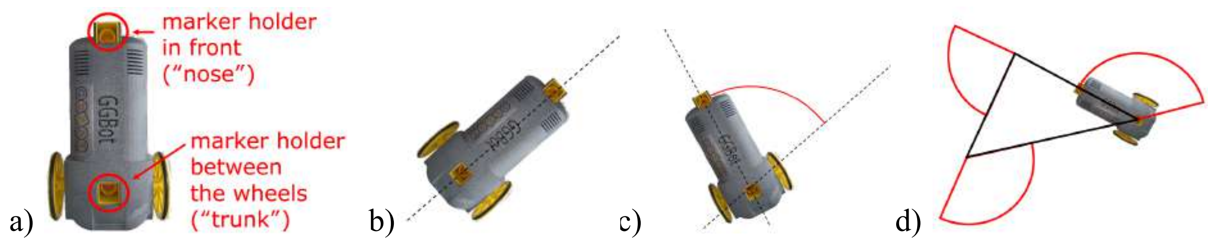

Figure 1: a) top view of GGBot with nose and trunk marker holders; b) translation movement direction; c) direction of departure and arrival after a rotation with vertex at the "trunk" marker, and trace mark left by a red marker in the "nose"; d) trace marks left if the GGBot is programmed to "do the path" of the black triangle.

GGBot has two marker holders, one in front ("nose") and one between the wheels ("trunk") so that it can "provide situated signs that can be elaborated into geometrical notions – such as segment, vertex, angle, rotation, polygon – while still carrying the situatedness given by the real movement of the physical artefact" (Baccaglini-Frank & Mariotti, 2022, p. 2660). The possible movements (translations and rotations) and trace marks are shown in Figure 1.

GGBot's semiotic potential with respect to the notion of angle consists in the relationships between the signs it can produce as it "turns changing direction" and seeing an angle as the rotation that takes one side of the angle onto the other (see Figures 1b and 1c). Concerning angles of polygons, GGBot's greatest semiotic potential is respect to "exterior angle(s)", seen as the change(s) in direction of a person walking along the contour of the figure (Figure 1d).

Such potential can only be realized through well-designed tasks. A dual pair of tasks that were used frequently during the didactic cycle in Sam's class, were the following: the *planning task*, or *figure-to-code task* (students are given the name of a figure and asked to produce a code with the blocks, working in pairs, to make GGBot draw the required figure on the paper); the *prediction task*, or *code-to-figure task* (students are given a code and asked to predict the trace mark GGBot will leave on the paper when executing such a code).

As Baccaglini-Frank & Mariotti (2022) have highlighted,

> "an essential feature of the semiotic potential of this artefact is its building on the relationship between the global movement and its breaking up into steps and turning points and the geometrical meaning of a polygon at a global and an analytical level. From a cognitive point of view […], the task consists in breaking down a path that is imagined to be generated through physical continuous motion, into geometrical elements […] of a different nature: they are static and discrete." (p. 2662).

Specifically, concerning the notion of angle, the artifact GGBot can support the development of geometrical reasoning stemming from a "temporal" and dynamic conception, similar to the one described by Freudenthal in his Goniometry. Moreover, the dual tasks described above have the potential of eliciting analytical reasoning, since in both cases a "whole" figure needs to be decomposed into (or recomposed from) smaller components: in the case of the angle these are single (parts of) sides, with a vertex in common and with a specific reciprocal inclination, given by "how much" GGBot needs to turn to face first one direction and then the other.

Research questions

Our questions are grounded in the theoretical framework above, and we ask specifically: Is it possible for GGBot to act as a psychological tool for solving primary school geometry tasks? If so, how can such a tool support students' analytical (recognizing components and using them to compose/decompose a figure) and geometric reasoning (recognizing and using specific representations of) concerning angle?

METHODS

We analyze here part of a task-based interview with Sam, a 5th grade student, whose class had worked with GGBot since the beginning of 4th grade, following the proposals of the *PerContare* project (teacher guides designed by many researchers including the first two authors, see Bartolini Bussi & Baccaglini-Frank, 2015; Baccaglini-Frank et al., 2023). Initially the activities only made use of the marker in the trunk and of right-angle rotations, for both planning tasks and prediction tasks; subsequently, also the marker in the nose was inserted, and through open problems the students were encouraged to discover new commands with parameters for controlling the amplitude of the rotations (turns) and the length of the translations (steps).

Sam's interview was part of a larger data collection aimed at assessing students' learning at the end of a 3-year cycle of teachers' use of the *PerContare* materials. After assigning and collecting a written set of tasks to 112 5th graders in 5 classes, two researchers (the second author is one of them) visited each classroom and individually interviewed selected students in a quiet room outside the students' regular classroom. Sam was one of the students selected because of the mathematical depth of some of their answers; others were selected because of their mistakes or reference to specific artifacts that helped them think. During the interview, the students were shown their previous written work and asked to further explain their thinking. Moreover, the researchers would ask additional questions when they noted clues of interiorization of a particular artifact. This was very clear in the case of Sam. Figure 2 shows: (2a) the original question ("How many internal angles does this figure have?", followed by the request to explain own reasoning; (2b) Sam's first and (2c) second drawings in response to researchers' questions.

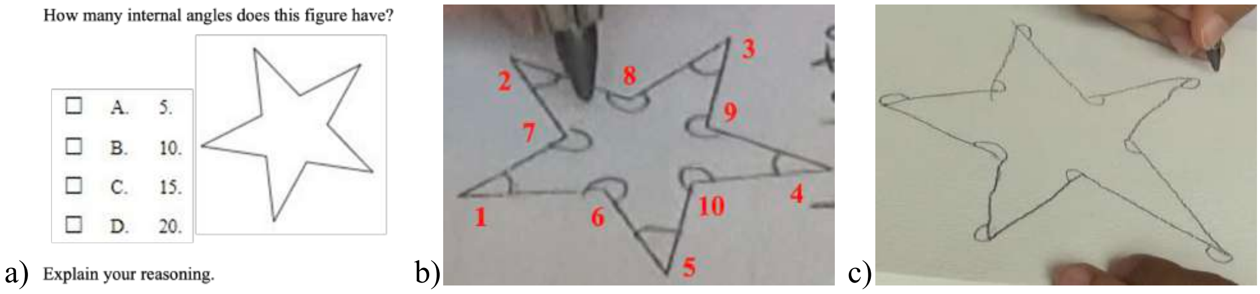

Figure 2: a) original task; b) first figure Sam used to explain his reasoning with numbers indicating the order of his counting; c) Sam's second figure, showing what "the nose would draw"

We selected this task and Sam's answers corresponding to figures 2b and 2c because of Sam's explanation of the relationship between angles drawn with markers in the trunk and in the nose of GGBot, and because the original question introduces the slippery term "internal angles" that frequently induces incorrect answers (this was an item of Italian national assessment test for grade 5 in 2009: in the national student sample, the percentage of A answers was 66%). GGBot was not available in the physical or virtual form during the interview.

To perform the analyses we did the following. We transcribed the interview and searched in the text and video for artifact signs and mathematical signs (verbal utterances, written signs on the paper used, gestures). Since the GGBot was never available during the interview, we interpreted the artifact signs as hints that the internalization of GGBot as a psychological tool. We then related such artifact signs to the mathematical signs produced by Sam, and we proceeded to identify instances of analytical thinking (when the signs are used to distinguish components or possible decompositions or re-compositions of angle) and geometrical thinking (more in general, when the signs are used to talk and reason about representations of angle), till we reached agreement.

ANALYSIS

We selected 3 excerpts that are particularly illuminating with respect to Sam's internalization of the GGBot relative to the notion of angle. We provide summaries of skipped parts around each excerpt.

Excerpt 1: How would GGBot do it?

In the written task, asking for the number of interior angles in the star (Fig. 2a), Sam had answered "10" and explained on the sheet of paper: "I calculated all the internal angles (without falling in the trap)". When one of the interviewers asked to say something more about the "trap", he explained the way he counted the angles producing the Figure 2b. Referring to such angles, to better explain how he reasoned, Sam mentioned "turns" (Italian: "svolte"): the interviewers immediately asked him more about what these referred to, and he spoke of the movements of GGBot (original excerpt here).

Sam: So let's assume it starts from here: it has to go straight and turn, so this is already an angle. Then it has to go straight and turn, and it's 2 angles. [He repeats the refrain "go straight and turn" four times] Straight, turn, 7. Then it has to go straight again and then turn, and it's 8. It has to come, uh, it has to go straight and then turn, that's 9. Go straight and turn, and it's 10… and then come back to where it started from.

Table 1 shows correspondences between GGBot's movements, Sam's verbal utterances and his gestures associated to the figural units he identifies in the star (segments and angles).

Movement	Verbal utterances	Hand gestures
Step forward	"go straight" or only "straight"	He sweeps a segment with the right index finger
Rotation	"turn"	He points the index finger at a vertex and rotates it; after counting angle number 5, the finger rotation is less emphatic

Table 1: Overview of GGBot's predicated movements, Sam's verbal utterances and his gestures

Sam temporalizes the figure by looking at its contour as a path to be followed – in this attempt we see an instance of internalization: Sam talks as if an imagined GGBot is tracing the star. Internalization is also evident in the turning gestures: they change during the excerpt, becoming less emphasized as if the movement is just imagined or unessential. The small arcs Sam had previously drawn (see Figure 2b) are different from those that the GGBot would draw, but this confirms our point: internalization is more than a replication of a past experience. Rather, Sam uses GGBot's movement as a (psychological) tool for supporting analytical reasoning and detecting angles in the spatial and dynamic essence of its figural units: just a turn preceded and followed by a straight path.

Excerpt 2: What about the GGBot's nose?

The interviewers, curious about the marks that Sam uses to count the times GGBot turns, and so the angles of the star, and their correspondence to marks that either of GGBot's markers would leave, ask Sam what GGBot would draw if it had markers both in the trunk and in the nose (original excerpt here). Sam produced Figure 2c and the signs analyzed in Table 2.

Again we can see how GGBot supports analytical reasoning to decompose the path into a combination of translations and rotations, but there is more. The predicted movements of the GGBot's nose are used as psychological means to solve a new task and not just to reproduce a known procedure. Indeed, the deixis (see "it makes *this*" when Sam is drawing the arc, Table 2) manifests a deeper identification between Sam and the robot: Sam's gestures are no longer just a reference to the movement of the GGBot, but gestures and GGbot's motion overlap intimately. Sam talks as if the hand holding the pencil *was* the GGBot. Moreover, Sam does not trace the actual angle that GGBot "makes with the nose", but his verbal and gestural productions suggest that it is just imagined. When Sam claims that GGBot "lies here on the tip", the gesture points to the lengthening (just mimicked) of a side (see

Figure 4) which corresponds to the direction taken by a fictional GGBot. In absence of a physical artefact, Sam's gestures are tools that realize the body syntonicity with the imagined GGBot.

Predicted nose trace mark	Verbal utterances	Pencil gestures
Figure 3 	"go straight" or only "straight"	The pencil runs along one side
Figure 4 	"it makes this" or "it lies here on the tip and it makes this with the nose"	The pencil draws an arc from this vertex to an inner point on the consecutive side; it moves back and forth along the side around the vertex, and then along the drawn arc.

Table 2: Predicated nose trace mark, Sam's verbal utterances, and pencil gestures from excerpt 2

Excerpt 3 – Comparing different signs for angles

Intrigued, the researchers ask Sam to better explain the difference between the angles he drew in Figures 2b and 2c (original excerpt here). He explains:

Sam: Because GGBot instead of making the angles as they are on the protractor, it works with the supplementary angles, if I am not mistaken, which are on the basis of 180. Because if we made a 90-degree angle [he draws a right angle]… oh god, maybe with 90 you don't understand so much....

and, invited by a researcher to show a drawing, he produces the signs in Table 3.

Predicted trace marks	Verbal utterances	Pencil gestures
Figure 5 	"an obtuse [acute] angle"	He draws an acute angle
Figure 6 	"instead of going straight", "it will go here on the brink"	He traces the extension of one side of the angle; points at the acute angle; runs along one of its sides and stops at its vertex
Figure 7 	"with the nose, to turn, it will do this", "on the basis of the supplementary angles", "let's say this is 30 […] and this 50 [degrees]"	He draws an arc from the vertex of the acute angle to its other side; then he draws more lightly also an arc for the obtuse angle

Table 3: Predicated nose trace mark, Sam's verbal utterances, and pencil gestures from excerpt 3

Sam is now using formal mathematical signs and the GGBot's movement to focus on the features of different representations of angle, demonstrating sophisticated geometrical reasoning. He speaks of "supplementary angles" to distinguish angles made by GGBot from those of the protractor (another artifact that he had the opportunity to experience). Interestingly, he uses dynamic expressions (*do, make, works*) to indicate movements when referring to the former, and static ones (*are on*) when speaking about the latter. His control over the mathematical concept of supplementary angles seems quite strong: he starts with an example (right angle) but quickly changes his mind, acknowledging that in such a special case, his example would have been less understandable, since both the exterior angle and the interior one would have both been right angles. Sam then continues to construct his example, mixing artifact signs and mathematical signs (Table 3): the verbal utterances about GGBot's movement are intertwined with more general and atemporal forms ("obtuse angle", "on the basis of the supplementary angles", "let's say this is 30"). Moreover, this example allows him to better explain what he was referring to in excerpt 2 (Table 2), since after that he re-draws the arc associated to the action of "turning" starting from a side of the angle, not its vertex: this is coherent with the "nose" marker's trace, i.e. with the actual angle of rotation of the GGBot.

Shortly after, Sam identifies a pair of supplementary angles, and he says:

> Sam: …ah no, sorry. This one is 30 and this one is 150. He is turning, he makes an angle of 150 degrees [he retraces the small arc]. Internally there is 30 [he retraces the small arc] and together they make 180 [he marks the two horizontal sides a whole]. Because maybe we see the initial angle like this [he retraces the sides of the acute angle], but the angle he sees is this [he retraces the sides of the obtuse angle].

Sam talks about the two angles drawn earlier (Figure 7) by mixing different signs: the obtuse angle is drawn turning (It: "svoltando"*)*, while the acute angle *is* inside. The verbal expression "there is" addresses a static dimension that is less related to the process of drawing the angle, while the gesture of retracing the small arc in the acute angle recalls the trace mark GGBot would make when rotating. Possibly, these different takes on angles mirror two way of looking at – and reasoning about – angles: one way is static in nature – angles *are somewhere* and, for example, "they are on the protractor" – and the other is dynamic and recalls the angle that the GGBot "makes turning". The dynamism is partially overcome by the whole figure representing supplementary angles that Sam now sees as two adjacent angles, as the gesture reveals. In the last utterance the dynamism has been completely overcome, as if the original angle that "we see" and the one that "he [GGBot] sees" have blended together, giving rise to something new, indicated through a single repeated gesture with the pencil along their sides. This interpretation is strengthened by Sam's use of "see", instead of "do".

CONCLUSION AND DISCUSSION

We set out to explore whether GGBot could act as a psychological tool solving primary school geometry tasks, and found compelling evidence that, yes, it is indeed possible. The case of Sam allowed us to gain insight into how such a tool supported his analytical and geometric reasoning on a specific task. Digging deeper into what Sam had internalized, we discovered how GGBot had supported his conceptualization of the notion of angle, and in particular of supplementary angles, proving that it plays the role of tool to recall and productively use geometrical notions, developing ever more effective objects-to-think-with (or, in our case, without).

Our analyses revealed how working with GGBot led Sam to re-thinking a figure, decomposing it and using analytic reasoning to relate the commands, movements, trace marks and figural units in a global whole. This is possible not only through the use of GGBot, but thanks to the dyad composed of the artifact and the dual tasks we have designed, which elicit processes of (de)composition and anticipation. Specifically, we discovered that the sequence of activities with GGBot that was proposed to Sam led him to conceive angles as rotations, but also as geometrical figures comprising two sides and an "inclination" between them, recognizing different kinds of angles (right, acute, obtuse), their mutual relationships (two supplementary angles) and angles of closed figures (internal and external). In particular, Sam constructed a theoretical lens that let him interpret the expression "internal angles" in the initial task coherently with formal geometry and correctly solve it, differently from many other Italian students in the national assessment. Moreover, thanks to synergies with other artifacts such as the protractor, Sam merges dynamic and static experiences, reaching a generalized static and atemporal (but with dynamic features encapsulated within) conception of angle.

In this perspective, the learning path through the *PerContare* activities with GGBot fully exploited the artifact's semiotic potential, leading Sam to develop a psychological tool that he can use in different ways, unpacking as necessary the interiorized experiences when necessary, for example, to "see" a geometric figure as a part of the plane enclosed by consecutive segments, or supplementary angles as the "pair" of angles, one "seen from the figure" and the other "seen by GGBot" as it traces the contour. Of course there remain many open questions and issues to study in more depth. Just to mention one in which we are particularly interested, we are curious to explore how the dual tasks presented can be harbingers of similar activities in dynamic geometry (with step-by-step construction open problems) useful for enhancing key geometric skills (Leung et al., 2023).

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A STRUCTURED PRACTICAL ACTIVITY TO ENHANCE UNDERSTANDING OF CIRCLE CIRCUMFERENCE AND π APPROXIMATION

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This paper presents a structured practical activity designed to enhance middle school students' understanding of circles in geometry. The activity focuses on promoting "relational understanding" and deepening students' engagement with concepts such as the number π and the formula for calculating the circumference of circle. The findings showed the potential of practical structured activities in enhancing students' conceptual understanding of mathematics. This research contributes to the teaching practices in geometry education, emphasizing the importance of experiential learning. To conclude, the structured practical activity demonstrates its effectiveness in advancing students' understanding of circle-related concepts.

INTRODUCTION

The circle, a simple closed curve, has held its place in human knowledge since long before recorded history. As far back as 1700 BC, the Rhind Papyrus provided methods for calculating the area of circular fields. As well, the history of mathematics reveals the profound influence of the circle, inspiring the development of geometric and calculus concepts (Koestler, 1990). The fundamental nature of a circle has made it as a cornerstone for mathematical understanding and has become as essential part of school mathematics curricula worldwide. This paper reports on a segment of a larger study that aimed to explore the ways for enhancing middle grades students' comprehension of geometrical concepts. Specifically, this study focused on a structured practical activity designed to promote "relational understanding" in the context of calculating the circumference of a circle . Traditionally, students were introduced to the approximation of π as 3.14, a key constant in mathematics that allows for the calculation of a circle's circumference as π times its diameter. While this formula has often taught by rote learning, sometimes to justify its correctness, students are asked to measure the diameter and circumference of various circular objects such as coins and rings and calculate their ratio. The findings will help to design structured practical activities for improving middle grade students' understanding of circle and its properties. The focus of this study was to explore how these activities might enhance students' conceptual knowledge and understanding related to circle. By implementing these structured practical activities and assessing their effectiveness, the research seeks to provide insights to take instructional approaches for teaching circle-related concepts. The theoretical foundation for the development of these activities is rooted in the role of structured practical activities in Skemp's framework for "relational understanding" in mathematics (1989).

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STRUCTURED PRACTICAL ACTIVITIES

Previous studies have emphasized the benefits of incorporating structured practical activities in mathematics education. Such activities provide students with hands-on experiences, allowing them to actively engage with mathematical concepts and develop a deeper understanding (Lesh, 2003; Hiebert et al., 2007). The teaching of circle-related concepts, including circumference and the approximation of π, has often relied on rote learning approaches. However, recent research has advocated for more meaningful and experiential learning experiences to promote conceptual understanding (Battista & Clements, 1996; Lehrer & Chazan, 1998). By engaging students in measuring the diameter and circumference of real-world objects, they can explore the relationship between these measures and discover the role of π in calculating circumference (Gravemeijer, 1994.)

Relational understanding framework

Structured practical activities presented in the model of intelligent learning (Skemp, 1989), allow students to experience mathematics at various levels increasing their power of prediction and to find suitable solutions for their real-life problems. Structured practical activities are designed to help learners to do the followings:

- to build structured mathematical knowledge and to make predictable predictions.
- to take pleasure from finding their predictions confirmed by events.
- to put students in a position to immediately correct their own thinking rather than wait for a teacher to tell them where they went wrong.
- to get control in directing their own learning process.
- to provide shared sensory experiences which ensure that there is a common ground for co-operative learning by exchanging ideas and discussion to extend learning into abstract areas of thinking.
- to make it possible for the natural creativity of the learners to come into action (paraphrase Skemp, 1989).

Skemp's framework of "relational understanding" emphasizes the importance of connecting mathematical concepts and procedures to develop a deeper comprehension of mathematics (1989). In the context of circle-related concepts, this framework suggests that structured practical activities can facilitate students' construction of mental schemas, enabling them to relate the formula for the circumference of circle to underlying geometric properties of circle. The importance of these kinds of experiences has been recognized by Polya (2004, p. 113) who asserted that "before obtaining certainty, we must be satisfied with a more or less plausible guess". These experiences give learners a greater control in directing their own learning processes as opposed to situations in which, they depend on their teachers or other knowledge authorities to tell them whether they have answered the questions correctly or what they did wrong. These guesses "can be a tool to help learner to transmission from 'external conviction proof scheme' to other desirable proof schemas" (Harel, 2008, p. 492). Overall, literature supports the use of structured practical activities to enhance students' understanding of circle-related concepts. These activities provide students with opportunities to explore, measure, and reason about circle, fostering relational understanding. By incorporating Skemp's framework this study aims to contribute to the understanding of effective instructional approaches for teaching circle-related concepts in middle school mathematics education.

THE STUDY

The purpose of the study was to investigate the effect of "structured practical activities" on Grade 8 students' understanding of the circumference of circle in Iran. To conduct the study, a seven-step activity was designed and developed as an extra-curricular activity in a middle school for gifted students. The activity provided a problematic situation to stimulate "intellectual need" of students in a way that Harel (2013) has described as the following:

- To establish common definitions, notations and conventions, and to describe mathematical objects unambiguously categorized as "need to communication."
- "Need to quantify and to calculate values of quantities and relations among them", categorized as "need for computation."

METHODOLOGY

14 Grade 8 students voluntarily participated in the study. The study was conducted in 2016 school year and the duration of instruction was 80 minutes. The first author was the mathematics teacher of that class. He divided the 14 students into four groups of three to four members with no specific order or criteria for the formation of groups. To carry out the activity, enough number of plastic strips with different colors where distributed among the groups, and students could form them to circles of various sizes (Figure 1). The teacher/researcher acted as facilitator and guide in class, while each group was involved in doing the activity, and the data were collected through students' work.

DESCRIPTION OF THE ACTIVITY

The length of the circumference of a circle *C* is related to its radius *r* and diameter *d*, by the formula $C = 2\pi r = \pi d$. This formula depends on two key ideas:

- The proportion of circumference of circle to its diameter is constant.
- Knowing this constant, one can calculate the circumference of any given circle when the length of its diameter is known.

The activity for finding the circumference of circle consisted of seven stages to bring about an opportunity for students to observe the relation between quantities; in this example, the circumference of circle with different diameters. This is what Harel (2013) calls "need for computation" as students go forward in various computations to ensure that the concept is internalized, organized and consequently, retainable for them.

DATA COLLECTION

To implement the activity, a set of circles made from flexible plastic strips in various colors was used, and data was collected by utilizing these plastic rings.

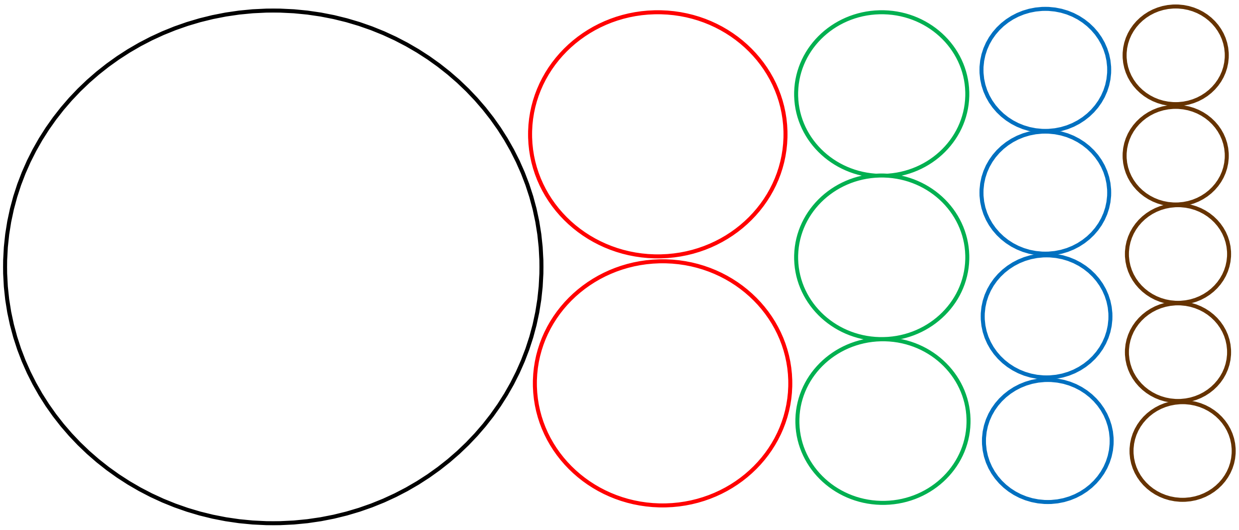

Figure 1

The relation between the diameters of colored rings to the big black ring as referent circle, was as follows:

- The diameter of every red circle is 1/2 of the diameter of black one.
- The diameter of every green circle is 1/3 of the diameter of black one.
- The diameter of every blue circle is 1/4 of the diameter of black one.
- The diameter of every brown circle is 1/5 of the diameter of black one.

DATA ANALYSIS

The data collected for this study was analyzed according to the seven stages designed for the activity and the findings presented accordingly.

Stage 1

At this stage, students were encouraged to cut the biggest (black) circle and straighten it out and make it into a strip as figure 2. The circumference of the biggest circle is equal to the length of this strip. Then they should cut the green circles, straighten them out and put them besides of black strip. The purpose of this stage was to draw students' attention to the relation between circumference of the black circle and each green circle. The expectation was that students could see the length of each green strip is half of the length of the black strip and all students realized and observed it.

Figure 2

Stage 2

At this stage, the teacher asked students to cut the green circles, straighten them out and put them under the red strip as Figure 3. The purpose was to draw students' attention to the relation between circumference of the black, red, and green circles. As the teacher expected, all students noticed that the length of each green strip is 1/3 of the length of the red one.

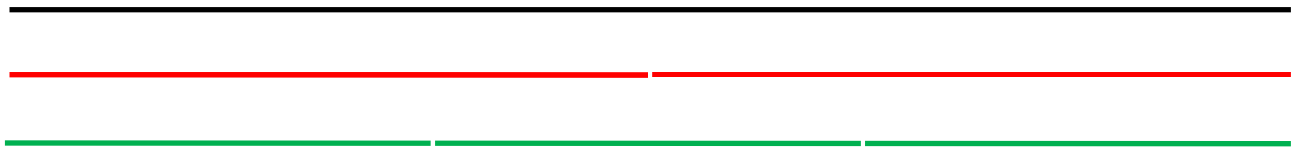

Figure 3

Stage 3

At this stage, students were asked to answer the following question:

> **Question 1:** *How many blue circles are needed to obtain strip lengths equal to black, red and green ones?*

All students successfully noticed the relation between the circumferences of colored circles with black color circle as reference circle. After wards they had to examine their guesses by cutting the blue circles, straightening them out, and putting them under the green strips as Figure 4. In this case, there was a situation for students to make predictions, examine them, and find out whether their predictions could be confirmed or not.

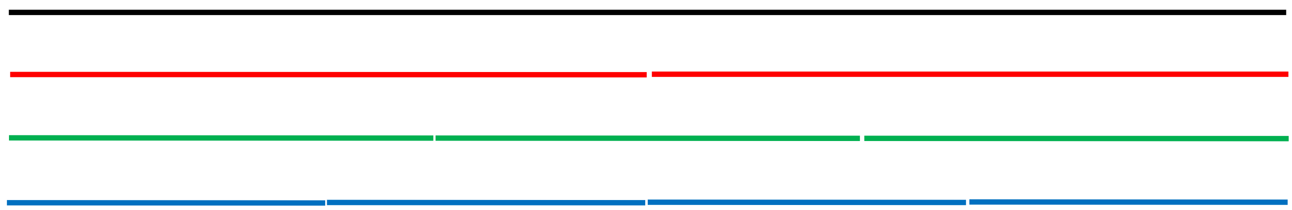

Figure 4

Stage 4

At Stage4, students were expected to answer the second question:

> **Question 2:** *How many brown circles are needed to obtain a strip that its length is equal to black, red, green, and brown ones?*

To answer this question, students needed to examine their guesses by cutting the brown circles, set them straight and putting them under the previous strips as in Figure 5. Similar to the previous stage, students made predictions, and investigated them to approve or reject their predictions.

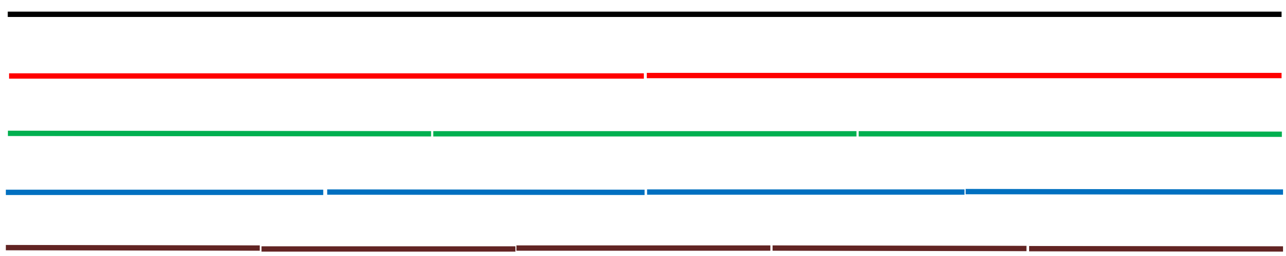

Figure 5

Stage 5

The goal of stage 5 was to provide an opportunity for students to explore that when the diameter of a circle is multiplied by any natural number, its circumference will be multiplied by that number accordingly. The visible observation was that almost all students successfully generalized the findings from the last four stages and expressed their findings as the subsequent conjecture:

"*If the diameter of circle A is multiplied by m, its circumference will be multiplied by m as well"*. The following formula is representing this conjecture as "if diameter of circle $A = m\times$ (diameter of circle B), then circumference of circle will be $A = m\times$ (circumference of circle B)."

Stage 6

At this stage, there was a chance to find a large group of circles as:

- Circles with diameter 1/2, 1/3, 1/4, and so on.
- Circles with diameter m/n multiplies of diameter of a given circle where m/n is a positive rational number.

By reflecting on the results gained from stage 6, students were able to see the emergence of the key idea as:

If the circumference of only one circle is known, the circumference of any circle could be found by knowing the ratio of its diameter to the diameter of that circle.

Furthermore, the students asked interesting and deep questions including the followings:

- Is there any circle that other people (for example mathematicians) had tried to calculate its circumference?
- If there is such a circle that mathematicians calculated the circumference of it, how could we calculate it as well?

Stage 7

At this stage, students were asked to draw a line on a sheet of paper and divide it into seven equal segments that the length of each is the same as the diameter of brown circles. Then cut some brown circles, flatten them out and put them over a line as is shown in figure following:

Figure 6

The goal was to attract the students' attention to the relation between circumference and diameter of circles. At this stage, they saw that the sum of the lengths of seven brown circles was equivalent to the sum of the lengths of the 22 brown circles.

FINDINGS

The analysis of the data showed that "structured practical activities" are suitable means to enhance students' understanding of mathematical concepts and specifically, the circumference of circle. The data collected from14 Grade 8 students arranged in four groups. The analysis of the data revealed that all four groups found the number of blue circles needed for a strip with the same length as black strip, and they verified this number by doing a structured activity. At the fourth stage, all groups guessed the number of brown circles for constructing a strip with equal length as earlier strips, examined their guesses and verified the numbers. At the end of this stage, they could generalize their findings from previous stages and concluded that if the diameter of circle is multiplied by any number, its circumference will be multiplied with that number as well. At the end of stage 6, students were able to infer that for calculating the circumference of any circle, it is sufficient to find the circumference of a specific circle such as a circle with unit diameter. The students came up with this important outcome by reflecting on the results obtained from earlier stages. At this point, the teacher/researcher, as facilitator, challenged the students by his enquiries to become assured that it is a useful decision to give a name for the circumference of such a circle and prepared a situation to introduce number π as a constant ratio of the circumference of circle to its diameter. Finally, by going through stage 7, students realized that when they folded a sheet of paper and divided it into seven equal segments, they made a strip that its approximate length was 22 times the diameter of the same circle. In fact, at the end of stage 7, they came up with a good approximation for π as 22/7.

DISCUSSION

Conventionally, students are told that the ratio of circumference of every circle to its diameter is constant. However, in this study, a structured activity was designed with seven stages and carried out with 14 volunteer Grade8th students, to make a guess regarding the relation between circumference and diameter of circle. The findings indicated that the students examined their guess and came up with a more generalized version of it at the final stage of the activity. Thus, the conclusion is that:

- Students found out that for calculating the circumference of every circle, finding the circumference of one circle is sufficient. Therefore, there is a suitable situation to inform them historically that the circle with radius 1 has been chosen to calculate its circumference and the length of circumference of this circle is represented by symbol π. In fact, the structured activities are potentially appropriate to arise the students' intellectual needs for establishing common definitions, notations, and conventions to fulfil the "need to communication" in mathematics learning.
- Structured activities could create serious need for computation as the students involving in this activity did. They tried to find a way of calculating the circumference of circle with radius 1 or as Harel (2013) puts it, to fulfil the "need for computation."

Finally, the structured activity designed for this study, created a suitable and challenging situation for Grade 8students to acquire conceptual knowledge of the concept of π, construct the formula of circumference of circle, and find the relation between diameter and circumference of circle. This activity gave students a chance to enhance their conceptual understanding and enabled them to apply their knowledge "autonomously" and "spontaneously" and to "reorganize" the knowledge that they are able to construct them as required by situations created by structured activities. Harel (2013) emphasizes, these activities could create learning situations for students to meet their intellectual needs.

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THE DEVELOPMENT OF SEQUENTIAL AND DISCURSIVE APPREHENSION IN KINDERGARTEN STUDENTS WHEN THEY BUILD POLYGONS

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To develop the geometrical thinking is important to carry out relevant interactive learning tasks that connect the representation of the figures (perceptual) with the geometrical concept (conceptual). Two aspects to develop the geometrical thinking are the sequential and discursive apprehensions. A good practice to develop these apprehensions are building tasks with didactic material. The aim of this study is to generate information about the sequential and discursive apprehension of kindergarten students when they build polygons with manipulative material. The participants were 17 kindergarten students of 3-year-olds that performed a task of build polygons with manipulative material after an instruction designed ad hoc. The results obtained show that students began to develop the dimensional deconstruction after this instruction. That is, students began to be able to make mathematical sense of the built figures through both apprehensions, anticipating and building figures (sequential apprehensions) and justifying the properties of built figures according to the geometric concepts (discursive apprehension).

INTRODUCTION

The geometrical thinking is an important aspect of the mathematical development in children (Clements, 2004). This type of thinking is developed before children enter school, and when they enter, in some cases, is not explored in appropriate ways (Jatisunda et al., 2021). Most of the tasks in kindergarten are related to circle or name certain figures, which are usually prototypical figures (the prototypical figures are the representations of a concept most commonly used, e.g. a square in prototypical position as an example of quadrilateral), without using the attributes that define them (Bernabeu, 2022; Clements et al., 1999). Due to the few examples of geometric figures shown in class (Verdine et al., 2016), the low cognitive level tasks (Bernabeu, 2022; Clements et al., 1999) and the lack of knowledge of many teachers or pre-service teachers about geometry (Bernabeu et al., 2019), many students have a limited concept image. The concept image (Tall & Vinner, 1981) is considered as the overall cognitive structure associated with a concept, which includes the mental representations and associated properties. So, the concept image can be limited or rigid including only certain prototypical examples and examples that contradict the definition (Maier & Benz, 2020). But the concept image' cognitive structure is dynamic, being constructed and changing over time through the different experiences that children encounter (Bernabeu, 2022). For this reason, the teachers recognize the need for relevant interactive learning tasks that connect the representation of the figures with the geometrical concept (Kinzer et al., 2016), that is, tasks that develop the *figural concept* (symbiosis between the perceptual and the conceptual, (Fischbein, 1993)). A good practice to develop figural concept, and amplify the concept image, of children is through building tasks. For building, didactic material offers a wide combination of geometrical figures' representations (not only prototypical examples), which is very important in geometry teaching and learning (Herbst et al., 2017). However, most of the research developed about kindergarten' geometry is based on the recognition of geometrical figures (Clements et al., 1999; Maier & Benz, 2020; Verdine et al., 2016), building of 3D shapes, normally with blocks (Casey et al., 2008; Kinzer et al., 2016), and few ones are based on characterizing the ability of pre-school students to build geometric figures (2D) with manipulative materials. For this reason, in this study, we will characterize the way kindergarten students build polygons with manipulative material as an ability to connect the representation of the figures (perceptual) with the geometrical concept (conceptual).

FRAMEWORK

To build a geometric figure with manipulative material it is necessary to give mathematical meaning to the pieces that constitute the material, assuming that these are the parts of the figure to be represented. This manner of seeing the material in a mathematical way by making use of geometric concepts to give mathematical meaning to their parts is what is meant by *dimensional deconstruction* (Duval, 2017). The dimensional deconstruction can be evidenced through the *sequential apprehension* which is the ability to construct a configuration or to describe its construction (Duval, 1995). Thus, for example, in tasks of building geometric figures with manipulative material, students need to make effective use of the material anticipating the parts that compose the figure to be obtained. That is, students need to anticipate the deconstruction of the 2D configuration to be constructed into 1D figural units (Duval, 2017). For example, to build a quadrilateral (2D configuration), a student must anticipate that four segments joined (1D figural units) are needed. In this sense, students need to endow mathematical meaning to the material, for example, the sticks of Meccano. Therefore, the student will consider the Meccano sticks as if they were sides of the quadrilateral (1D figural units), so the student will have to join the sticks to build a four-sided closed plane figure. As sequential apprehension is also describing the building process, the child can anticipate through the words what material is needed to build a certain figure. For example, to make a triangle "I need three Meccano sticks". In this way, dimensional deconstruction is also evident through the student's discourse, as the children give mathematical meaning to the parts that will compose the figure to be represented.

Furthermore, when a child is asked what a partner has built, providing the properties of the building, the dimensional deconstruction is evidenced through *discursive apprehension*, which is the ability to relate the configuration with mathematical definitions or properties to solve a geometric task (Duval, 1995). Thus, for example, a child, when faced with the building of a triangle with Meccano, may justify that it is a triangle because it has three sides or three sticks.

Considering these theoretical references, we aim to generate information about the manifestation of dimensional deconstruction through sequential and discursive apprehension when kindergarten students build polygons with manipulative material, promoting the expansion of the concept image and the development of the figural concept of the students.

METHOD

Participants and context

The participants in this study were 17 kindergarten students (3-year-olds). In relation to the curricular context, the geometry expectations of the NCTM (2000) to grades Pre-K-2 (from prekindergarten to grade 2), all students should (i) [...] build [...] two-dimensional shapes and (ii) describe attributes and parts of two-dimensional shapes. Therefore, in this research, we will focus on making sense of the concept of the polygon (a closed plane geometric figure with straight, non-crossed sides) and the class of polygon according to the number of sides by building figures with manipulative material and describe the attributes of built geometric figures. The students who participated in this research had not received any instruction on geometry during the 2021-2022 school year except for what they knew from their own extracurricular experience.

Instruction and data collection instrument

The instruction was carried out by a member of the research group. The instruction consisted of 4 sessions to address the polygon concept and polygons according to the number of sides, and a final test through individual interviews. In this study we will focus on session 3 (polygons according to the number of sides) (instruction), to explain how the concept of this class of polygon was constructed, and on the task of building from the final test (data collection instrument), which will show our results.

In session 3, once the polygon concept had been constructed (sessions 1 and 2), we carried out tasks to know the classes of polygons according to the number of sides, specifically, triangles, quadrilaterals, and pentagons. For this, we performed a task to recognize differences between different pairs of cards containing polygons with different numbers of sides (Task 1 of Figure 1). In this task, students had to recognize that, for example, one was a polygon with three sides and the other with four sides. Following this task, we performed another task in which we presented a set of polygons with a common attribute, e.g., having four sides, and asked the students what the showed polygons had in common (Task 2 of Figure 1). Once they had identified the common attribute (having four sides), the teacher-researcher would say what term the four-sided polygons were called, i.e., quadrilaterals. The same applies to triangles and pentagons. After this, a task of classifying polygons according to the number of sides was carried out (Task 3 of Figure 1). Lastly, a task of building polygons according to the number of sides, using the manipulative material Meccano, was performed (Task 4 of Figure 1). First, the students were left free to build their polygons (sequential apprehension) and if they had not built any of the polygons learnt in class, they were instructed to build a pentagon, for example. After the building of each polygon, another classmate was asked which polygon had been built and why (discursive apprehension).

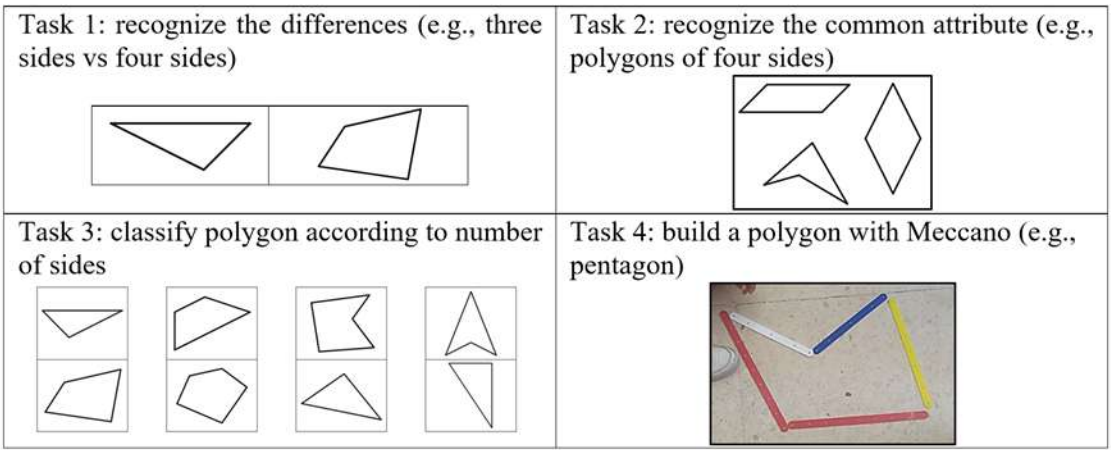


Figure 1: Tasks of the session 3

In the interview of the final questionnaire, one of the tasks was focused on sequential and discursive apprehension. Specifically, the task asked: "Build the following figures with Meccano: (a) polygon, (b) triangle, and (c) pentagon". In the interview, the teacher-researcher asked the students what they needed to build these polygons and the students anticipated the Meccano sticks they would need (whether the students anticipated through oral discourse what material they needed or whether they took the necessary material to build the requested polygon) (sequential apprehension). Then, the students built the demanded polygon (sequential apprehension) and finally, the teacher-researcher asked students by the attributes of the built polygons (discursive apprehension).

Analysis

The segments of the videos from the building task of the final interview were viewed by the members of the research group. For this, it was analyzed for each of the students whether they had solved the different commands of the task correctly. That is, if the student (i) had built the geometric figure requested, (ii) had anticipated what material they needed (sequential apprehension), (iii) built the figure requested (sequential apprehension) and (iv) justified the attributes of the figure they had built (discursive apprehension). At the end of the heading line, we added a column to note some observations about the students' way of solving the task. In the case of building a "polygon", we considered as non-polygons the constructions that remained open or had crossed sides (since in class it was reinforced that to be a polygon, the sticks of the Meccano had to be the ends of the sticks together and not cut each other), which we will consider as almost correct. In addition, in the building of the polygon, if they built a triangle, they were asked to build it again after the pentagon.

Once the fragment of a student's interview was viewed, the annotations of the members of the research group were discussed and if there were discrepancies, the video was watched again to reach an agreement. In this way, we obtained the characteristics of geometric thinking that students showed when building polygons with manipulative material.

RESULTS

The results are divided into two sections. First, we will show the ability of students to build with manipulative material examples of polygons, pentagon, and triangle. Second, we will show the evidence of the sequential and discursive apprehension when students built triangles and pentagons.

The success levels when building polygons

The Table 1 shows the frequency of correct, almost correct, and incorrect buildings of polygon, triangles and pentagons made by children.

n=17	Polygon	Pentagon	Triangle
Correct	6	4	9
Almost correct	7	0	2
Incorrect	4	13	6

Table 1: Frequency of correct, almost correct, and incorrect of the buildings

According to polygons, 6 students built a polygon correctly, joining and avoiding the Meccano's sticks being crossed or cutting each other. From these 6 students, 4 built triangles, of which three of them were prototypical (Figure 2a) and the other 2 built irregular convex quadrilateral. Other 7 students built the polygons almost correctly, but we did not consider them as correct because sometimes the buildings were open and/or one (or more) side was cutting off another (Figure 2b). From these 7 students, 5 students built convex quadrilateral, of which 4 were figures similar to rectangles (prototypical example) (Figure 2b) and the other 2 students built a hexagon and a heptagon. The other 4 students built polygons incorrectly because they put the sticks randomly, without forming any figure similar to a polygon (Figure 2c).

Concerning pentagons, the polygon with the lowest frequency of success, only 4 students were able to build it correctly. The other 13 students were not able to build an example of a pentagon because they did not know to combine the five sticks to make a polygon, where some of them put the sticks randomly, with the sticks crossing each other.

Regarding triangles, 9 students built a triangle correctly, which all were in the prototypical position (the base of the triangle parallel to the border of the table) and most were isosceles. Only 2 students built a triangle almost correctly because one side is cutting off another. The other 6 students did not know to build a triangle with Meccano.

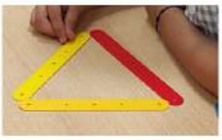
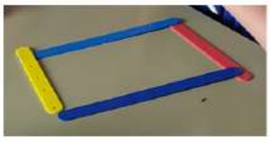
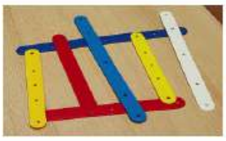
		
a. Correct polygon building: prototypical triangle	b. Almost correct polygon building: two sides are cutting off another	c. Incorrect polygon building: sticks put randomly

Figure 2: Examples of students' buildings

Evidence of apprehensions when building polygons

In this study, we will only focus on the analysis of the pentagon and triangle buildings. Table 2 shows the frequency of the evidence or no evidence of the apprehensions when students built pentagons and triangles with Meccano. Specifically, whether students show that they anticipated the building through oral discourse or whether they took the necessary sticks to build the requested polygon (sequential apprehension (SA)); whether they built the requested polygon, considering those that were almost polygons because the buildings were open or had one side cutting off another (sequential apprehension (SA)); and whether they justified the properties that the built polygons had (discursive apprehension (DA)). We considered as evidence of discursive apprehension whether students justified the properties of the polygon if the building was correct.

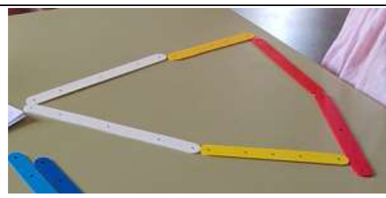
	Pentagon			**Triangle**		
	SA (anticipate)	SA (build)	DA (justify)	SA (anticipate)	SA (build)	DA (justify)
Evidence	7	4	4	11	11	10
No evidence	10	13	13	6	6	7

Table 2: Frequency of the evidence of the sequential and discursive apprehensions

Regarding the building of pentagons, in relation to the sequential apprehension, on the one hand, 7 students evidenced the ability to anticipate the needed material to build a pentagon, of which 5 said orally the material they needed and took it and 2 took the necessary material. On the other hand, 4 students evidenced the ability to build a polygon with manipulative material. According to discursive apprehension, 4 students provided a justification of the attributes included in the built pentagons (the same 4 students that built correctly).

Of the 7 students who anticipated what they needed to build the pentagon, 4 were not able to build it. In addition, one student (Student 16), who was able to build and justify the pentagon correctly when she had to anticipate, selected six Meccano sticks to build a pentagon, so she did not anticipate correctly the necessary material. When the student built the pentagon, she realized that the building did not have five sticks and removed one, building correctly the demanded pentagon (Figure 3).

Teacher-researcher: How many vertices does it have? How many points does it have?
Student 16: One, two, three... nine, ten (count the endpoints of the sticks).



Teacher-researcher: How many sticks are there?
Student 16: One, two, three ... five.
Teacher-researcher: Is it a polygon of five sides?
Student 16: (Shakes head).
Teacher-researcher: I want a polygon of five sides.
Student 16: (Transforms the polygon).

Teacher-researcher: What is it now?
Student 16: I remove one (transforms the polygon).

Teacher-researcher: What is it now? [...] Is it a polygon? How many sides?
Student 16: One, two, three, four, five.

Figure 3: Discourse between the teacher-researcher and Student 16 during the building of pentagon

According to the building of triangles, the pieces of evidence of the apprehensions were higher than with the pentagons. In relation to sequential apprehension, on the one hand, 11 students anticipated the needed material to build a triangle, of which 9 said orally the material they needed and took it and 2 only took the necessary material. On the other hand, the same 11 students built the triangles, of which 9 built correctly, and 2 built almost correctly. In relation to discursive apprehension, 10 students evidenced this apprehension because they provided a justification of the attributes included in the built triangles.

Paying attention of combination of both buildings, on the one hand, the 3 students who evidenced the two apprehensions in the building of the pentagon (plus the exception of the case of the Student 16, who removed a stick to correct her building so did not evidence a part of the sequential apprehension (anticipate)), also evidenced these apprehensions in the building of the triangle. In addition, we consider that Student 16, who removed a stick to correct her building, would also form part of this group (being a total of 4 students) since, although she did not show evidence of half of the sequential apprehension (anticipate), she showed evidence of the other half (build) and of the discursive apprehension during the building of the pentagon. On the other hand, the 4 students who correctly anticipated the building of the pentagon, but they did not build it correctly and therefore their justification was not considered correct, showed the two apprehensions in the building of the triangle.

CONCLUSIONS AND DISCUSSION

In this study, we aim to generate information about the manifestation of dimensional deconstruction through sequential and discursive apprehension when kindergarten students build polygons with manipulative material. The results obtained in the first section of the results based on the frequencies of success (correct and almost correct) building polygons and triangle with manipulative material (13/17 and 11/17, respectively) show that students began to develop the dimensional deconstruction (Duval, 2017). The students were able to give mathematical meaning to the sticks of Meccano to join them and build a polygon or triangle, assuming that the sticks were the sides of the polygons. According to the results obtained in the second section, the pieces of evidence of the sequential and discursive apprehensions when students build triangles and pentagons, greater in the building of the triangle than in the pentagon, also manifests a use of dimensional deconstruction to solve the geometric task. These results show that students, after classroom instruction, began to be able to make mathematical sense of the parts of the figures to be constructed through sequential apprehension (anticipating and building) and subsequently, they were able to see the built figures mathematically to justify the properties of these figures according to the geometric concepts, through discursive apprehension. Therefore, the building tasks help the development of the figural concept (Kinzer et al., 2016; Fischbein, 1993), since students can associate the attributes belonging to the concepts with the representations of geometric figures through the manipulative material and modify a representation if it does not meet the properties of the concept to be represented (as in the case of Student 16). On the contrary, the low frequencies obtained with the pentagons lead us to think that the development of dimensional deconstruction depends on the attribute considered (Bernabeu & Llinares, 2017), in this case, the number of sides of the polygons.

Also, when building polygons and triangles, most of them were representations of prototypical examples such as isosceles or similar isosceles triangles (with similarly measured sticks) in prototypical position and non-polygons similar to rectangles (considered as almost correct because they had sides that cut off other sides). In this latter case, students forced the sides to cut off to build a familiar figure for them, the rectangle (see example b in Figure 2). This may be due to the students' concept image of polygons, which has a limited variety of examples and is still very much rooted in the examples that students are used to seeing in any school or out-of-school context (Maier & Benz, 2020; Tall & Vinner, 1981; Verdine et al., 2016).

In addition, the results lead us to infer that anticipating is easier than building as it only involves saying what material is needed or/and taking it but building requires manipulating it so that it fulfils the attributes presented by the concept (such as being closed and not having crossed sides to be a polygon). On the other hand, we can infer that if students know how to anticipate and build, they will possibly be able to justify the attributes of the constructed figures, demonstrating the importance of combining the perceptual with the conceptual to develop geometric thinking.

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KNOWLEDGE AND REASONING IN CIRCULATION DURING A SITUATION OF FIGURES REPRODUCTION BY FOLDING

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This work is part of a research project aimed at identifying the evolution of knowledge and reasoning of French elementary school students (aged 7-8) in the implementation of a situation involving the reproduction of figures by folding. The analysis of a verbalization phase is operationalized using a semiotic approach. It reveals reasoning based on spatial knowledge, as well as the circulation of geometric knowledge, but also highlights numerous semiotic dissonances.

This work is part of a research project that aims to identify the evolution of knowledge and reasoning of French elementary school students (aged 7-8) during the implementation of a figure reproduction situation by folding, the ‘Pliox situation’ (Guille-Biel Winder, 2014). Based on several experiments, Guille-Biel Winder (2021) highlighted the importance of the teacher's language during collective phases, in relations between students and the milieu (Brousseau, 1997). We take a semiotic approach to this study, focusing here on a verbalization phase. After outlining our theoretical and methodological bases, we present the ‘Pliox situation’. We then provide elements for *a priori* and *a posteriori* analyses of the selected phase.

THEORETICAL AND METHODOLOGICAL BACKGROUND

In the context of geometry, we begin by drawing on the Duval’s work, in particular the notion of figural unit (0D, 1D, 2D, 3D), and the apprehension levels of geometric figures - perceptive, operative (mereologic, optic, place) and discursive apprehensions (Duval, 1999). This work also falls within the general framework of the theory of didactic situations (TDS) that takes into account both knowledge to be taught and learned, and conditions of its appearance (Brousseau, 1997). In what follows, we return more specifically to the notion of verbalization.

Notion of verbalization

Within the TDS framework, the knowledge modeling is based on situations of action, formulation and validation (Brousseau, 1997). A process, organized around different dialectics (and not a fixed succession of situations), enables the construction of knowledge. This process is not necessarily linear and, in its contingency, depends on the management of the didactic sequence planned by the teacher. For this reason, we use the term ‘phase’ to refer to a stage clearly identified or organized by the teacher in the implementation of a situation. Based on the work of Blanquart (2020), we conceive of verbalization phase as an intermediary between action phase and formulation phase. When knowledge is verbalized, its function is to designate precisely the (instrumented) actions performed (or to be performed), or to describe their purpose. This action is contextualized, situated in time and space, and marked by the one acting. However, we differentiate between verbalization and effective action, because, from the point of view of knowledge, verbalization already enables a first reflexive

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 343-350). ICMI.

feedback on action: the subject verbalizes for others (or for himself). To that end, he/she selects the information to be communicated, the order in which he/she states it, and the signs useful for this communication (oral language, gestures, written symbols). The subject must take into account an interlocutor who can provide him/her feedback on the form of the message, and who is therefore part of his/her environment. He/she must be able to point to the objects he/she is talking about and their relationships (e.g. in geometry, through a geometric technical language (Petitfour, 2016)). This can encourage the acquisition of linguistic knowledge and the production of reasoning, and thus help to link visualization and discursive activity, which is essential in the field of geometry. We define reasoning as any voluntary action or formulation by a subject that enables progress towards a defined goal, constructed with his/her knowledge, and which he/she can make explicit, even summarily (Blanquart, 2020). The analysis of reasoning is, in particular, based on semiotic analysis.

Semiotic approach

Drawing on Vygotskian sociocultural theory of learning (Vygotsky, 1978) and following Radford (2009), we consider that in mathematics, 'thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools' (*ibid*., p. 113). We then rely on the concept of semiotic bundle (Arzarello, 2006) to grasp the multimodality of the learning process and to consider the dynamic development of interactions between the different systems of signs, such as language, coordinated with embodied aspects (gestures, body postures, glances, actions with tools) and written (words, drawings, mathematical symbols). We use a semiotic table (Petitfour & Houdement, 2022) to reveal the multimodality of teaching and learning phenomena and to interpret the different signs activated during interactions between the actors in the classroom (e.g. Figure 6). Finally, we use the term *semiotic dissonance* (*ibid*.) to name a discrepancy between two interpretations of the same signs, or a quirky personal interpretation of a mathematical sign.

THE 'PLIOX SITUATION'

The study takes place in the context of the development of the 'Pliox situation', a situation involving reproduction of figures by folding. In order to dissociate the geometrical shape from the artifact (Rabardel, 1995), we design by 'Pliox' a square sheet of paper, shared by four colored square areas (yellow, red, green and blue) (Figure 1). The generic 'Pliox situation' consists in a didactic situation (according to Brousseau sense): it is the reproduction of a model figure by folding a Pliox. The possible folds correspond to the symmetric axes of square (diagonals and medians) of the so-called 'secondary squares' (*i.e.* the four colored squares) (Figure 1(c)).

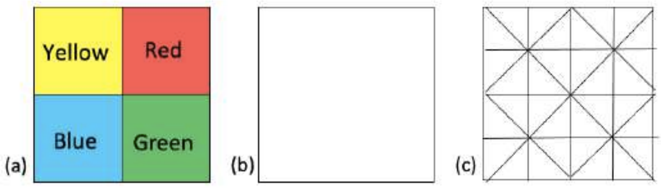


Figure 1: *Recto* (a) and *verso* (b) of the Pliox; possible folds (c)

The 'Pliox situation' can be seen as an origami in the sensitive space, that is to say it corresponds to a spatial problem. The Pliox is accessible to vision; turning, returning, folding and unfolding the Pliox are the possible manipulations (3D gestures with tactile and visual coordination); all movements of the Pliox in regard to the subject are possible, as well as those of the subject; the subject is outside the considered space, which contains the object. Therefore the corresponding space is the 3D micro-space. Moreover, the 'Pliox situation' can be considered according to a geometrical point of view. The Pliox is indeed very flat, so that it can be related to a representation of a two-dimensional geometrical figure (possibly shared into other sub-figures). In primary school, we believe that a figure is a drawing whose properties can be specified within the theoretical geometrical framework by considering it as a representation of a geometrical figure (Duval et al., 2005). In this case, a figure can be obtained by juxtaposition or superposition of shapes. Consequently, we consider reproduction of figures by folding a Pliox as geometrical problems. The progress of the situation consists of several sessions. The first one is devoted to the manufacturing of the artifact by students, the others to reproduction of several figures by folding. We focus here on the reproduction of model figure (a) (see Figure 2(a)).

A PRIORI ANALYSIS

According to Berthelot & Salin (1995), 'the usual teaching geometry tasks call up skills from usual microspace relations' (p. 23). The 'Pliox situation' involves indeed spatial knowledge (because of the orientation and the positions of figures that appear into model figures), but also geometrical one (mainly about the square, the rectangle, the right-angled triangle and their relationships, and specific straight lines (diagonals, medians)). Moreover, two points of view on a same figure can be involved in this situation: what is immediately recognized (the colored zones and the global form according to Gestalt Theory) and the figures who can be identified after taking account of some folds. For this case, knowledge and recognition of elementary two-dimensional figural units require perceptual apprehension and operative (mereologic, optic and place) apprehension (Duval, 1999)[16]. Reproducing figure (a) requires identification of the external figure (isosceles right-angled triangle) and to take care of the three colored zones (yellow, red and green) and their relative positions (but not necessarily to recognize them as triangles or square). This model highlights a relationship between square and isosceles right-angled triangle: dividing a square into two parts can lead to two isosceles right-angled triangles. The reproduction also requires folding according to a diagonal of the square. Hence the folds allow recognizing two red triangles, a yellow and red triangle, but also a red and green square or a yellow, red and green rectangle, … (Figure 2(b)).

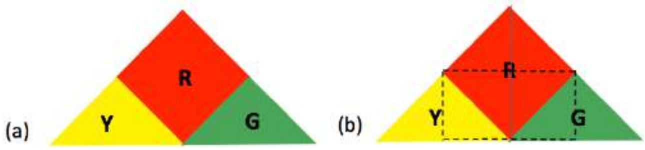


Figure 2: A model figure (a) and various decompositions (b)

The 'Pliox situation' is an action situation (Brousseau, 1997) within which different phases are set up: (1) presentation of the model figure; (2) short collective analysis of the model figure; (3) individual reproduction of the model figure by folding; (4) collective highlighting of procedures and validation; (5) mathematical synthesis. We focus here on phase (2) that corresponds, for our sense, to a verbalization phase. During this phase, the mereologic decomposition induced by color as well as the spatial organization of the figural units can be highlighted. The request to justify the identification of a figure leads gradually the students to become aware of some properties (the number of sides and vertices of the square, of the rectangle…); some relationships between some of the shapes can also

[16] For more details, see (Guille-Biel Winder, 2014).

been highlighted; the spatial vocabulary (over, down, behind…) and the geometric vocabulary (square, triangle, diagonal, point…) can appear as necessary to the communication. Let's see what knowledge and reasoning can be identified when implementing this phase in a classroom.

A POSTERIOI ANALYSIS

The sequence takes place in a class of 23 students (14 boys, 9 girls) aged 8-9. A beginner teacher with a scientific background, who had done university work on the 'Pliox situation', giving us access to her preparations and stated intentions, led it. The sequence (6 sessions) was filmed by the teacher, then transcribed by us. Each session was split into episodes (unit on the task, on the nature of the activity). Session 1 was devoted to diagnostic assessment. Session 2 aimed to build the Pliox; it led to instrumental recognition of the square and rectangle (checking right angles with a square and equal lengths with a graduated ruler); students reproduced various model figures in later sessions. The analyzed verbalization phase (session 3, episode 2) corresponds to the presentation and analysis of the model figure (a) that is the students' first reproduction work. During this phase, the model figure is displayed on the blackboard in this orientation. The teacher asks the students to say what they see, and progressively invites them to show on the Pliox the figurative elements they are talking about. Based on the synopsis, we identified within three extracts of episode 2 'remarkable events' (Leutenegger, 2009) *i.e.* significant in terms of students' knowledge and/or reasoning. We present and analyze them in what follows, supplementing them when necessary with elements of analysis from other sessions of the sequence.

Figures apprehension

In the first extract (1'02" to 2'43"), a student locates and names triangles, differentiating them by their relative size ('one large', 'two small triangles') while outlining them with her finger. Another pupil, Clara, then announces 'a square on top'. She outlines it, while other students dispute: one says it's not a square, another that 'you can see it's a rhombus'. Clara explains how she recognized a square by making a rotating gesture with her hand, while saying 'Well, you put it a little upside down, like this'. We can see here that the students all rely on their perceptive understanding of figures to recognize and name the three colored juxtaposed sub-figures as well as the overall figure they compose. Clara's intervention reveals a line of reasoning that demonstrates mastery of spatial knowledge: a geometric figure does not change its name (and therefore its nature) according to its orientation on the board; we can choose the orientation of a figure (by mentally rotating it) to better recognize it in comparison with the 'prototypical figure' stored in memory. As shown by the students' reactions to Clara's proposal ('it's not a square', 'it's a rhombus'), and as confirmed by the teacher in her professional writing, many of the students in the class do not possess this knowledge at this point of the session, since they consider the orientation of the red figure to be decisive in naming it square or rhombus. This is how Mat explained it during the diagnostic evaluation (session 1), based on an actual rotation of the PLIOX (Figure 3) (on this figure, T indicates the teacher). Furthermore, to recognize the square, his (erroneous) reasoning was based on the horizontality and verticality of the figure's sides, showing these directions and expressing it by 'seeing straight' in a spatial sense (Figure 4).

	3'17"	3'19"	3'21"	3'24"
T			Right, so it depends on how you look at it	
Mat	If you stand like that, it's a rhombus	if you stand like that, it's a square.		Yes, yeah
			 *Mat outlines the square.*	

Figure 3: Square and rhombus recognition (Mat, session 1)

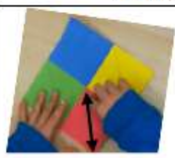
	3'39"	3'40"	3'43"	3'45"	3'48"
Mat	You see straight on, first	and then,	you've got things that aren't like that.	You don't look ... like that	You look at the side.
					

Figure 4: Spatial characterization of the square (Mat, session 1)

Thus, the students' reasoning (right or wrong) to recognize this figure involves spatial knowledge without explicit reference to geometric properties of the square.

Vocabulary and geometric knowledge

In the second extract (2'43" to 4'26"), which follows a disagreement between students about the name of the red figure, the teacher asks them to recall what they saw in session 2 about the square. One student talks about 'faces and sides', another about 'faces and edges'. The teacher discards these terms associated with solids. Luis says there are three sides in the square, but points to three of the square vertices on the Pliox. The teacher invalidates this proposal by asking one pupil to show the sides of this figure (the pupil first points to them, and, at the teacher's request, follows them with her finger), then another to show the vertices (which are pointed to). The teacher then summarizes the square characteristics: four sides and four vertices. In this extract, we identify the students' use of geometric terms specific to the characterization of a solid ('faces', 'edges') and therefore not suitable for the description of a plane figure. Maybe this misuse of terms is linked to specificity of the Pliox as a two-sided material object. Moreover, Luis' intervention highlights another lack of geometric knowledge. On the one hand, he announces an incorrect number of sides for the square (three instead of four), and on the other, he uses the term 'side' that is at odds with the designated graphic object (a vertex). This same semiotic dissonance will be repeated later (session 4) for another student who asserts that the square has four sides (a correct assertion), but points to the vertices to show which sides he's talking about. We can therefore conclude that, even if students do settle into a geometric language register to express properties of the square, the terms chosen are not always appropriate and the meaning given to the terms is not always adequate.

Difficulties with right angles

In the next extract (4'26" to 7'03"), Katia points out that the square also has four right angles. She points to a side to indicate a right angle (figure 5, 4'52"), and answers the teacher that this can be checked with the ruler. Following the teacher's negative response, she immediately suggests the set square. She then positions the set square on the Pliox (figure 5, 5'28") and points to the vertex of the right angle of the set square (figure 5, 5'29"), saying ‘There's one there’. The teacher asks her to be more precise (‘Position it well, look, you have to put it on the edge like we did yesterday’). Katia readjusts the set square and specifies that the angle is right. She then changes the orientation of the Pliox and quickly checks a second angle with the right positioning of the square (figure 5, 5'45").

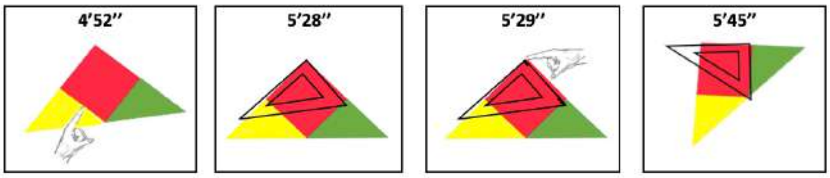


Figure 5: Instrumented verification (Katia, session 3)

Katia then struggles to position the set square to check a third angle. The teacher (T) comes to her help: she positions the tool herself and leaves it to Katia to say whether the angle is right or not (figure 6).

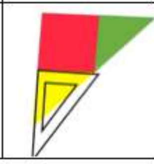
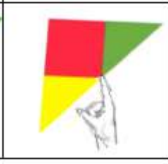
	5’51’’	5’52’’	5’56’’	6’01’’	6’02’’
T		Be careful, you’re not		Look, to make this one	We’ll put the setsquare here
				T points to a vertex of the square.	*T positions the setsquare.*
Katia	*Katia positions the setsquare*		*Katia repositions the setsquare.*	*Katia observes.*	
Pliox posted					

Figure 6: Positioning the set square (Katia and Teacher, session 3)

Katia doesn't recognize that the angle is right-angled, so the teacher (T) helps her to spot it (figure 7). The teacher then asks Katia for the number of right angles. When she replies ‘three’, the teacher sets up the set square to check the fourth angle of the figure. Katia doesn't recognize it as right, so the teacher asserts that it is, relying on the verification work done the day before.

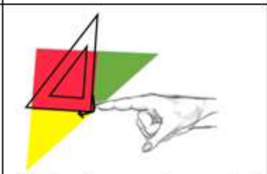
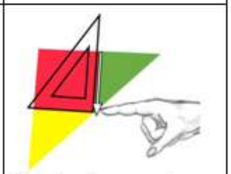
	6’13’’	6’14’’	6’18’’	6’19’’
T		Ah no but that’s normal! This is the right angle		Ah but that’s an extension!
Katia	Here		And that’s straight too	
Pliox posted	 *Katia points an angle.*	 *The teacher runs along part of the sides of the right angle.*	 *Katia runs along a side of the right angle.*	 *The teacher runs along the same side.*

Figure 7: Checking the third right angle (Katia and Teacher, session 3)

In this episode, Katia states a correct geometric property of the square – it has four right angles – but as in episode 2, we observe a semiotic dissonance between the terms used (‘right angle’) and the graphic object pointed to (side) (Figure 5, 4'52"). It's possible that Katia associates the qualifier ‘right’ with the straightness of the square's sides[17], since she initially proposes the ruler as an instrument for verifying the property. Her remark ‘And here too, it's straight’, made as she follows with the finger

[17] In French, the same word (‘*droit*’) means both ‘straight’ and ‘right’.

along one side of the square (Figure 7, 6'18"), corroborates this hypothesis. It's also possible that 'straight' is used in the spatial sense of verticality, as Mat did in session 1 (Figure 4, 3'39"). Since the ruler as an instrument for verifying a right angle doesn't suit the teacher, Katia suggests the square. The use of this instrument leads her to look for right angles. Let's note that Katia points to the vertex of the angle (object 0D) to designate the right angle (object 2D) (Figure 5, 5'29"), as does the teacher (Figure 6, 6'01"). This assimilation of an angle to its vertex perhaps contributes to Katia's difficulty in positioning the set square correctly on the angle to be checked. For the first positioning, the teacher notes a lack of precision: the edges of the right angle of the set square are not adjusted on the sides of the angle to be checked, only the vertex of the right angle of the square is positioned on the vertex of the angle to be checked. We later note a lack of *technical knowledge* (Petitfour, 2016) for the third positioning: Katia places just one side of the right angle of the square on one side of the square (Figure 6, 5'51" and 5'56"). Furthermore, she does not correctly interpret the nature of an angle (right or not) when the square is correctly positioned, as we can see when checking the last two angles of the figure (Figure 7, 6'13").

SYNTHESIS AND CONCLUSION

In this verbalization phase, we noted difficulties in recognizing a square in a non-prototypical situation. This finding resonates with the works of Battista (2007) and Marchand (2020). We observe that the reasoning (right or wrong) produced by the students is spontaneously based on spatial knowledge, perhaps encouraged by the artifact's manipulative possibilities. In our view, these spatial reasoning deserve to be discussed in class, or even institutionalized, as they contribute to the construction of geometric knowledge (Blanquart, 2023). Following on from the work of Berthelot and Salin (1995), this study highlights the importance of taking into account the interweaving of the spatial and the geometric knowledge. It also raises the question of the need to formulate spatial knowledge, while remaining vigilant to the polysemy of certain terms (such as the word '*droit*' in French in the last episode, that means 'right' or 'straight'). In addition, the teacher's request to associate, through gestures, the oral and material or graphic designations of geometric objects enabled us to identify numerous semiotic dissonances. These show that the use of a geometric lexicon by students is no guarantee of the acquisition of geometric knowledge. This highlights the value of semiotic analysis for the researcher, and suggests the importance for the teacher of not focusing exclusively on oral language or vocabulary.

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THE DIGITAL INTERACTIVE MATHEMATICAL MAP FOR GEOMETRY

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We present results from the design-based research and development process of a digital didactical tool connecting secondary and tertiary mathematics education among others by historical aspects and elements from Mathematics, Didactics, History, Literature and Technology especially for prospective mathematics teachers: Digital Interactive Mathematical Maps (DIMM), developed at the University of Passau, Germany. The DIMM for Geometry was integrated into a second semester course on Euclidean and non-Euclidean geometry in mathematics education at Karlstad University, Sweden, designed collaboratively by university mathematics lecturers and researchers in mathematics education from the cooperating universities. While, according to the results of evaluation, the (perceived) usefulness of individual functionalities of the digital tool was confirmed, the use of the DIMM in the courses seems to promote favorable beliefs about mathematics, too.

DIGITAL INTERACTIVE MATHEMATICAL MAPS (DIMM)

In this article, we present a new digital tool for teaching geometry and other areas of mathematics. Within this tool we combine a historical-genetic didactical approach with a technological tool for the visualization of historical and biographical aspects of mathematics and mathematicians, respectively (Brandl, 2009; Brandl & Vinerean, 2023; Datzmann et al., 2020; Przybilla et al., 2022; Vinerean et al., in press), to support students with a meaningful connection regarding the development of mathematical knowledge by “the trials, the triumphs, the tribulations” (I.I. Rabi in Holton et al., 1970, as cited in Kubli, 2002, p. 57) of the people behind this science. One of the most important things is here the humanistic element in the learning process: “Portraying scientists as human beings and giving students the opportunity to become effectively involved in the *story* of science are worthy goals in themselves” (Klassen, 2006, p. 51), whereas in many common school books “only the bare decontextualized scientific facts, theories, and laws along with the exemplar problems that demonstrate them” (ibid., p. 33) are presented. The motivation is therefore to overcome historically developed fragmentation of mathematical contents in curriculum at higher secondary schools and to offer an integration of the world of higher mathematics in the sense of Felix Klein (Klein, 2016/1924: “double discontinuity”), which is still a problem today (Hefendehl-Hebeker, 2013; Pinto & Cooper, 2022; Winsløw & Grønbæk, 2014). The double discontinuity implies that teachers often base their school teaching on their own pre-university experiences and do not rely on their academic mathematics knowledge gained at university (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013). Mathematics teaching becomes therefore fragmented, and students have fewer opportunities to notice connections and develop conceptual understanding (Winsløw & Grønbæk, 2014). To overcome Klein's double discontinuity, students are required to notice connections between the ‘axiomatic-formal' world of university mathematics and the 'perceptual-symbolic' or 'conceptual-embodied' world of school mathematics (cf. Tall, 2008).

The digital tool "Digital Interactive Mathematical Maps" (DIMM) that we present here is based on first ideas in Brandl (2009) and has been developed at the professorship for Didactics of Mathematics at the University of Passau (Germany). It was tested in different teaching and learning contexts at Karlstad University (Sweden) and University of Passau. The DIMM are freely accessible[2] and offer a visual representation of the historical development of mathematical knowledge in time as well as the interrelation of different knowledge items according to each other. The core of the maps are three-dimensional nets, starting from one initial problem and expanding like a tree from node to node. The nodes represent pieces of mathematical knowledge, often a famous discovery resulting in a central theorem (see Figure 1). One dimension of the maps represents time, while the other two dimensions represent inner-mathematical dependency and relationship. The relationship between the nodes is computed by an algorithm from graph theory using certain weights given when putting a new content into the database (for details e.g. see Przybilla et al., 2021, 2022). Clicking on a node opens the corresponding content on a linked timeline, where milestones of the considered mathematical area are clearly and consecutively displayed (see e.g. Figure 10 in Brandl & Vinerean, 2023).

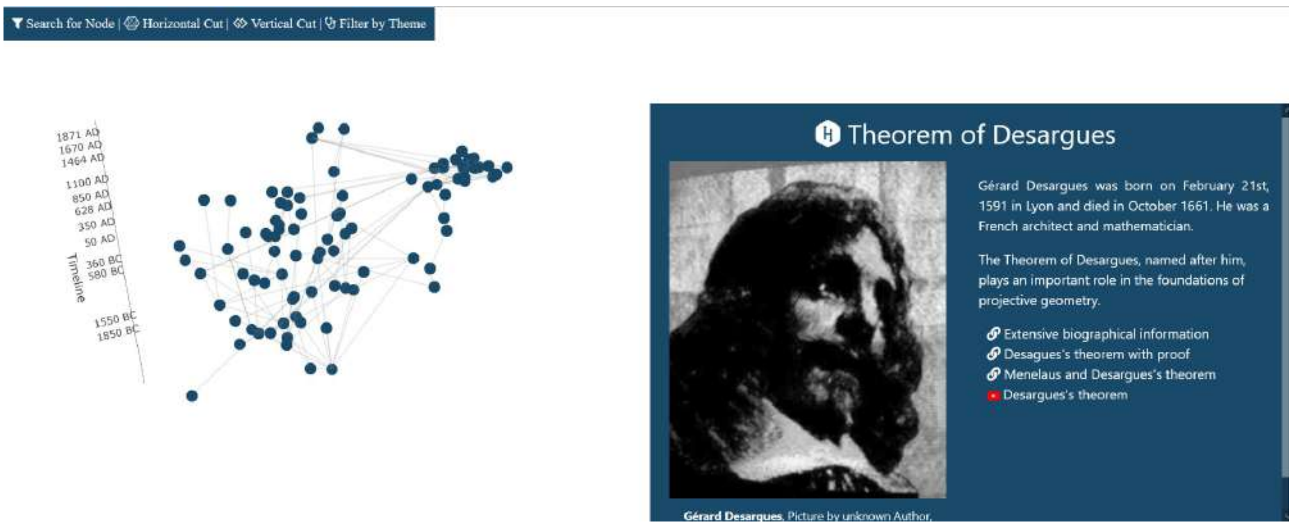

Figure 6: Screenshot of the Geometry Map (14 September 2023) and preview of a node

Functionalities of the DIMM

Each node of the map or element of the timeline, respectively, comprises information about the mathematical discovery and the person responsible for that, flanked by providing biographical and factual information via texts, links to further publicly available sites and free videos. Where feasible and suitable, these links to webpages contain besides instructive texts to go deeper in the material also instructive tasks or problems from well-known mathematics competitions such as Bundeswettbewerb Mathematik and Landeswettbewerb Mathematik Bayern from Germany or the European Girls' Mathematical Olympiad. These contents open up the possibility to use the DIMM also in courses for gifted students and for the preparation for mathematical competitions.

The content elements of the map or the timeline, respectively, can be collected by a shopping cart system and exported for further use in lessons or tasks. Other functionalities are so-called "horizontal cuts" and "vertical cuts". A "horizontal cut" is a projection into the plane of all nodes selected in a certain period of time and possibly filtered by a specific topic. It allows for an optimal visualization of inner-mathematical relationships between the different content-nodes (Figure 2 right). Contrarily, flanking this functionality is the "vertical cut" mechanism resulting in a reduction of all nodes to those representing the historical development of the mathematical idea or result in form of a two-dimensional directed graph (Figure 2 left). Clicking on the nodes then allows for an efficient and meaningful connected way through the genesis of the mathematical aspect and its explorers. Furthermore, a summary of all nodes being part of the evolutionary line can be created and downloaded, in order to be distributed to other learners and worked with in offline mode.

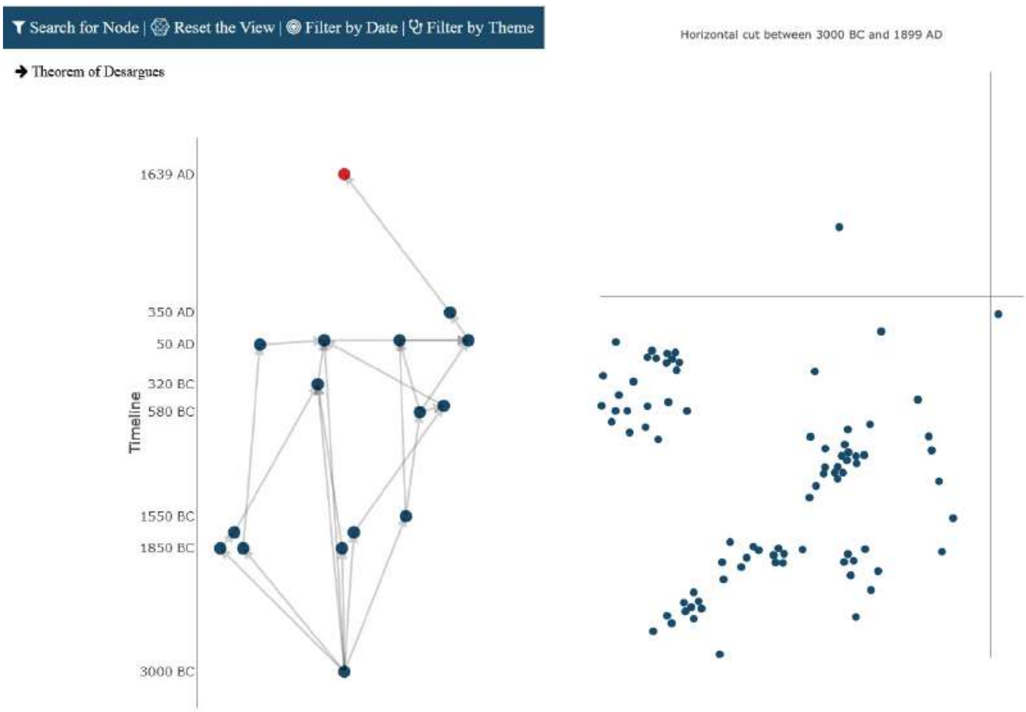

Figure 7: Exemplary vertical (left) and horizontal (right) cut (14 September 2023)

THE DIMM IN GEOMETRY EDUCATION

There are several possibilities to use the DIMM in teaching. One possibility is their use in combination with elements from narrative didactics as the offered content elements of the map nodes serve as providers of basis bricks for the construction of motivational narratives for school lessons (see Brandl & Vinerean, 2023). Especially, the vertical cut function of the map can help to create narratives covering several points in time. The "main goal of the vertical cut is to promote an accurate picture of mathematics as an *emerging science*. It is a consequence of its very nature that mistakes and misunderstandings can and probably will occur when *doing* or *creating* mathematics." (Przybilla et al., 2022, p. 4796). A further possibility offers the integration of competition tasks for a certain amount of nodes. By a way through the map as described in Table 1, mathematically gifted students can work on the tasks in a manner that on the one hand fits their special needs and on the other hand connects different contents with each other via historical development (Brandl & Kaiser, 2022; Brandl & Szabo, 2019, 2023):

Year	Mathematician	Mathematical Content
90 AD	Menelaus	Theorem of Menelaus
320 BC	Euclid	Elements of Euclid
520 BC	Pythagoras	Congruence
580 BC	Thales	Intercept Theorem
1850 BC	Moscow Papyrus	Area of a triangle

Table 1: Way through the Geometry DIMM with tasks from competitions (as presented in a similar version in Brandl & Szabo, 2023, as Table 1)

METHODOLOGICAL APPROACH: QUALITATIVE EVALUATION OF A GEOMETRY COURSE

In a second semester course on Euclidean and non-Euclidean geometry in mathematics education at Karlstad University (Sweden), the centuries-long discussion about Euclid's parallel postulate was picked as a suitable example, which also connects school mathematics (Euclidean geometry) with university mathematics (non-Euclidean geometry). The course was part of the second semester of the teacher-training program (300 ECTS credits) and could also be attended by teachers who wished to receive further mathematical training. Mathematics teachers and researchers of mathematics education from Karlstad University and the University of Passau had designed the components of the course cooperatively prior to the study (Brandl & Vinerean, 2023; Przybilla et al., 2022; Vinerean et al., in press).

The mathematical content was flanked by the necessity for fostering competencies with regard to the evaluation of digital learning tools. In sum, 44 students (27 teachers & 17 prospective teachers) actively took part in the course. Weekly lectures were followed up by weekly work assignments[1]. They concerned the timeline (TL), narrative didactics as an area of application of the contents in the map, the three-dimensional mathematical map (3D Map), the vertical cut (VC), the horizontal cut (HC), the discussion around the parallel postulates, and a general evaluation of the didactical tool (cf. ibid.).

For the technical evaluation of the tool we used the technology acceptance model (TAM) of Davis (1985), which is widely used in the community and well supported both theoretically and empirically (cf. Scherer et al., 2019). The goal was to analyze students' answers to questions on the perceived usefulness and ease of use of the technology (cf. Przybilla et al., 2022; Vinerean et al., in press). Therefore we added quizzes in connection with the weekly assignments concerning the functionalities of the map. Using the TAM as described in Przybilla et al. (2022), we addressed two major beliefs - *perceived usefulness* and *perceived ease of use* - and the fact that the design features in the tool (in our case, the individual components and functionalities of the map: 3D Map, TL, VC and HC) directly influence the two beliefs (Figure 3).

Regarding the evaluation of elements from narrative didactics, the methodological procedures are explained in detail in Brandl & Vinerean (2023, pp. 9 – 10) being executed in two ways:

A) Formulation of a short historical-orientated narrative motivation for a free-of-choice topic from the school curriculum by using information provided in the nodes or the timeline of the Digital Interactive Mathematical Map

B) Description of the historical evolvement of non-Euclidean geometry from Euclidean geometry by using information provided in the nodes or the timeline of the Digital Interactive Mathematical Map

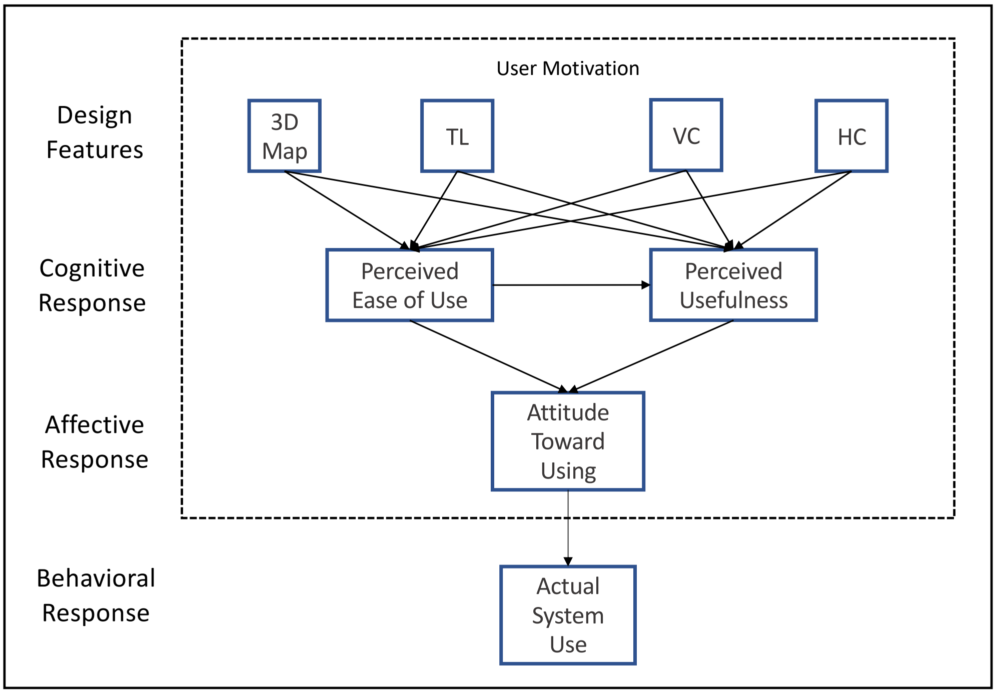

Figure 3: Adaption of the TAM (Davis, 1985) for the technical evaluation of the DIMM

RESULTS

The students largely agreed or strongly agreed with the *perceived ease of use* (cf. Table 2 in Vinerean et al., in press): 85% considered that the timeline is simple to use, 77% that the selection function is useful, 74% that the preview function helps to navigate through the map, and 84% that bookmarking videos is useful.

Similarly, the students largely agreed or strongly agreed with the *perceived usefulness of the tool* both as students and as future teachers: 85% confirmed that they use the search function and the filter function to find a node in the map that interest them. In addition, 93% of the students agreed or strongly agreed that the 'filter date' functionality helps them to get deeper insights about the development of mathematics and the circumstances of this development, and 75% that they can imagine themselves in the future, when learning new content at university, informing themselves about its origins and development. However, 22% could not decide on the last statement, suggesting that one intervention might not be enough to make a real change in the attitude of the students.

When asked in their role as future teachers, 91% of the students agreed or strongly agreed that they see the vertical cut functionality as a useful and clear way to show historical developments, and 93% that they see the horizontal cut functionality as a useful and clear way to show thematical connections. In addition, 82% confirmed that they think that knowledge of the historical lines of development of mathematical concepts will help them in their future profession as a teacher. However, 18% could not decide on the last statement, leaving room for improvement in the way the map influence them in this direction. Additionally, we cite an excerpt of the conclusion from Vinerean et al. (in press):

> Overall, few of the students (6%) agreed or strongly agreed that they had previously experienced the historical-genetic principle (i.e., before the course). Of the students, 22% agreed or strongly agreed that they had experienced the teaching of mathematics at the university as oriented on the historical-genetic principle. […] Overall, the students largely agreed or strongly agreed with the perceived usefulness of the tool both as students and as (becoming) teachers.

Regarding the analysis of the evaluation part concerning narrative didactics we decided on an approach via a single-case study and a qualitative analysis of the text. Detailed results can be found in Brandl & Vinerean (2023), where also an exemplary excerpt from a student's assignment and answer to the feedback possibilities on the narrative didactics aspect is presented in order to illustrate successful and not successful use of the desired elements. "Student 5" gives a short introduction concerning the motivation and intention to use narrative elements and pieces of biographies collected with the help of the map in an imaginary mathematics lesson (ibid., p. 10, translated from Swedish to English by Brandl and Vinerean):

> To introduce 4-5 graders to coordinates and the number plane I think that the story about Descartes seeing the flies on the ceiling can really help to visualize and understand it. By now, the students should already be familiar with the number line. To expand on that knowledge I will tell a very abridged version of the story of Descartes and tie it to previous discoveries by other people.

A further result of a qualitative content analysis of the students' answers indicated the following with respect to influence on beliefs related to the nature of mathematics (Vinerean et al., 2023, p. 575):

> [T]he map seems to open up for viewing mathematics as an emerging science. […] Dynamic understanding of mathematics can be seen in, for instance, students' reasoning about the struggle when mathematicians develop mathematical concepts. […] Most students show a process-related and application-related orientation in relation to their future role as teachers when using the map. […] Overall, students' reflections suggest that the use of the digital tool Interactive Mathematical Map(s) promotes favorable beliefs related to the nature of mathematics.

Finally, we present some results concerning students' experience with the combination of the two scaffolding tools narrative didactics and Digital Interactive Mathematical Map (with emphases on narrative didactics) from the evaluating quizzes in Przybilla et al. (2022), Brandl & Vinerean (2023) and Vinerean et al. (in press): Almost 80% of the students found the contents in the map useful to write a narrative motivation that could be used in their teaching; only half of the students had experienced teaching with narrative elements before. Moreover, 95% of the students would like to incorporate narrative elements in their future teaching, and almost 70% of the students would appreciate narrative approaches in university courses. In addition, students find it "rewarding […] to be able to produce a text oneself!", and most of the students found the information in the map relevant and easy to use for the assignment purpose and for their own understanding.

SUMMARY

To summarize, in this article, we present a digital tool, Digital Interactive Mathematical Maps (DIMM), for teaching geometry and other areas of mathematics. It brings together mathematics itself with didactical methods, historical events, narrative elements, and technological aspects. Both the logico-scientific and the narrative mode of teaching are addressed (Brandl & Vinerean, 2023; Bruner, 1986). The tool allows for more successful learning processes as it visualize connections between various contexts. The connections serve as anchor points in a scaffolding structure (Brandl & Vinerean, 2023). Evaluations of usefulness of the map and its bridging teaching concept showed to be promising when striving for a holistic design of mathematics lessons, that is, bringing together areas of science and arts, as well as dimensions of rational thought and emotional feelings in a learning process. Further, the map seemed to promote both a process-oriented and an application-oriented approach as well as favorable beliefs, such as mathematics being an emerging science promoting a view of *doing* mathematics, in which an open error culture can be established (Przybilla et al., 2022).

Notes

[1] The weekly assignments are freely available at https://docs.google.com/document/d/e/2PACX-1vTq_w6A2MqresPr5PQLGujDt3NIS6i7aiL7ShsmaR841TtI-hrsyTM2zLY6UEhbULBumIx0zEtTOz6V/pub.

[2] The DIMM are freely accessible under the address https://math-map.fim.uni-passau.de.

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GRIDS AS OBJECTS AND TOOLS FOR THE GEOMETRY CURRICULUM

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Although not typically considered as basic objects of the geometry curriculum, grids are significant objects-to-think-with that, we argue, simultaneously require and enable geometric seeing, acting and reasoning. Grids function both as intrinsically rich geometric structures and *as geometric ways of structuring the world. We situate our work within the research on spatial reasoning. We also draw on theories of embodied cognition that emphasize the importance of bodily actions, and gestures in particular, for exploring and communicating geometric ideas. Through two examples, we show the complexity and conceptual richness of grids. One example involves young learners engaging with paper-and-pencil grid tasks. The other describes a multi-touch app that enables users to create and manipulate grids for ordering or analyzing their surroundings. Although the examples are quite different in nature, together they suggest how working with grids could constitute a significant component of the geometry curriculum that would not only support students' spatial reasoning, but also equip them with a powerful tool for seeing, acting and thinking.*

LEVERAGING SPATIAL REASONING TO THINK-WITH AND ACT-ON GRIDS AS DYNAMIC SPATIAL STRUCTURES

In many countries including Canada and the United States, geometry is increasingly overshadowed as a curriculum topic by arithmetic and early algebra. Even topics that are geometric in nature are typically treated numerically or quantitively. Much of the focus on geometry is devoted to learning the names of shapes—what Duval (2006) calls geometry as botany. Despite the preponderance of *numerical* approaches in the curriculum—and their perceived significance to real world situations—research over the past decade has shown a stunning correlation between *spatial* reasoning and mathematical achievement. Further, research has also shown that spatial reasoning is malleable: it can be improved with practice. Several Canadian jurisdictions (Ontario and British Columbia) have thus begun emphasizing spatial reasoning in the curriculum. It is in this spirit, of "spatializing" the curriculum, that Francis et al. (2023) propose number lines and mutable grids as tools that are spatial in nature and relevant across the curriculum (including in geometry, number, and measurement strands). In this paper, we focus on the grid, which we put forth as an object-to-think-with (to use Papert's (1980) term), that functions both as a geometrically rich structure in its own right and as a geometric tool that can be used to structure, and mathematise, the world.

Grids are used in coordinate systems, but also in 2D and 3D measurement, in geometric transformations, and in multiplication. In these cases, grids often appear incidentally, as background fields or frames of reference rather than as objects defined, emphasized, or analyzed in their own right. Grids, their attributes, and applications can all be introduced more explicitly, as tools, or as ways of producing structures. Placing a grid on a 2D shape, for example, can decompose it into equally sized units that enable measurement. Gridding a drawing can make it easier to copy and/or enlarge. Imposing a grid provides a structure that invites qualitative, proportional focused thinking

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 359-366). ICMI.

rather than (necessarily) numerical thinking. For example, placing a grid on a human body makes it easy to see how large the head is compared to the whole body.

The work we present in this paper builds on prior studies that Bruce, Sinclair and the Spatial Reasoning Study Group (comprised of researchers from Australia, Canada and the United States) have carried out on the development of young children's spatial reasoning and its role in mathematics learning (Sinclair & Bruce, 2015). It also extends the work of Jackiw and Sinclair (2014) who have designed gesture-driven multi-touch apps for young learners to create and manipulate numbers.

GRIDS AS SPATIAL OBJECTS

We see grids as spatial tools. They can be tangible physical objects such as graph paper or even lines drawn through the sand, or they can be imagined objects that serve as mental constructions allowing us to conceptualize mathematical metaphors such as continuous and infinite space, parallel lines and infinite points along those lines (Lakoff & Nuñez, 2000). They can be laid beneath or on top of other objects; they can be stretched, skewed, rotated and even wrapped around other surfaces. They can be scaled up and down, zoomed in and out, stretched and twisted, parallel or convergent. Structured grids typically feature cells of identical shapes, but we also find unstructured non-uniform grids, polar grids, perspective grids of broad variety, spherical and space-packing and other 3D grids, and many more uncommon members of the grid world.

Despite the possibilities that grids enable, the traditional square grid that students encounter throughout their geometry learning is presented as static in nature, with the perspective and structure being fixed. Interestingly, what is left unexplored in most school geometry is the conceptual underpinning that grid lines represent divisions in continuous space (Lakoff & Núñez, 2000), which is of concern given that the fundamental properties of grid structures have been reported as being difficult for students to generalize (Battista et al., 1998, Outhred & Mitchelmore, 2000). Grids can help us organize and partition space, track and describe movements and locations in coding and mapping, as well as compare objects and distances visually. The particular way in which grids structure space provides a system for thinking about many fundamental ideas in geometry, algebra and in mathematics more broadly. It follows that students would benefit from having explicit opportunities to develop a robust relationship with grids as spatial objects to think-with and act-on.

While textbooks and worksheets usually present square, static grids that have already been made, digital technologies can make it easier for students to create their own grids and to explore grids in terms of their anatomy (geometric properties) and their flexibility (how they can be applied and manipulated in various contexts) in making sense of space. In our explorations with children, we have used both static canonical grids with low-tech materials such as black grid lines on clear plastic sheets as well as dynamic grids that can be rotated, shifted, stretched and compressed using iPads and software developed by Jackiw (in press). In both cases, we are interested in how learners act-on and think-with grids, paying particular attention to the way they see grids and the gestures they make to describe, create and manipulate grids.

INSTANTIATIONS OF GRIDS AS OBJECTS-TO-THINK-WITH AND ACT-ON

We now turn to two cases from our research that point to the potential conceptual richness and power of grids. We focus not only on grids as visual objects, but also on the gestures involved in acting on them. We posit that new ways of moving and acting (of moving one's fingers, for example) produce new ways of thinking, which is in line with theories of embodied cognition that refute the Cartesian mind-body binary (e.g., Nemirovsky et al., 2013; de Freitas & Sinclair, 2014). Given the importance that gestures play both in mathematics learning and communicating (Edwards et al., 2017), we are particularly interested in both spontaneous and intentionally designed gestural experiences.

Case one: Early encounters with a static, square grid

In an effort to capture some of the earliest conceptions of rectilinear grids, Bodnar (2023) conducted a series of exploratory one-to-one interviews with children ages 3 to 7. Emir, age 5, was shown a 5x5 square grid that had some of the cells covered with coloured paper. He was asked to think of the coloured area as a blanket that was covering some of the squares on the 5x5 grid and that his job was to determine how many squares were "hiding under the blanket" (see Fig. 1a). When shown the 5x5 grid with a 3x3 coloured 'blanket', Emir explained how he was thinking about the 'covered area':

Interviewer:	How many squares are hiding under the blanket?
Emir:	Nine
Interviewer:	How did you know it was nine?
Emir:	I was thinking one here, two here, three here, (points to where each cell might be in the top row that is covered) four here, five here, six here (points to where the second row of cells would be), but then there was another row I didn't really notice… Nine! Cause six plus three is nine…is this whole square.

Emir tapped each cell with his pointer finger as he counted squares and said, "one here, two here" and so on across the first two rows in sequence, which may suggest that he sees the relationship between cells as non-continuous. That is, each cell was independent and was not bound to a higher structural system, such as a row. Interestingly, Emir paused his count after the first two rows, then used his left hand to cover the area of six squares that he had just counted, and then smoothly ran his right pointer finger back and forth horizontally along the bottom row. He did not count each cell in this bottom row but rather named and gestured to the whole row as a unit of three, related cells. After having some more time to consider this grid, Emir said: "this whole square" while he used both hands to cover the 3x3 area. The square of nine came together as one whole unit, one whole shape that Emir dis-embedded from the surrounding 5x5 grid.

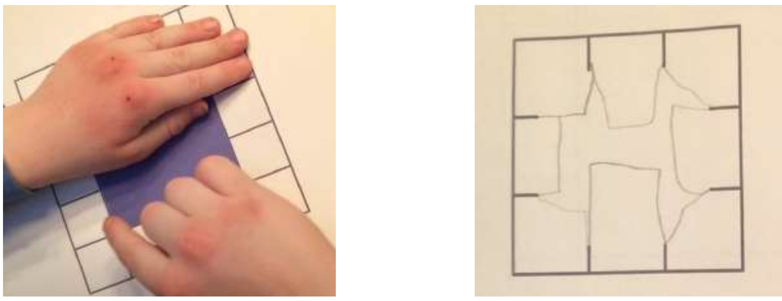

Figure 1: (a) Emir breaks apart a 3x3 covered grid into an area of six with the addition of a row of three to reach 9; (b) Emir's attempt to complete a 3x3 grid structure

In another task, the interviewer asked Emir to complete a square 3x3 grid that was partially generated (adapted from Battista et al., 1998), as in Fig 1b. Emir was asked first to give an estimate of how many square tiles he thought it would take to cover the entire grid (showing one plastic tile equivalent to one cell as a visual point of reference). He estimated it would require 10 tiles. After giving his estimate he was then asked to "complete the grid" with a pencil. Emir began to draw individual cells within the grid. He started in the top left corner by connecting the partial grid line indicators provided to make one 'square' cell. He then continued in a one-by-one approach, generating the cells as a series of discreet squares with no evidence of continuity or relationship.

In comparing Emir's responses to the two tasks, we can see that in the first, Emir had some reasonable strategies for thinking about a series of 9 cells in the formation of 3 rows of 3 cells each, as evidenced in his gestures and language. However, when it was Emir's job to construct a grid with limited information (a perimeter and some partial partition lines), Emir opted for a single cell approach, with no particular attention to rows or columns. We observed this same phenomenon consistently: young children found the blanket task far more accessible compared to the grid construction task. Emir does not appear to see the grid as a structure in which cells are components of both rows and columns that are produced by vertical and horizontal lines that intersect. This suggests that the grid is a complex visual and haptic object for young learners that may require intentional pedagogical mediation, such as shifting learners' attention from the 2D spaces produced by gridlines to the 1D lines themselves, which Duval (2006) calls dimensional decomposition.

Case two: Griddler application

The Griddler (Jackiw, in press) is an app we are developing to explore ideas about grids outside of plotting and early-number contexts. The broad purpose of the app is to let users superimpose dynamic grids on their worlds, and to use these grids to assess geometric and spatial patterns and relationships present in those worlds. The specific design has evolved through cycles of user evaluation. Users, in this case, are considered without particular prerequisite age or educational background; and while both the app's emphasis on "gridding one's world" and the specific gestures it encourages users to apply are potentially provocative pedagogically, the app features no explicit pedagogic questions or formal mathematical language in its (minimal) interface. Rather, the intended broader learning design of the software in any pedagogic context puts such potential questions and language in conversations outside the tool itself, where they can follow a trajectory meaningful to a specific learner rather than to some entire class of generic "app users."

In choosing a view of "their world" to 'begrid', users may opt alternately to manipulate *what is directly in front of them* (represented in the app by the live video feed of their device's camera); or *what is important, precious or interesting* to them (represented by the collection of images they've stored on their camera's photo roll), or finally *where they are*, represented by a re-scalable map showing their position in their neighbourhood, city, or continent. Finally, a last view lets a teacher suggest a canonical image for exploration (based on a small set of images provided for this purpose), so that multiple users, with multiple devices, can explore a common image. This fourth view breaks the premise of strict user-centricity in favour of a group-centricity that supports whole-class exploration or precise instructional moments.

Imposing a dynamic grid on any of these world views is accomplished through a set of touch-screen gestures. A first finger touch on any world view causes a square grid to sprout and expand from the touched location, rapidly covering the entire view (Fig. 2a). Subsequent dragging with one finger translates this grid so that the cells line up as the user wishes over the image (2b). Two fingers simultaneously dragged twist and shrink or enlarge the grid, allowing the user to quickly resize the square grid to exactly match any visible feature of the view (2c). Introducing a third finger (perhaps on a spare hand) stretches the square grid—pegged down in two places by the prior fingers—to expand into a rectangular one, or shears it into a grid of oblique parallelograms. A fourth finger further deforms the original grid into a receding perspective grid. In Figure 2, a user rapidly begrids the child's head, and observes 1:5 ratio of head:body height.

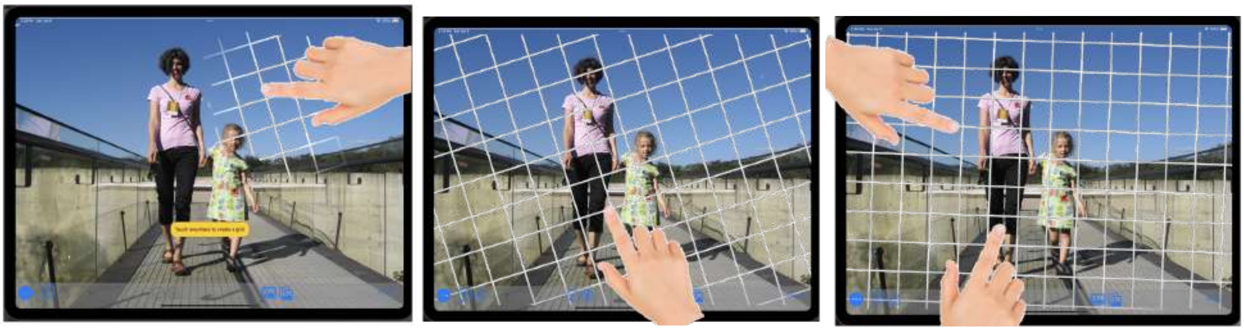

Figure 2: (a) Touching the screen to create a grid; (b) One-finger dragging to translate;
(c) Two-finger dragging to rotate and dilate

This gestural language for grid manipulation draws on our previous work in pedagogically provocative gesture design (e.g. Jackiw & Sinclair, 2014) and reflects several design goals for users' interactions with, and early conceptualizations of, "grid experience". First, it positions all possible grids in a dynamic, continuous evolution of an initial square grid, rather than in a typology of fundamentally separate grids. Learners begin with a conceptually simple, generic structure and gradually modify it, through direct manipulation, into more specific forms appropriate to gridding specific aspects of their world. Second, it associates unique physical gestures with different types of manipulations, offering embodied and haptic instantiations of operations that are also experienced visually (in their effect on the grid) and propositionally (in any conversation one has around the grid, such as "move it over here"). Third, it presents a pedagogically laddered trajectory of gestures which begin with the obvious, accessible act of dragging, for basic operations, and develops in a structured fashion (adding "one more finger" at a time) into less obvious or less familiar gestures for more advanced operations, revealing new capabilities as users gain proficiencies with more elementary operations. Fourth, as an overall grammar of grid transformation, it features strong underlying mathematical coherence: if a finger position on the screen represents an idealized geometric point on the plane, then a finger dragging from one location to another represents a point-to-point mapping. Transformationally, such a single point-to-point mapping defines a translation of the plane: the grid *moves*. Two point-to-point mappings define a similarity: with two fingers, the grid also *dilates* and *rotates*. Three point-to-point mappings (equivalently, two triangles) define an affinity (the grid now also *shears*) and the mapping of two convex quadrilaterals (four points, four fingers) defines a perspectivity (gridlines now *converge*).

The impact of these four design goals is, of course, not easily isolatable into four distinct dimensions of user experience. Instead, they operate together, and interact with users' prior touch-screen experience as well. For example, most newcomers interpret *The Griddler's* one-finger and two finger transformations as already-familiar gestures with expected behaviours: "dragging" and "pinching" are common manipulation skills in diverse mobile apps. Applying them to fit a grid cell exactly around some feature of an image may require their use at a more precise or thoughtful level than applying them simply to "zoom in" on an image, but they remain accessible starting rungs of the laddered trajectory of gesture. By contrast, the next ladder rung—the three-finger drag, which is easiest with two hands—falls outside most users' gestural vernacular. This physical unfamiliarity creates a mild "access gate" keeping the conceptually unfamiliar idea of the parallelogram grid somewhat beyond users' naïve grid manipulations. It also perhaps promotes a more intentional, reflective consideration of the gesture's effect once it is encountered or discovered. In our piloting of *The Griddler*, where most new users already know the impact of one- and two-fingered grid transformations, we see them having to actively develop proficiency in controlling or predicting the effect of three-finger transformations. In *The Griddler*, this intentional design mapping of gestural affordance to mathematical definition makes gestures not only communicative but also functional epistemically (Jackiw & Sinclair, 2017).

DISCUSSION

Recently there are some strong examples of the incorporation of spatial reasoning within and beyond geometry-focused learning experiences (see Lowrie & Logan, 2023; Mulligan et al., 2020; Moss et al., 2015; Ng & Sinclair, 2015) as a dynamic visual-spatial approach. These studies recognise both the fundamental role that imagery plays in mathematical thinking, even outside of geometry, and the many models, representations and diagrams that can be used to mediate visual-spatial conceptualisations of mathematical concepts. Our focus on the grid contributes to these efforts, but extends prior work on spatializing concepts to also consider spatializing tools. We argue that grids function both epistemologically (as objects to be learned through acting-on them) and ontologically (as tools to use—and hence objects-to-think-with) in ways that cut across much of the curriculum.

In the two cases described above, we highlighted the potential of explicitly leveraging grids for mathematics thinking as an underpinning spatial structure that supports children in their spatial sense – their making sense of space and their geometry thinking. In this discussion, we focus on two implications that challenge typical conceptions of how geometry is treated in school mathematics. The first is that the application of objects-to-think-with and act-on have the effect of folding together geometric thinking and action instead of artificially splitting them apart as is typical classroom practice in North America and other parts of the globe. The second implication is that we can reconceive of a geometry curriculum in a world where objects become the basis for exploration of mathematics principles – that grids, for example, as dynamic tools and spatial structures, can be the curriculum starting point and the site of investigation, with the mathematics tumbling strategically from these investigations. With Hansen (2009), "…we take the position that *geometry is about grasping space*, not only with the eyes and hands, but also with the mind" (p. 240).

Spatial objects as an opportunity to fold thinking and action together

In a push against the artificial disconnect between conceptual thinking and procedural actions of school mathematics, researchers have become interested in pedagogical artefacts that can fold together thinking and doing (Bruce et al., 2023; Niss & Højgaard, 2019). In the previous example of Emir, the very act of partitioning a figure to make a square grid appears to involve reasoning spatially about concepts such as equal spacing, continuousness, intersecting space and the spatial integration of cells, as well as density. In the example of *The Griddler*, we see that placing and moving two fingers on a touch surface can facilitate thinking about and acting on scaling and rotation of structured grids to fit various contexts while maintaining the integrity of the shape of the cells. Acting on the grid by placing more fingers on the grid enables the eyes, hands and mind to explore affine and perspective transformations.

Reconceiving geometry curriculum

As a thought experiment, we consider how this focus on objects-to-think-with and act-on might reposition geometry curriculum for young students as an active dynamic learning space, unbounded by typical textbook-driven approaches of naming and sorting canonical shapes. We conceive of a geometry curriculum where the objects explicitly reveal and support student conceptions of geometric principles and ideas as a primary goal rather than simply as the means to understanding a rule. In other words, exploration with grids ***is*** the mathematics from which we can draw key ideas about geometric principles, configurations, manipulations and uses of grids. In such a curriculum the goal can be flipped: rather than having the curriculum goal of teaching children that rectangles have four sides and four right angles, the creation and manipulation of dynamic rectangular figures made with a tool such as *The Geometer's Sketchpad* (Jackiw, 1991), and grids applied and manipulated using a tool such as *The Griddler*, become a highly focused and exploratory method for revealing the 'rules' they follow in space and how they can help us think mathematically. Such a repositioning of dynamic play with grids as a curriculum objective to draw out geometric principles and spatial reasoning, notably requires a high level of teacher confidence and understanding.

Classroom implications

In building teaching facility for a geometry curriculum that is truly exploratory and dynamic in nature, researchers, geometers and teachers will need to work together. As researchers, our efforts are squarely focused in this arena. The research described (in case 1) will enable us to design lessons that support students' structural awareness. Similarly, the tool described (in case 2) can provide a basis for classroom activities involve gridding (some of which we have already developed and will share at the conference). In both cases however, efforts will be needed to prevent use of these resources from relapsing into procedural approaches giving rise to the botanical metaphor used by Duval—which in our case would involve simply naming and classifying static grids and their parts. We think such efforts might shift teachers' conceptions of geometry away from a rigid set of given concepts and theorems towards what Herbst et al. (2017) describe as more flexible ways of modelling and understanding shape and space.

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VIRTUAL ENVIRONMENT FOR SPATIAL KNOWLEDGE: OPPORTUNITIES AND LEARNING OBJECTIVES

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This text presents a research project in which a didactical engineering on spatial orientation using a virtual environment has been conceived and experimented in five classes with the same students over 3 years in grades 2 to 4. After a general presentation of the project and some theoretical elements, an example of a task on path planning is presented and analyzed in order to illustrate some possibilities of the technological environment.

RESEARCH CONTEXT

Nowadays it is practically impossible to organize our daily lives without technology. Indeed, technologies are taking up more and more space in several of our actions, including spatial orientation when traveling. Clearly less and less people still uses a map to move around a city, on foot or by car, since Global Positioning Systems (GPS) are now easily accessible and have totally changed our orientation skills (Ruginski & al., 2019). Although children rarely use GPS when traveling, three-dimensional technologies are nevertheless present in many videogames and have inevitably an influence on their construction of space orientation skills (Spence & Feng, 2010). These are important issues regarding the evolution of spatial orientation and more generally our representation of space. As educators, and researchers in education, we would like to better estimate their effect on the learning of spatial knowledge and see how we could use virtual environments and possibly virtual reality to design school activities more meaningful for students at the beginning of primary school regarding spatial orientation within the mathematics curriculum.

These reflections motivated a research project called SPAGEO[18] which connects three research fields: cognitive psychology, educational technologies, and mathematics didactics. Each of these fields contributed to the whole project which combines three facets each being led by one the three components. The first facet, led by the cognitive psychologists aims to document the spatial abilities of a large population of children aged 7 to 10. The second facet, led by the mathematics didacticians aims to design a teaching sequence over several levels of primary school on spatial orientation using a virtual environment. The third, led by the educational technologists aims to conceive and develop of a virtual environment, in the shape of a city on a computer, that we called SPAGEO City. A demo of the immersive view is available on the website[19]. It provides access to the city's various neighborhoods, including the city center and the residential area (Coutat & al., to be published).

In this paper, we will only refer to some aspects of the second facet, on which we were the leading team. Our focus was first to conceive and experiment some spatial orientation activities and show

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[19] https://tecfa.unige.ch/tecfa/research/spageo/

what can be done with the help of a virtual city. We led our experiment in all the classes of two schools in Geneva, around 100 students, that we followed over three years, while in Grades 2, 3 and 4 (7 to 10 years old)[20]. For each year, a didactic engineering of about six to eight sessions of 1 to 1.5 hours each was designed and tested. Of course, we worked in close collaboration with the technologists in order to implement in SPAGEO city, the different aspects and tools that our activities required. Moreover, the collaboration with the psychologists was important in order to have a more precise idea of our students' spatial abilities. In this sense, the development of the virtual environment has been thought out using the competences from the three components. The discussions and interactions were focused on the opportunities and challenges that could be used by the technology to achieve our research and teaching objectives taking into account the various works on spatial abilities by psychologists.

This text presents a digital technology for teaching and learning spatial knowledge with a specific focus the opportunities and challenges that are presented by the use of technology in geometry education.

SPATIAL KNOWLEDGE IN GEOMETRY

In this study, the mathematics at stake is related to spatial knowledge. Berthelot and Salin (1999) in their study on the teaching of space and geometry defined spatial knowledge as "the knowledge that allows a subject to control his relations to sensitive space" (Our translation from the original). Marchand (2020) refined this definition by referring to mental transformations or manipulations that a subject uses to anticipate or control objects from the sensible space (Our translation from the original):

> We refer to the knowledge which, physically (concrete action) or mentally (MI [Mental Imageries]), leads the learner to control, anticipate and communicate the states, transformations or deformations of objects in the sensitive space, the graphic space and, hypothetically, geometric space (Marchand, 2020, p. 142) .

This mental consideration appears also in cognitive psychology, with spatial abilities or skills which are constructed using different factors like spatial visualization, spatial orientation, mental rotation, and spatial navigation (Carroll, 1993). Using spatial knowledge and spatial abilities in the analyses can contribute to a better understanding of tasks related to spatial knowledge or geometry (Coutat & al., 2022; Coutat & Berney, 2023). In our research we are particularly interested in spatial navigation, as the ability to locate and move in real or virtual space by using perception and comprehension of the organization of this space (Pick & al., 1999). According to Seigel and White (1975) three stages of developmental spatial knowledge are identified: landmark knowledge, route knowledge and survey knowledge. Landmark knowledge comprises environmental objects perceived and recognized, they can be points of reference, global (seen from far) or local (for more precise movement). The use and recognition of landmarks are essential features of spatial representations. They can become strategical elements of a route, to or from which a person moves (Golledge, 1999). Indeed, landmarks can be used to identify a strategic position (start or arrival) where specific actions may be needed. In this sense, route knowledge is made of the relations a subject is able to construct between landmarks. Duroisin (2015) considers itineraries like an ordered succession of decision points linked together by

[20] In Grades 3 and 4 there were 5 classes which reduced to 4 in Grade 5.

sections of path along which there are different landmarks. Finally, the survey knowledge is the development of a personal and cognitive map.

Navigation in a real or virtual space relies also on spatial orientation and mobilization of different frames of reference, in which we distinguish between egocentric and allocentric points of view. In an egocentric frame of reference relations between objects of the environment are encoded from the person (Chen & Stanney, 2002), while in an allocentric one, these relations are encoded from a point of the space independent of the person (Pennel & al., 2001). The control of these two types of frame of reference and their coordination is crucial in real life and in several professions, e.g. in construction, for builders, plumbers or electricians (Bessot et al., 1992). According to Bell (2002) their efficient control brings a structuring of spatial information and contribute to the development of spatial knowledge.

On another level, it is important to categorize the large variety of spaces to which spatial knowledge may refer as well as our relation to these spaces. Brousseau (1983) followed by Berthelot and Salin (1999) define three different types of space according to the size of the space seen from the perspective of an observer . The micro-space is characterized by the fact that the observer can see it straight at one glance (like a worksheet or small objects). The macro-space, on the other hand, refers to an area that the observer can only see entirely through movement and displacement. Thus, in order to build a representation of the whole space, one needs to do recollections that require processing partial reference points as well as stored and memorized information (neighborhood, school). The meso-space refers to an in-between, a space too wide to be seen at one glance, but which only requires a shift of the observer's gaze. It is typically the classroom or the school courtyard that can be fully seen by turning the head. Developing activities in the macro-space is a key issue, in order to suitably develop good bases for landmark, route and survey knowledge. Yet, to design activities in a macro-space necessitates to take students outside school, which is usually difficult or even impossible, beyond the nearby area of school in the usual conditions of teaching. Therefore, the use of technology for the development of spatial knowledge can be a relevant option for the use of a virtual macro-space to create activities training navigation or rotation abilities. This was our motivation in the use of SPAGEO city, in the part related to didactics of mathematics in this project.

The next part of our contribution presents some characteristics of SPAGEO City and our questioning regarding a specific activity tested during the end of the third year of our study. One can find more examples in Frauchiger et al. (to be published), Matri et al. (2023), Coutat et al. (2022).

OUR QUESTIONING

The technological environment allows different representations of a same position in SPAGEO City. The immersive aspect of the environment offers an immersive 3D view with an egocentric frame of reference where it is possible to spin around (Fig. 1). In that view, all the landmarks around the position are visible. It is possible to identify relations with visible landmarks and the position. This is the perspective closest to a real situation of a person walking in a city. The device however allows another point of view using a map of SPAGEO City, with an allocentric frame of reference (Fig. 2). This view is a model of the city and like all modeling, it has its own codes and gives only partial information. So that, it is necessary to know the different codes or representations to be able to use the map. For example, it is crucial to be able to distinguish the road from constructions. On another hand, some landmarks have to or could deliberately be omitted on the map, like traffic lights or pedestrian crossings. The use of an allocentric frame of reference could influence or support the elaboration of the cognitive map because it gives the relations between the different areas of the city, which are not as easy to see from in the immersive egocentric view. Therefore, the use of the virtual environment without a map, only from an egocentric frame of reference, forces students to construct their own relations between the different locations, an important component of their cognitive map. In our didactical sequences we decided to use only the immersive view during the first year (Grade 2). This was coherent with the curriculum in mathematics in Geneva which does not introduce maps until Grade 3. During the second year, the map (on paper) and the immersive view (on the computer) were accessible at the same time. Finally, in the last year, we introduced a map on the computer, but we chose that it would be impossible to access to the map and the immersive view simultaneously. On the computer screen, either the map or the immersive view of the city appears. Students switch from one view to the other by pressing a specific button on the joystick. The objective was to constrain memorization of information from a frame of reference to use it in the other. A specific activity was proposed in the third year. They had to plan a path between 2 positions with the help of the map and realize it in the immersive view. The study of one of this activity is guide by these questions: how do the students plan a path between two positions and how students use the access of a map when they have to plan a path between two positions?

THE TASK AND THE DIDACTICAL CHOICES

Among the many tasks proposed to students throughout the didactical sequences during the 3 years, we chose to present and analyze in this paper a path planning task, proposed during the final evaluation of the third year (grade 4). A similar task, with different path to plan, had been worked with the students during the second year with simultaneous access to the map (on paper) and the immersive view (on the computer). During third year, this path planning task had been worked two times. Students are expected to make progress on this task over the years. On a practical viewpoint (see Fig. 1 & 2), when one launches the task in the environment, the map appears. It indicates the starting position (yellow dot) and a position (here numbered 1) that one must find. The instructions are written on the top of the page: "Go into the city, stand in exactly the same place as the place indicated on the map". Two situations were proposed, in the first one (Figure 1 1) the target position is in the city center (with lots of landmarks) while the second one (Figure 2) is located in the residential area with very few landmarks and identical houses.

Figure 1: Immersive view on the computer

Figure 2: Map of the residential area

We made some didactical choices to constrain students in planning their path. First of all, we chose to indicate the starting position on the map, but not the orientation. In the city center case, students are facing north (the top of the map), in the residential area, they are facing south. So, once in the immersive view, students must choose the correct direction to start their path, without knowing on the map which is their initial orientation. Then, when students start to move, we have chosen not to update their position on the map. Therefore, they have to memorize their position using the various landmarks which are present on the map. This is fairly easy in the city center case, but quite a bit more complicated in the residential area, where counting the interaction is about the only possible strategy, as well as making use of global landmarks, like the mountains visible in the far distance.

To reach the target position, students can switch from the immersive view to the map back and forth at any time. However, to force a minimal memorization, we have chosen to restrict to 4 switches (students were warned). If this number is reached, students could not access the map anymore and had to remain in the immersive view of the city. When students think they have reached the target point, they have to validate the position and they got a retroaction: if it is correct, they can move to the next task, if not they are brought back to the starting point (yellow dot) and must start again. After 3 failures, a path is drawn on the ground in the immersive view so that they are guided to the target position. The maximum of 4 switches to the map were counted for all the trials.

This task aims to get students to plan a path between two given positions on the map and carried it out in the immersive environment. We are therefore very close to a real situation when using a map in a real city to go to target place. The use of technology here transposes the task in a virtual environment, but it also allows us some didactical artifices: the fact that the map is not visible simultaneously to the immersive view and the number of switches is limited top 4. Those choices operate as constrains on the didactical situation, proper to create better conditions for acquiring the knowledge we aim at.

SOME OF STUDENTS' ANSWERS

This task was proposed to 80 students, out of which 77 were finally exploitable. The software keep data from student's activity like the time spend on the map, number of switches, position every second. 58 (respectively 61) students succeeded on the first go in the city center (respectively in the residential area). So altogether, at the end of our engineering, we can see that such a task is successful. Moreover, we want to analyze deeper the results, to do this, we're looking at how students use the map.

First, Table 1 gives the number of wrong answers crossed with the number of switches to the map used over all the trials.

	Number of wrong					Number of wrong				
Nb	0	1	2	3	Total	0	1	2	3	Total
0	12				12	8				8
1	12	2			14	18		1		19
2	15	1	1		17	22	4			26
3	11	2			13	11	1	1	2	15
4	8	5	2	6	21	2	2	1	4	9
Total	58	10	3	6	77	61	7	3	6	77

Table 1 : Scores for finding a position indicated on the map

For both positions 6 students commit 3 failures. However, they are not the same students. Only 1 student commits 3 failures in the 2 cases. In the city center, 12 students managed to position themselves in the right place on the first go without having to switch again to the map. Among these 12 students, 10 also manage to find the position in the residential area on the first go but with switch at least once on the map. On the opposite, among the 8 students who found the position 2 without error and without switching to the map, there were 6 who also found position 1 without error. Only 2 students manage to find the two positions without error and without switching.

Now, if we look at the number of switches between the map and the immersive view, 21 students used 4 switches for position 1 while only 9 needed 4 switches for position 2.

The two tasks, although different, are performed equally well by the students. But do they have identical strategies between the two districts? If we look at the video recordings from the action cameras that the students had on their head, we can analyze some gestural information. Only 57 students out of 77 have a complete video recording of the exercise. Of these 57 students, it appears that 30 students point out intersections at least in one of the two cases (city center and residential area). We can then assume that these students used a numerical strategy, counting intersections to code their path. On another hand, the videos show that 3 students draw the route with their finger on the map. This shows that these students plan their route on the map before they execute it in the immersive environment. Yet, it is not possible to know how they memorize at which crossroad they have to change direction and in which sense. The 24 remaining students are motionless and silent in front of the map for a few minutes then switch to the immersive view to initiate a movement. For these students no hypothesis about their strategy can be made based on their actions.

Let us now go in deeper analyses for the students with a counting strategy: It seems that memorizing information is not the same for all of them and also differ for both situations. Indeed, looking at how these students use the switch to the map, it seems that distinctions appear. Among the 30 students who count the crossroads, only 8 mobilize their first switch to the map under the same conditions in both situations: at the starting point (2 students), on the path (1 student) at a crossroads with change of direction (4 students) or at the target position (1 student). For the 22 other students, the choices are different with almost all possible combinations. For example, 5 students do not need a switch to the map before getting to the target point in the city center, but they need it in the residential area at the start point (2 students), on the path (2 students) or at a crossroads with change of direction (1 student).

To identify whether these variations are also present in the other students, we organized the students according to their use of the switch to the map. We thus measured the difference between the total number of switches mobilized in the city center and the total number of switches in the residential area. It turns out that 3 profiles appear. A first category of 35 students out of the 77 make more switches to the map when they are in the city center. Among these students, at least 15 used counting at least in one case. The second category is made of 27 students who make more switches to the map in the city center than in the residential area. Among them, at least 13 counted the intersections (from what we can infer from the videos). Finally, 15 students use the same number of switches in the two cases. At least 2 of them count the intersections. Thus the 30 students who use a counting strategy are divided into the three categories of students. We can assume that other elements complete this strategy of coding the path when it is carried out in the immersive view.

CONCLUSION

As we mentioned it, one of our main initial questions was to know how students plan a path between two positions and how they use the access to the map. It is difficult to answer these questions in the sense that in our experiment students were on their own and therefore did not speak. However, regarding the use of the map, it seems that for the vast majority, it is a real support to plan their route and then realize it in the immersive environment. Moreover, it seems that students are able to use the map adequately to reach the target position. It also seems that having access to the map contributes to the realization of the planned path, even if their position is not updated on the map. It can thus be assumed that students are able to memorize their position as they move in the immersive environment. However, other parameters can also come into play.

Concerning the path planning procedure, even if our data are limited, it seems that several students use the map to plan their route and code it by counting the intersections. Moreover, even if the counting strategy is valid both in the city center and in the residential area, it seems that the specificities of the area and the number of landmarks influence the realization of the planned path and the use of this counting strategy. One might wonder what influence play the landmarks in the city center and to what extent they can be distractors or aids when traveling?

For the students who mobilized more switches to the map in the city center, one could assume that this provided a first experience which influenced what they did in the second case. They are thus more prepared and anticipate their path planning better when in the residential area.

The city center has many landmarks that can be used to "control" the path. Students can thus use switches to the map at specific landmarks to make sure that they are on the correct path. Moreover, the absence of local landmarks in the residential area can have two opposing effects on students. On one side, students may feel less tempted to frequently check if they are on the right path by switching to the map. On the other side, since students do not have reference points in order to control their path, they can use more switches to the map to control their progress. Thus, they compensate the lack of landmarks to control their path by switching more often to the map.

This gives a very limited view of all what we tested over the three years in our experiment. This study on path planning especially echoes the study of another task, that we experimented with our students; indicating on the map landmarks that were seen in the immersive. These two tasks are dual in the sense that they are similar but with the inversion of the role of the reference frames. We are now finalizing the analyses of most of the various activities we experimented over the three years. Our next goal will be to select the most significant activities in order to design a sequence that could be implemented in classes and to organize an in-service teachers' training to help teachers invest in the use of SPAGEO city in the real conditions of everyday teaching,

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ADVANCES IN GEOMETRY EDUCATION
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LEARNING COORDINATE GEOMETRY WITH SCRATCH: TASK DESIGN FROM AN EMBODIED AND APOS APPROACH

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This paper presents the designs of a series programming-based mathematical tasks targeted on coordinate geometry learning. Grounded in APOS theory and embodied cognition, we elaborate on how the task activities via Scratch, a block-based visual programming environment, support students' mathematical development alongside programming concepts and skills. We highlight the embodied experience prompted by the task design, as well as the progression from concrete actions to abstract mathematical thinking, thereby fostering a rich and holistic understanding of coordinate geometry.

INTRODUCTION

Our work builds on the second author's previously developed conception of "learning as Making" (Ng & Chan, 2019; Ng & Ferrara 2020) to envision a computationally enhanced mathematics education. In particular, we were interested in screen-based programming as a form of digital making to support mathematical problem-solving in fundamental ways. Programming-based mathematical activities involves students' active creation of programming artefacts; it transforms mathematical problem-solving from merely using formulae and performing arithmetic calculations procedurally to endangering CT concepts and practices such as sequences, variable-naming, abstraction, algorithmic thinking, decomposing, and iterating, through which scientific inquiry, mathematical thinking, and engineering design can also be exhibited as integrated STEM learning (Ng & Cui, 2021).

In the past three years, the research team as led by the second author has made significant efforts and progress in designing problem-based mathematical activities with content appropriate to senior primary and junior secondary mathematics curricula in Hong Kong. This paper aims to instantiate such an exploration by delving into the intersection between coordinate geometry learning and computer programming. Specifically, we discuss our recent and novel design of regarding Scratch, a block-based programming environment, as a boundary object (Ng et al., 2023a), which mediates between coordinate geometry and computer science concepts. In this paper, we discuss the designs of these programming-based mathematical tasks, which aimed to support students' mathematical development alongside programming concepts and skills while using Scratch as a necessary cognitive and physical tool in the task solutions. Our aim for this paper is to facilitate discussion regarding the potential for a pedagogical approach which conceptualizes embodiment and the progression from concrete actions to abstract mathematical thinking in coordinate geometry via Scratch.

THEORETICAL FRAMEWORK

APOS for the integration of programming and mathematics

The APOS theory, an acronym for Action, Process, Object, and Schema, is a cognitive framework from the constructivist tradition that elucidates the stages of thinking and learning mathematical concepts. Originating from the research of Dubinsky (1991), this approach postulates that a learner's comprehension of mathematical ideas evolves through distinct phases. It suggests that, initially, individuals perform algorithmic actions (Action) without intrinsic understanding. As they progress, these actions are internalized into coherent, repeatable patterns (Process). Further abstraction sees these processes conceived as holistic, manipulable entities (Object). Ultimately, learners integrate these objects into broader interconnected networks of understanding (Schema). Further supported by studies such as Arnon et al. (2014), this theory offers educators a robust framework for assessing and facilitating mathematical learning.

Taking consideration of a programming context, we suggest that the APOS approach is particularly suitable for the integration of programming and mathematics. We take the block-based programming environment, Scratch, as an example. Programming always starts with concrete actions. Users drag and drop blocks, each of the motion corresponding to specific commands. Users can see the output of these commands without further explaining, which is exactly the action stage. By exploring different commands and construct different codes, the users may see some common features or patterns between the commands and their outputs, which align with the transition from using individual commands (actions) to forming the flow and logic of codes (process). Meanwhile, pattern recognition is an important construct of computational thinking (Weintrop & Wilensky, 2015); it supports the development to process stage. Advanced features in Scratch, such as creating custom blocks (My Block) or cloning sprites allow users to encapsulate scripts or behaviors as singular objects. This movement from seeing codes as a process to understanding it as modular, reusable objects mirrors the Object stage in APOS. As users become proficient in programming, they begin to see the interconnectedness of various programming concepts, such as the integration of loops, and conditionals with events and variables. Moreover, they might advance their thinking to conceptualize broader projects through "remixing" (Brennan & Resnick, 2012). This holistic understanding aligns with the Schema stage of APOS.

Embodied cognition

Embodied cognition is a transformative perspective in cognitive science which posits that the mind is not just a product of the brain but is deeply intertwined with the body's interactions with its environment. This theory challenges traditional cognitive views that treat mental processes as abstract and detached from physical experiences. Grounded in the works of scholars like Varela et al. (1991) and furthered by Clark (1999), the essence of embodied cognition is the assertion that our bodily experiences, sensorimotor activities, and environmental interactions play a crucial role in shaping cognitive processes, from perception to reasoning.

Recent studies highlight the successful incorporation of embodied learning in various classroom environments, including mathematics classrooms. For example, scholars have introduced both individualized and collective interventions that encourage bodily engagements in the learning process (e.g., Nemirovsky et al., 2020), and to explore abstract concepts physically (e.g., Ng & Ferrara, 2020). Designing learning environments and tasks that effectively create the conditions for embodied learning is receiving increasing attentions. Abrahamson (2014) introduced an action-based embodied design for mathematics learning by intertwining both intuitive kinesthetic interactions and scientific forms. This approach aligns with Dourish (2001)'s views on the third wave human-computer interaction (HCI), focusing on embodied interaction with a new wave of digital tools, such as wearable devices and augmented reality. In this study, we adopt the perspectives of embodied cognition and HCI to create and provide learning conditions fundamentally anchored in real-time and space, where embodiment is not just central but forms the very essence of the learning experience.

TASK DESIGN WITH AN EMBODIED AND APOS APPROACH

We will first briefly introduce the features of Scratch which are relevant to the study. Then, we present four sequential examples of our designed tasks of integrating Scratch into coordinate geometry learning and explain them from both embodied and APOS perspectives.

Introduction to Scratch

In the vast realm of programming languages and environments, Scratch stands out with its unique approach to introducing computational concepts to novices. Central to its distinctiveness are two interactive features (Figure 1): the stage and the characters (often referred to as "sprites"). These features not only set Scratch apart from traditional text-based programming languages but also provide an intuitive platform that resonates with the way humans naturally process information—visually and narratively. While most programming environments focus on abstract code structures and command-line outputs, Scratch embraces a more tangible (drag-and-drop), visual, and story-driven paradigm.

Stage is a space where commands can be executed as visual repercussions. It is not just a blank canvas; instead, it is expressed as a coordinate system. Centered at (0,0), the stage spans its x-coordinates horizontally ($-240 \leq x \leq 240$) and y-coordinates vertically $-180 \leq y \leq 180$. Importantly, the visual and dynamic representations of programming (Ng et al., 2023b) combined with the coordinate system inherently makes it possible to teaching and learning coordinate geometry.

On the other hand, the sprites, or the use of characters, can make the programming outputs animate and actionable. These characters can be anything—a cat, a ball, a person, or even custom creations. Each sprite operates like an independent actor by executing the scripts that it corresponds. They can interact with one another, react to user inputs, or even change their appearance and behavior based on the code behind them. By programming these sprites, learners are essentially directing a play, making coding an act of storytelling. The affordances of sprites lie not just in their dynamic nature, but crucially, in their ability to prompt embodied learning. As learners code the behaviours of sprites, they are not merely executing abstract commands; they're engaging in a form of digital puppetry, imbuing characters with life and narrative. This act of animating sprites fosters a deeper, more visceral understanding of code, as users see and "feel" the effects of their programming.

Figure 1: Stage and character of Scratch

Task 1: From movement to coordinate system

In the following sections, we illustrate four task designs that uniquely support mathematical thinking in the area of coordinate geometry, specifically in the junior high school level. The aim of the first task is to familiarize learners with the Scratch environment, particularly the stage, and to introduce the concept of movement in relation to the coordinate system.

Action. First, we would guide the student to use three different ways of cause movement of a sprite (Figure 2a): move towards certain direction; change x or y; go to x, y.

Process. There are two sub-tasks. First, we challenge students to predict where the sprite will land if given a set of coordinates. Besides, we ask them to describe the sprite's movement in relation to the stage's grid. They will observe how the three ways to move are interchangeable. Second, students are guided to use as many ways as possible to draw a right triangle and a square (Figure 2b-c).

Object and schema extensions. We introduce “My block” to chunk the drawing of a right triangle and a square, so that we can easily draw these geometric shapes by recalling the function, “My Block,” and can draw them at different location of the canvas. We would encourage students to visualize the entire journey of the sprite, understanding its path as a holistic representation of a sequence of coordinates. Finally, we prompt students to abstract the concept that any feature of the geometric shapes in the stage are related to and can be represented with coordinates.

Embodiment embedded. Throughout the task, we would encourage students to map the movement of the “Sprite” with their physical movement such as gesture or moving in the classroom. For example, in the task of using multiple ways to draw a square, figure 2d presents three different common approaches. Students can simulate the action of move and turn (such as walking) and moving left and right but keep their heading unchanged (change x or y).

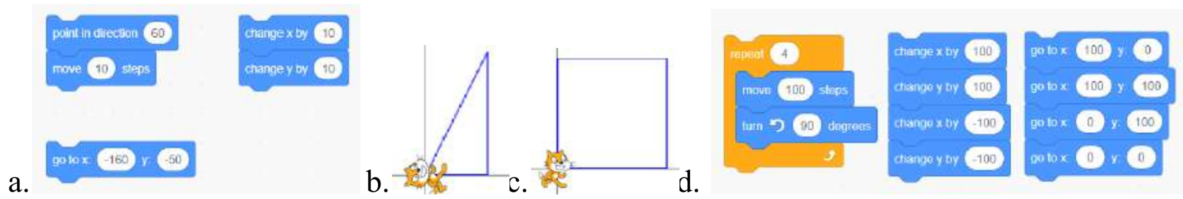

Figure 2: Different ways of movement and construct simple geometric shapes

Task 2: Parallel lines and slope as paths

Drawing upon the foundational understanding from Task 1, our next exploration dives deeper into the nuances of slopes and parallel lines in coordinate geometry through an embodied programming experience in the *xy*-coordinate stage in Scratch.

Action. We will ask students to draw a right triangle using two characters (Figure 3a); for example, the cat represents the right-angle edge and the bear for the hypotenuse. The key code for the bear is “point towards [cat]”, which can determine the direction of the bear’s movement.

Process. There are three sub-tasks. First, we will prompt the students to manipulate the triangles they have created. The students will need to change the starting and final positions of the sprites by using the codes “go to x, y” and “move [ ] steps”. They will observe how the direction of the bear changes according to the cat’s movement. We will also prompt them to try negative input values (Figure 3b).

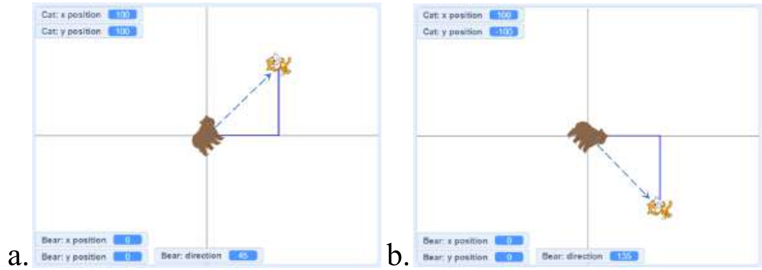

Figure 3. Different movement settings and directions of hypotenuse

Second, the students will draw another right triangle upon step 1 (Figure 4a). There are two requirements for the second triangle: (1) the path lengths must be different from the first one, and (2) the bear should move in the same direction as the first. A sample set of codes is presented in Figure 4b. When following the codes, there will be two cycles of sequential movements of "cat run and rise" and "bear climbing." Students may try different value settings and observe the "bear direction". Finally, the students are challenged to draw two bears' movement at different starting locations such that the two bears will never meet if they continue the movements in the same direction (Figure 4c).

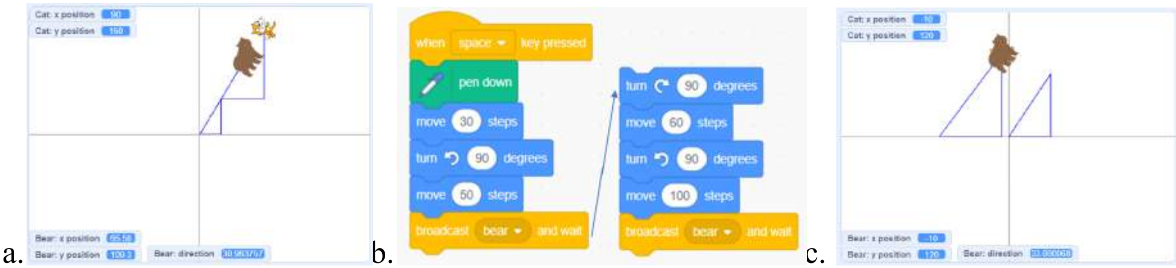

Figure 4. Different size and position of triangles with same slope

These tasks were meant to prompt the students to reflect on what lengths of paths travelled (the "rise" and "run") would make the triangles similar (or the lines parallel). In other words, they are intended to foster the mental construction of the concept of slope in a few ways: (a) Slope determines the direction of a line. Defining a "rise" and a "run" determines the slope of a line; (b) three points are collinear if they have the same slope; (c) line segments of different lengths at different locations can have the same slope, making them parallel to each other; (d) slope is determined by the ratio of the changes in y and x, not the absolute magnitude; and (e) positive and negative changes in y and x make positive and negative slopes. A final extension task can be to explore the relationship between slopes of horizontal and vertical lines, as well as slopes of perpendicular lines. They can do so by being tasked with drawing different paths of the bear that are horizontal, vertical, and perpendicular.

Embodiment embedded. We will connect the cat's movement with real-life situation such as staircases, in which "rise" and "run" are embedded. Also, we will connect to the bear's movement with real-life situation such as walking up/down a hill, in which "rise" and "run" are embedded. We will draw attention to the embodied experience of how they go up or dawn, forward or backward, as constructing the real-life meanings of slope, and what makes it positive or negative.

Task 3: Distance formula and pythagorean theorem as a racing game

As we continue the journey through the world of coordinate geometry using Scratch, the next waypoint focuses on two fundamental concepts: position and distance. This task aims to introduce and deepen learners' understanding of the relationship between position on the Scratch stage and the distance between sprites or points. Through this, students will also begin to grasp foundational concepts in analytic geometry.

Action and Process. There are two sub-tasks. The first sub-task is plotting points and distance. we instruct students to plot two separate points and use the function "distance to" to observe the distance between them (Figure 5a). Then, they shall devise two ways to begin the sprites' journeys (Figure 5b). The first way is changing x and y, and draw the right-angled sides of the triangle. The other way is to use "point direction" and "move steps". We will ask students to manipulate the values and observe the distance from the starting point (for the first approach), and the final x and y positions (for point and move approach).

The second sub-task is shading the squares. We ask students to draw a shaded square with repeated move and turn (Figure 5c), during which they will see the actions as a repeated process of moving and turning, and competent in the process of a 2-dimensional shaded region as a set of paths.

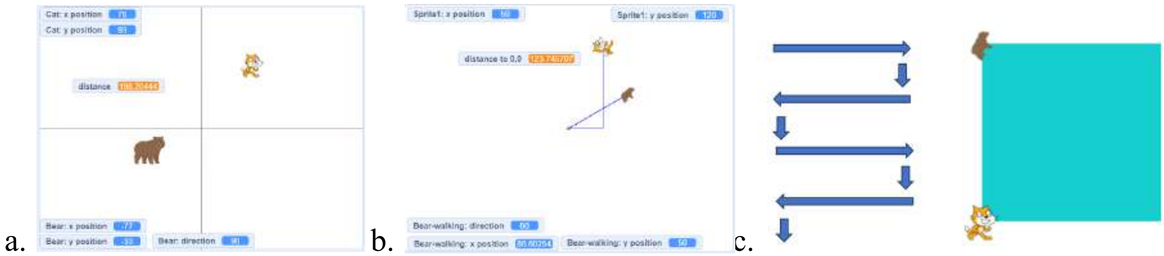

Figure 5. Move and turn, distance between points

Object: First, we will have the students encapsulate the steps into a single program using "My Block" so that they can use the self-constructed function to paint a square in any position, direction, and size. Having encapsulated the process of shading squares as an object, "shaded square," we can then pose a challenge to the students to shade three solid squares formed by each of the three sides of the right triangle, and to observe the progress of the journeys (Figure 6). They will be asked why the painting will finish at the same time in relation to the squares' area. Meanwhile, we will have the students relate the side lengths of the triangles to generate the distance formula, and use the relationship founded in the observed time taken for the sprites' journeys to generate the Pythagorean Theorem.

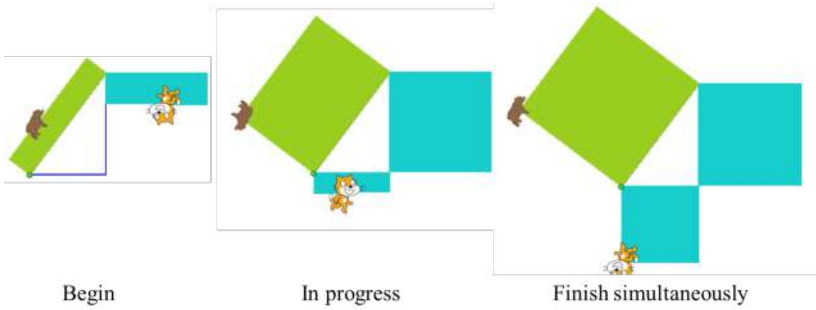

Figure 6. The "painting race"

Embodiment embedded. Students will use gestures and physical movement to simulate the turning and movement in the painting solid square task. Observe the "racing" between the two characters provides the situated experiences of how the sum of the two smaller squares equals to the largest square rather than just an abstract formula.

Task 4: Plotting and shading as stamping

Building upon the earlier tasks, the last exploration links the coordinate system with functions. Conditional will be used to serve as the basis of plotting points or graphing. The aim of this task is to introduce students to the representation of mathematical conditions and functions on the Scratch stage, guiding them to visually and interactively experience the relationship between functions and graphing.

Action. Students will revise their previous program to shade a solid square to paint the whole stage of Scratch. Then, they will use coordinate rather than turn and move to realize the function of paint the whole stage (Figure 7a). Instead of painting the whole stage, the students will be prompted to paint certain areas such as the one shown in Figure 7 (b-d). This will help the students to become familiarized with graphing with conditions. For examples, as the character is navigating throughout the stage, rather than leaving a "stamp" (or a mark) at every position of the stage, the character will only stamp when particular conditions are met (e.g., x position = 0 in Figure 7b).

Process. The students will explore different ways to shade/graph only half of the stage (Figure 7c) by creating conditions that achieve such outcome. They will manipulate their graphing into different shapes, positions (e.g., Figure 7c and 7d). Prompt them to observe how they are changing the value and construct of the conditions.

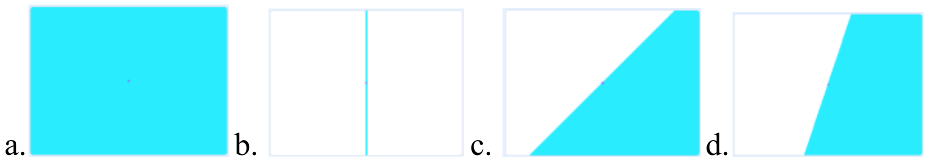

Figure 7. Painting the whole stage and painting certain areas

Object. First, we prompt the students to reconsider their navigating program as an object, in which it serves as a function to draw any figures according to multiple conditional statements. Then, we will have the students link the graphing tasks with the concept of slope that students just experienced. They shall connect the concept of slope with the conditions used in graphing. For example, to create the graph in Figure 7d, a typical condition could be "y position < 3 * x position", where the value of 3 represents the slope of the line which defines the graph. Finally, they will construct the linear functions in the form of $y = mx + b$ and manipulate the parameters m and b to observe their effects on the graphs.

Schema. Prompt the students to explore different conditions that represent the relationships between x and y positions in the conditions. The aim is to prompt students' thinking of how functions can represent different geometric shapes in coordinate systems.

Embodiment embedded. The action of "stamping" when meeting specific conditions situates the abstract concept of function into a motion of the characters in Scratch. Students will be prompted to think about "what should I do if I am the cat (bear)". These kind of situated simulation and embodied experience grounds the abstract idea of coordinates and functions in tangible actions, enhancing the embodied understanding.

DISCUSSION AND CONCLUSION

The exploration of integrating Scratch into mathematics education marks a shift from traditional mathematics instructional paradigms. We suggest that combining programming and mathematics doesn't merely offer an engaging platform for students; it also nurtures deeper cognitive links through embodied experiences and the gradual construction of abstract concepts. The melding of the APOS framework with embodied cognition in a digital environment like Scratch promotes algorithmic thinking and grounds abstract mathematical concepts in tangible, real-world actions. However, while the potential of this approach is vast, effective implementation necessitates an understanding of both teachers' and students' capabilities. Further empirical research is needed to evaluate the effectiveness of such designs.

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CONTRUBITION OF GESTURES TO THE ACQUISITION OF GEOMETRIC PROPERTIES

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This paper examines how gestures made by pupils aged between 5 and 11 during the process of drawing straight lines can help them discover the characteristics of plane figures. We also explore how this knowledge evolves throughout elementary school. The aim of our research is to show that it is possible to design problems that involve the manipulation of 2D or 3D geometric objects, thereby contributing to the conceptualization of their properties.

PRESENTATION

This text proposes how to more specifically answer the following research questions: how it is possible to ensure that students' knowledge derived from their actions toward objects contributes to their learning of figure properties in the sense of Fischbein (1993)? What problems can be used to support this conceptualization?

To address these points, we will draw on experimental results concerning the following:

- Knowledge of the properties of geometric objects, which ranges from shape comparison conducted in the “Grande Section” (GS in France: 5-6-year-old students) to figure analysis and reasoning based on answering the question "Can a triangle have two right angles?" in the “Cours Moyen” during the second year (CM2 in France: 10-11-year-old students).
- Knowledge of straight lines, which ranges from learning to draw straight lines in the “Cours Préparatoire” (CP in France: 6-7-year-old students) to answering the question "How many points are needed to draw a straight line?" in the “Cours Moyen” during the first year (CM1 in France: 9-10-year-old students).

OBSERVATIONS ON GEOMETRY TEACHING

Geometry teaching in primary and secondary schools has a dual purpose. On the one hand, it provides students with the knowledge they need to master their relationship with space and, in particular, to model phenomena covered in various other disciplines. On the other hand, it enables students to gradually discover this field of mathematics—with its objects of knowledge and its methods of proof (Balacheff, 2020)—the status of which will change over the course of their schooling (Houdement & Kuzniak, 2003). Our interests are in line with those expressed by Sinclair and Bruce (2015 p 320) who analyses a group of articles that highlight current trends of research: “(1) the role of spatial reasoning and its connection to school mathematics in general and school geometry in particular; (2) the function of drawing in the construction of geometric meaning;” and “(5) extending primary school geometry from its typical passive emphasis on vocabulary (naming and sorting shapes by properties) to a more active meaning-making orientation to geometry (including composing/ decomposing,

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 383-390). ICMI.

classifying, mapping and orienting, comparing and mentally manipulating two- and three-dimensional figures).”

A short analysis of easily available French resources for teaching geometry to young pupils shows that two main goals are pursued: teaching the vocabulary of geometry and drawing abilities. Spatial activities also exist, but their sole purpose seems to be manipulation itself, without this being a means of solving problems. These activities also seem unrelated to the pupil’s initial knowledge or mathematical reasoning. Teachers also frequently complain that activities are somewhat repetitive from one year to the next starting at the beginning of primary school and moving forward. We therefore have to identify the components of learning and the knowledge initially available to pupils.

RESEARCH BY THE ERMEL TEAM

The research conducted by the ERMEL team (a team associated with the French Institute of Education at the “École Normale Supérieure” in Lyon) on learning mathematics and problem-solving in primary school in France has two goals. Once the needs are identified, such research involves several steps: (1) An analysis of the mathematical knowledge (problems, properties...) at stake, as well as students’ knowledge and abilities; (2) An explanation of educational issues and the organizing of the study of the different notions throughout the years; (3) Development of teaching situations and tests in several classrooms. These last three components interact: the identification of students’ abilities is the outcome of experiments conducted. (4) Writing a book for teachers and trainers with an explanation of the issues of learning and teaching, a description of the learning situations identified, a reasoned choice of learning roadmap and syllabus planning accompanied by analyses and theoretical contributions (Douaire et al. 2020, 2023).

The results of these experiments, which have been carried out in ten and, in some cases, thirty classes over several years (1998 to 2019), have led us to question some of the assumptions on which certain learning activities are based and to modify both the related situations and the organization of learning. In this presentation, we do not detail the many changes needed in the description of the situations related to the successive and necessary experiments necessary for students to produce the specified procedures. Our goal is to build a proven, complete and reliable approach to teaching engineering. These publications are designed to offer a coherent vision of spatial and geometric learning in terms of both content and the role of problem-solving. All the tasks the teacher has to carry out have been precisely described.

TAKING STUDENT KNOWLEDGE INTO ACCOUNT

Since early childhood, pupils have acquired knowledge of objects in space, for example, through 3D construction games and 2D shape assemblies. The gestures they make while using these objects ("sliding", "turning", "flipping", "nesting"...) constitute actions that the child develops with the precise intention of completing a task. We will distinguish such gestures from those that are used to communicate (Petitfour, 2017), those used with instruments, or even "routine" gestures made when efficiently using familiar objects (e.g., unfolding a sheet of paper, etc.). These gestures apply to objects of the same type, for instance, flat polygonal shapes and polyhedral solids, which may have different appearances. Gestures are therefore common to several activities. Pupils use these gestures to successfully juxtapose, nest, and adjust the relative positions of two or more objects, as well as to compare or assemble them. These gestures are anticipated and followed by an assessment of their effects. These assessments of success or failure are generally not accompanied by any verbalization other than "it is working" or "it is not working". In this sense, these gestures constitute procedures. "The gesture - or action (sequence of gestures) - is defined not by the movements performed, but by the intention of its author, who decides on its execution as a function of its goal or purpose. He represents its effects and controls its execution… Every gesture has two components, one motor and the other cognitive. The cognitive aspects of gestures include all preparatory aspects of the action. But often only the intention to act is conscious: the subject knows and represents the result, the effects of his act... At this stage, the representation does not contain details of the precise way to execute the movements in each sequence". (Mazeau & Pouhet, 2014 p114). In the context of a didactic situation, gestures can be made explicit, and their effects can be analyzed. However, this knowledge is rarely considered in teaching methods at the start of elementary school; rather, gestures made with objects are often wrongly perceived as "manipulations" devoid of anticipation or intentionality. In the field of numerical learning, however, the knowledge that pupils have before tackling a new notion has been analyzed for several decades by numerous studies in the fields of psychology and didactics. Hence, in geometry, teachers are often at a loss to recognize pupils' potential, identify their procedures and exploit the observations they make about their productions.

A PROBLEM TO ENCOURAGE THE DISTINCTION BETWEEN SQUARES, RECTANGLES AND RHOMBUSES

Starting in kindergarten, pupils become familiar with real or pictured objects, and they use global perception to identify such objects with the support of a metaphorical vocabulary (e.g., "mountain" for pyramid, Coutat & Vendeira, 2019). Naming confusion is common between neighboring objects. In fact, numerous studies have shown that such confusion continues to persist following not only teaching that is focused on the simple and correct naming of the geometric shapes of physical objects or their graphic representations but also tasks involving the classification of objects according to instructions such as "put together what goes together". In other words, the geometric properties enunciated on these occasions do not actually become operative; the old criteria for recognizing shapes, based on similarities with real-world objects, remain. For example, we have found that a rectangle whose length and width are close to each other is still often referred to as a square by pupils in “Cours Élémentaire, first year” (CE1, students aged 7-8) and even in “Cours Élémentaire, second year” (CE2, students aged 8-9). The knowledge of the properties of figures and solids must, however, appear to pupils as necessity and not convention. “There is a growing amount of evidence both from psychology and mathematics education showing that children come to school with a great deal of informal geometric understanding”. (Sinclair & Bruce, 2015, p 322). To overcome the ambiguities of perception, one of our hypotheses is that highlighting the properties needed to distinguish between neighboring objects with differences that are difficult to discern is better than asking students to classify objects that are already perceptually different.

Square or quasi-squares’ situation

In the "square or quasisquare" situation, 6-7-year-old pupils have to produce an assembly of squares, rhombuses and rectangles whose dimensions are very similar (4 cm for square and rhombus sides, 4 cm ± 3 mm for rectangle sides, and an angle of 90 ± 4° for rhombus corners). Experiments have shown that these pieces are too similar to be distinguished when side by side; for the students, such differences are interpreted as cutting-related inaccuracies. One of our hypotheses is that by assembling these shapes, the characteristics of the square, rhombus and rectangle could be better identified (Douaire et al., 2023).

In the first problem, students are asked to produce an assembly of four pieces taken from a set of eight pieces, namely, four squares and four rhombuses, without leaving any gaps. They then make a record of the assembly by tracing its outline. In a second problem, the pupils have to try to construct an assembly of three squares and one rhombus. After several attempts, they find that it is impossible to produce an assembly that does not leave an empty space between the four shapes. Many students discover that the angles of the rhombus are not the same and that they differ from those of the square. They also find that if a square is turned, the assembly remained unchanged, whereas by turning a rhombus, a gap or overlap is formed. In a third problem, pupils are asked to align rectangles and squares to produce the longest possible assembly. Pupils express observations about either their actions, such as "When you turn the rectangle, it gets longer", or their properties, such as "There's a side where it is long, and a side where it is short". Some students discover differences between a square and a rectangle: "If you turn them [squares], they always look the same"; "A square is not like a rectangle, turning it does not lengthen the line (implied)". This makes it clear that, unlike a rectangle, the sides of a square are of equal length.

Findings

Four results can be stated: (1) Object properties are identified and formulated by the students. This problem contributes to a shift from the global recognition of shapes perceived as similar to their identification through the analysis of distinct geometric properties. (2) The production of solutions is made possible by the ability to place, juxtapose and swap pieces, acquired in games since kindergarten. (3) It was the assembling of similar-shaped pieces that made visible differences that the pupils had not initially perceived. (4) To overcome the ambiguities of perception, highlighting the properties needed to distinguish between neighboring objects whose differences are difficult to discern is more effective than asking students to classify objects that are already perceptually different.

A long-term evolution of knowledge

By solving these problems, the criteria for distinguishing between neighboring objects will evolve between the ages of 5 and 7 years old. First, designations may be based on a known object from the physical world ("a pyramid"); later, designations may be made by pointing out a local physical feature (e.g., "that one is pointed and the other one is not") and ultimately by geometric criteria. From the age of 7 years old and onward, the gradual technical mastery of drawing straight lines enables students to work on figures rather than shapes. Drawing these figures enables the study of relationships (perpendicularity, parallelism, symmetry, etc.) to complete a drawing so that it, e.g., represents a rectangle.

FROM PRODUCING LINES TO ANALYZING THE GEOMETRIC PROPERTIES OF THE STRAIGHT LINE

Approaching the concept of a straight line in elementary school involves several aspects related to the questions posed in this subtheme. At this level, we cannot yet talk about the "notion of a straight line", and the idea that a straight line can be "extended to infinity" is not yet accessible. At the beginning of elementary school, straight lines are often introduced as tools for reproducing geometric drawings; these practices do not always allow pupils' different initial conceptions to express themselves, whether they are erroneous or potentially useful. How can these conceptions be considered and taught in a way that challenges students if necessary? Should the concept of a straight line be approached on the basis of one of the spatial experiences that pupils may have encountered, e.g., a taut wire, the edge of a solid shape, a ray of light, a fold in a sheet of paper, the rectilinear trajectory of a moving object, all of which would be modeled by a line? Or is it better to define a straight line from two distinct points?

After conducting several experiments with 7-year-old students based on one or more of these choices, we found that the transition from solving spatial problems in which a straight line was the solution for aligning objects to modeling them on a sheet of paper presented several difficulties. On the one hand, this modeling was handled by the selected device and not by the student. In other words, it was the teacher who had to state that the e.g., taut wire, , was represented by a straight line. On the other hand, the students drew broken lines without realizing why their solutions were wrong. In computer simulations using 3D dynamic geometry software, the same problems were solved by students using the "straight line" tool. However, when these problems were repeated on pencil and paper, students were no longer successful. Thus, the straight line did not appear as the solution for aligning points based on the modeling of situations in space.

Using a tool that is too short

From CP onward (6 years old and older), if pupils are to understand that straight lines can be extended and learn to do so, we feel that it is necessary for them to be able to produce tracings to solve problems. These drawings can then be checked and improved. Thus, in a study, pupils were asked to complete a network of (parallel) lines on an A3 sheet of paper (29.7 x 42 cm) with a geometrical instrument no longer than 20 cm. They were therefore required to switch the ruler, the instruction being: "You must extend the lines drawn" (cf. Figure 1).

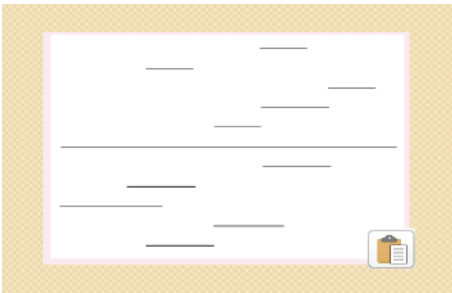

Figure 1. The support

We carried out experiments in approximately ten first-grade classes (6-year-old students) over several years, successively proposing several tracings of this type to the pupils in rectangular or round-cut sheet formats.

Findings: A wide variety of products

In this problem, we observed: freehand tracing; lines partially drawn with a ruler but sometimes extended by hand; lines drawn with a ruler made up of broken lines; and correct tracings, approximately one-third of the total number, with possible inaccuracies in the continuity of the lines.

During the pooling of the results, both the analysis of the productions and the formulation by the pupils of the conformity and the differences between the tracings enabled different judgment criteria to be made explicit, which constituted one of the components of this learning process. Such judgments were based on students' ability to perceive the difference between a curved and a straight line (Wu & Ma, 2005). These criteria included understanding the task and the technical mastery of ruler use and ruler transfer technology. To succeed, students had to be able to do the following:

- understand the goal to be achieved, namely, that a straight line must be drawn (the sight of either intersecting/diverging lines or lines drawn freehand was enough to disqualify some of the productions);
- master the drawing technique (position the ruler, hold it firmly in the middle without letting your fingers protrude, place the pencil, etc.);
- know how to switch the ruler when it is too short (i.e., place an intermediate point or slide the ruler over a sufficiently large part of the line), which presupposes that these techniques are formulated so that they can be shared and understood.

When successive problems were solved in the subsequent sessions, most pupils produced straight lines, often with technical inaccuracies concerning the switch of the ruler or the positioning of the pencil along it. Thus, such use of a ruler should be reinforced on numerous occasions, both mathematical and nonmathematical (underlining, framing, etc.).

THE RELATIONSHIP WITH SENSITIVE SPACE: A LONG-TERM EVOLUTION OF PROCEDURES AND EVIDENCE

Knowledge of a straight line and its properties continue in two directions in our proposals for elementary school.

From CE2 (7 years old) onward, straight lines will be used to solve problems involving the alignment of points. Drawings on a sheet of paper serve as solutions to problems associating points and straight lines and representing spatial situations (e.g., determining the location of a fold hidden under a sheet of paper where only two of its points are known and marked by pins).

In the “Cours Moyen” (9-11-year-old students), drawing straight lines are part of the solutions to problems of parallelism, perpendicularity and symmetry. The question "How many points do you need to draw a straight line?" often arises after solving problems in which pupils must draw parallels. Some students are still content to draw lines in a guessed direction from one point, while others claim that two points are enough, and still others insist on using at least three points based on the argument that doing so "is more precise". The point of such an exchange is to enable students to distinguish between the properties of the straight line, which is defined by two distinct points, and the technical requirement for precision in drawing, for which many students recognize their own limitations.

For the study of geometric objects and modeling, we therefore give priority to problem-solving involving the following:

- Engaging in actions (assemblies, comparisons...) regarding physical objects (shapes, solids) in grades GS to CE1 (from 5 to 7 years old), which enables pupils to apprehend the properties of such objects. Then, in CE1 and CE2 (7 and 8 years old), students start to construct figures.

- The simple evocation of spatial contexts from CM1 (9 years old) onward and engaging in problem-solving by tracing on a sheet of paper. Solutions are validated by students describing how they were reached.
- Decontextualizing problems in CM2 (10 years old), such as "Can a triangle have two right angles?" Discussions between pupils help them understand that their various attempts at tracing do not enable them to answer such questions. Validation of their solutions are thus based on the statement of properties ("If there are two right angles, two sides are perpendicular to the third, then these two sides are therefore parallel; thus, it is not a triangle") and not on the accuracy of a drawing.

We previously mentioned a shift from parts assembly problems to drawing problems (graphical problems); to that extent, the sensitive space is changed.

Research findings

The proofs produced by the pupils will therefore be constituted successively by the following:

- practical validation based on superposition;
- the formulation of the geometric characteristics of the shapes;
- the justification of their constructions made by describing each of their stages; and
- the production of reasoning based on known geometric properties.

This evolution is also expressed in language forms, as part of a language used by a student to control his or her actions or to communicate about a production (formulation of a procedure, checking the validation constraints of a solution, etc.).

ABOUT THE RESOURCE

As mentioned at the beginning of this text, the second goal of our team is to make the results of our research available to teachers and trainers. The tools (e.g., progressions, situations) we have developed promote knowledge construction based on problem-solving. They have a certain "robustness" due in particular to the fact that the results and procedures that will be produced by students in a class are described in the situations, allowing teachers, who are generally not mathematics specialists, to anticipate their decisions based on their own class productions. This presupposes that the teacher's decisions are made explicit and that he or she has a progression plan in place that is backed by answers to the questions he or she may have after implementing them. This reliability seems partly due to, first, the level of coherence between the concepts of learning and the proposed situations and, second, to their experimentation in many classes for many years.

Our results relate to the following three levels:

- that of research, the construction of analytical tools and the testing of hypotheses;
- that of didactic engineering (Artigue, 2002) and the relationship between students' initial knowledge, the targeted knowledge and the proposed teaching devices; and
- that of the resources produced and their appropriation by teachers who wish to modify their teaching.

CONCLUSION

The gestures made in relation to geometric objects in which we are interested are those used in purposeful, problem-solving actions, where a pupil can observe either success or failure; it is in this sense that we understand the use of the term "manipulation". In our view, this analysis of the procedures that have already developed by pupils in their experiences with objects from the physical world is an essential component of the study of the relationship between actions on materials and the acquisition of geometric notions. One of the functions of our learning proposals is to enable these procedures to be clarified and criticized. This empirical study, which is based on numerous experiments, shows that having students solve certain types of problems involving manipulation with objects or tracing can help them conceptualize mathematical objects through the progressive recognition of their properties. This engineering approach has been applied to other geometric objects and properties (parallelism, incidence relations in space, etc.) with the same results (Douaire et al., 2020).

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CONNECTING SECONDARY AND COLLEGE GEOMETRY: RESOLUTION OF PROBLEMS OF CONSTRUCTION OR MEASURE OF CURVES USING DGS

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In this paper, we advance a schematic proposal on how to approach the resolution of problems of construction or measure of curves by intersections of straight lines or curves, a core topic in Cartesian Geometry, and whose solutions are based on geometric constructions of continuous movement, facilitated in this case using Dynamic Geometry Software (DGS). This proposal is based on some of the Cartesian geometric constructions (see, Jullien, 1996; Boyer, 1956; Bos, 1981) with articulated devices, which include the revision and study of conics. The empirical (and curricular) proposal for teaching the conic sections of one experienced teacher on the subject is reviewed and incorporated into the discussion, as well as some of the executions of his students.

INTRODUCTION

In Mexico, in the Official Secondary Schools (located in the State of Mexico) the teaching and learning of geometry takes place in the second half of the 10th and the first half of the 11th grade in high school (SEP, 2017). In general terms, it can be said that the content of these two courses is reduced to learning some of the topics of trigonometry, the criteria of congruence and similarity of triangles (from a visual treatment of geometric properties), and analytic geometry basics. How are these topics related to the understanding and development of geometric thinking, which is one curricular axe in the mathematics education of the students at this educative level? (See Appendix 1).

In this paper, we advance a schematic proposal on how to approach the resolution of problems of finding the measure or construction of curves (by intersection of articulated straight lines or curves that slide in a plain, see Jullien, 1996, p.83), core topics in Cartesian Geometry, and whose solutions are based on geometric constructions of continuous movement, facilitated in this case using dynamic geometry software. And it is important to mention that the driving force that moved this proposal forward was a detailed review of the teaching experience at this educational level of the second author in this article.

The questions addressed in this paper are the following: (i) is there a connection between the topics proposed in the secondary geometry curriculum of SEMS (SEP, 2017), with the understanding and development of geometric thinking (an explicit study axis in this curriculum)? (ii) How are the topics proposed in this secondary geometry curriculum (see Appendix 1) related to the resolution of problems of finding the measure or construction of curves, core topics in Cartesian Geometry? (iii) How can the resolution of these problems be approached using dynamic geometry software? (iv) How does this approach contribute to the geometry teaching and learning proposed by the official study plans of SEMS (SEP, 2017)?

THEORETICAL FRAMEWORK

In the work of Wasserman and colleagues (2023), the functional structure of the university courses offered in universities to future secondary-level mathematics teachers is reviewed, seeking to elucidate or find answers to the questions that exist everywhere regarding the usefulness of university mathematics for teaching school mathematics at the upper secondary level. According to these authors, this situation constitutes a challenge for the mathematics education of teachers at this educational level (p. 719).

In Mexico, though we are apparently disconnected from such educational problems, this is not the case, since to the question: who or what are the educational institutions in charge of training upper secondary level mathematics teachers in Mexico? The answer is that the mathematics training that high school mathematics teachers received is precisely what they have acquired during their undergraduate studies at the university. That is, in Mexico, it also happens that the mathematics knowledge of secondary school teachers goes through a double discontinuity, a term described by Félix Klein (2016) since the first third of the last century.

According to Wasserman et al. (2023, p. 721), the double discontinuity lies in the following: the first disconnection is that what the mathematics teacher learned at the university had little resemblance to what he knew when he left high school, and the second disconnection occurs again in the transition from studying mathematics at university and the subsequent return to teaching mathematics in the school curriculum. Moreover, Gueudet et al. (2017) had identified that "Klein's notion of a double discontinuity between university mathematics and school mathematics… can be seen to constitute the core of teacher mathematics education in both theoretical and practical aspects." (Gueudet et al, 2017, p. 104).

By another hand, the mathematical topic addressed in this paper is the construction of curves by continuous movement that Descartes proposes in his *Géometrie*, it includes knowledge of basic geometric properties, such as the application of the Thales Theorem, and appears in the context of solving problems and theorems linked to the origins of analytical geometry, such as the resolution of the Pappus problem[1], and/or the algebraic representation of curves. The hypothesis advanced in this paper is that the historical context of the emergence of the algebraic representation of curves is a resource to work on establishing connections between university Geometry (and Calculus) and secondary Analytical Geometry. Moreover, the construction of curves by continuous movement suggested by Descartes could constitute an intuitive, or historical-evolutive approach to the study of conics from an evolutionary-historical perspective of the development of mathematical knowledge, as Moreno (2014) suggested by establishing that intuition and rigor in mathematics are coextensive.

Through an exploration of the construction of conics as continuous curves, as expressed by Descartes in his *Géometrie*, and the simulation – using dynamic geometry software – of Descartes' device, this teaching and learning proposal on conics will soon be implemented in one of the courses for the professional development of secondary teachers at the National Pedagogical University, in Mexico.

METHODOLOGY AND DATA

Construction of curves by continuous movement

In Descartes' *Géometrie*, a line is constructible if it is possible to draw it through continuous movement and/or through certain articulated mechanisms, where the points on the line are obtained from the intersection of other lines (curves, or even straight lines):

> [Descartes] has in mind the justification of a determining procedure in his *Géometrie*: the recourse to lines generated by moving intersection... As in a deductive chain, no matter how long it is, it can lead to a condition exact provided that the rules of the method have been respected, thus the generation of a curved line can be extremely composite provided that the rules of composition are respected. These rules refer in fact to only one: that the movement that leads from one curve to the other is continuous and entirely determined... By this means one can always have exact knowledge of its measurement [of the curve's measurement]. The measurement here should not be taken as a numerical notion but as a constructible magnitude, as it has been presented at the beginning of Book I. (Jullien, 1996, pp. 77-78, our translation)

Descartes sought to establish a rule of general one-to-one correspondence between construction criteria and invariant considerations of proportionality in the articulated system in motion: “These are 'moving planes' where the principle is the following: a starting line will move [in a parallel way] along an axis and its intersection with an articulated ruler will produce a 'more composed line'..." (Jullien, 1996, p.83).

But the combinations of movements that Descartes had in mind (Bos, 1981, p. 310) involved only straight lines as moving parts, making it possible to generate curves in the plane, in particular, conics. For example, Descartes described the following construction artifact (see Figure 1).

> A ruler GL pivots at G. It is linked at L with a device NKL which can be moved along the vertical axis while the direction of the line KN is kept constant. When L is moved along the vertical axis, the ruler turns around G and the line KN is moved downwards remaining parallel to itself. The intersection C of these two moving straight lines describes the curve GCE. (Bos, 1981, p. 311)

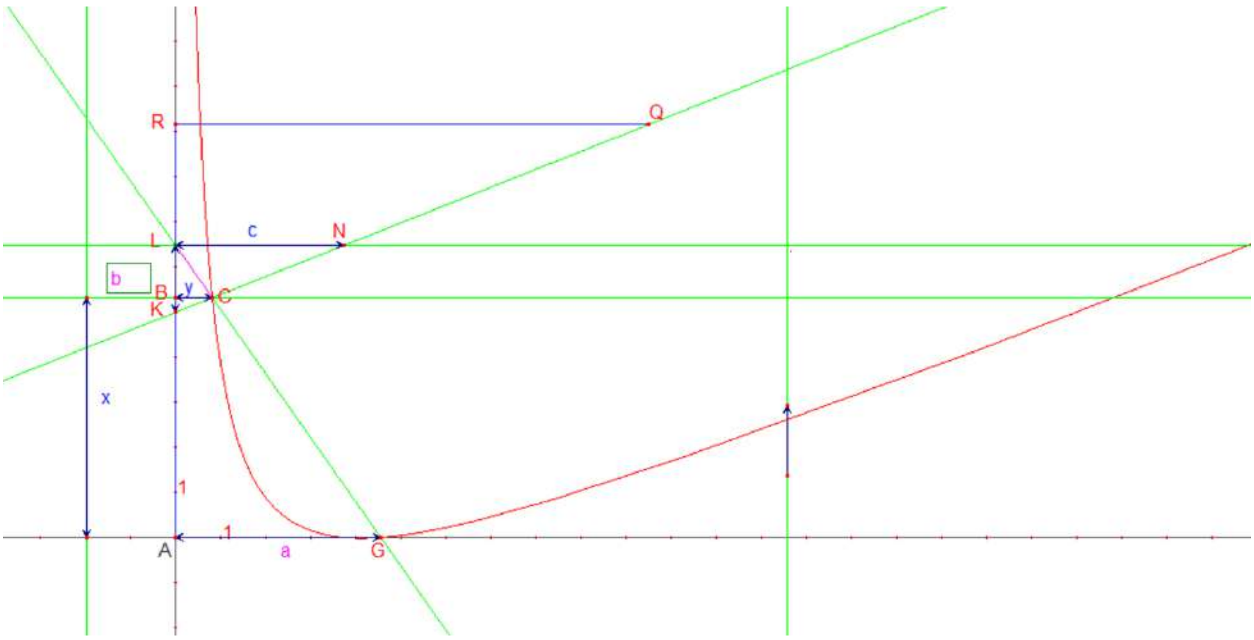

Figure 1: A diagram obtained through the use of DGS, shows the construction of the articulated system described by Descartes in his *Géometrie*, which, when sliding vertically, generates a hyperbola (see curve drawn in red)

In this way, to obtain the equation of the hyperboloid, it is enough to apply the Thales Theorem to the similar triangles formed by: (i) the vertices K, L, N & K, B, C, i.e., KLN ≅ KBC; (ii) the vertices L, A, G & L, B, C, i.e., LAG ≅ LBC (see Figure 1). Considering now GA= a, NL= c, AB= x, KL= b, CB= y (Bos, 1981, p. 311), the proportionality relationships between corresponding sides of the triangles lead to obtaining the equation: cy + (c/b) yx + ay – ac = y2 (you can see an equivalent diagram, perhaps clearer, in Figure 2).

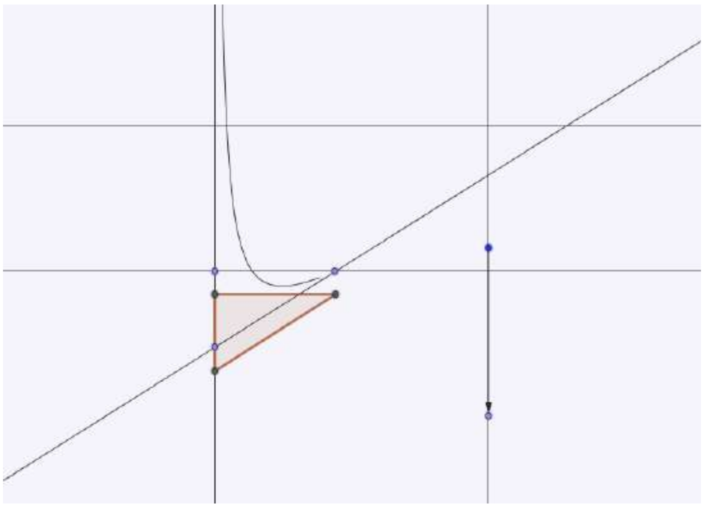

Figure 2: An equivalent diagram (perhaps clearer) of the use of the SGD to draw the hyperboloid of Descartes

The practice of teaching-learning conics, in grade 11 (students of 16 years old)

According to what was reported by Pecina (2023), the teaching of analytical geometry in the Official Preparatory Schools (EPO by their acronym in Spanish) of the State of Mexico (e.g., at the EPO 171) is carried out with an ordered planning of practical work in the classroom. In each of the sessions (see Table 1), and by the '55 methodology' (see Hernández, 2023), the learning of the subject is guaranteed by the students' strict attention to the teacher's explanations, provided by the teacher for the students can follow and replicate the use of the formulae extracted from his planned presentation of curricular content in play. This sequence is based on a rote strategy of repetition and exercise, as shown by the planning (see Table 1) and the record of the student's activities and productions in each of the class sessions (see Figure 2).

The following Table 1 shows the planning prepared by the teacher responsible of class #21:

<table>
<tr><td colspan="3">Unit II. The Conics</td></tr>
<tr><td>Session planning #21</td><td>Ellipse’s ordinary equation (the center out of the origin)</td><td>Estimated time: 210 min</td></tr>
<tr><td colspan="3">Instrumental objective: Analyzing the ellipse with a center out of the origin, utilizing its ordinary equation to obtain references and magnitudes of its different elements in the Cartesian plane.</td></tr>
<tr><td rowspan="9">Learning’s Task</td><td>Task</td><td>Materials</td></tr>
<tr><td>1. Revisiting previous session</td><td rowspan="8">Blackboard and marker; wooden colors; pencil; eraser; pen; notebook; geometry set; projector; DVD; extended power cable.</td></tr>
<tr><td>2. The instrumental objective, the checklist, and the observation guide for the session are announced.</td></tr>
<tr><td>3.The teacher makes examples of graphing ellipses with the center outside the origin given the ordinary equation.</td></tr>
<tr><td>4.Groups are asked to perform exercises given the ordinary equation of the ellipse with the center outside the origin in real or simulated situations.</td></tr>
<tr><td>5. Screening of the video "The Universe of History" to prepare a synthesis of the documentary.</td></tr>
<tr><td>6. Group conclusion.</td></tr>
<tr><td>7. Time to ask questions.</td></tr>
<tr><td>8. Account of the observation guide and checklist.</td></tr>
</table>

Table 1: Teacher planning to approach the study of the ellipse equation

CONCLUSIONS

This work sought to contribute to the line of research of establishing connections between mathematics at the university level and secondary mathematics (see Wasserman 2023; Gueudet et al., 2017), through the development of an intuitive, or historical-evolutive (see Moreno, 2014) proposal for teaching analytical geometry for the 11th grade (in high school) that looked to be complementary to what teachers have been implementing in practice in their Analytical Geometry courses at this educational level.

In his *Géometrie*, Descartes began addressing the problems of construction or measurement of curves based on the resolution of the Pappus problem. In his resolution, he arrived at one of the most celebrated mathematical discoveries, namely the algebraic representation of curves. However, another of the formidable ideas that underlie Descartes' resolution is the geometric idea of obtaining or constructing curves through the continuous movement (or by sliding in the plane) of articulated mechanisms, based on the intersection between straight lines, or between straight lines and curves, or between curves and curves.

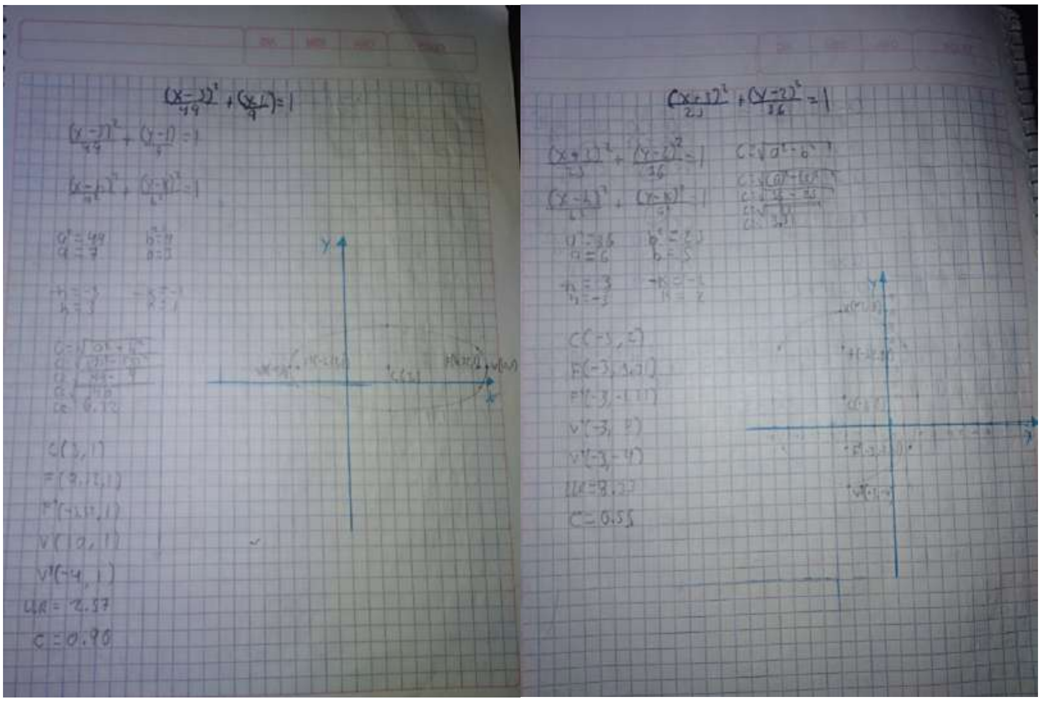

Figure 3. Students' products in the classroom, prepared after the teacher explained the topic, in this case, devoted to the study of the equation of the ellipse

Such simulation, provided by the use of a DGS, probably enables student observation and application of invariant geometric properties of the moving device. Furthermore, discerning and applying the invariant geometric properties in play, such as Thales' Theorem, to finally arrive at the corresponding conic equation (or a quadratic equation in two variables), adds a different meaning than the commonly associated with conics when approached only through repetition and exercise of its rigorous definition. This proposal focuses on the utilization of basic geometric properties, such as Thales' theorem, as an argument for the construction of conics by continuous motion.

Algebraic curves in the plane are algebraic varieties of dimension one and can be thought of as the entry to the study of multidimensional algebraic varieties, a topic of Geometry in university mathematics (see Ramírez-Galarza, 2011). Or also as a complementary background to the study of the historical development of calculus (see Edwards, 1979).

The hypothesis that was advanced in this paper is that through the Cartesian approach to the construction or measurement of plane curves (see Jullien, 1996) it is possible to establish connections between the geometry at the university with the analytical geometry at the secondary level. Furthermore, this constitutes an intuitive, or historical-evolutionary approach (see Moreno, 2014) to the study of conics, and complements an approach of rote teaching by repetition and exercise of the definition of conics that is being implemented in practice (see Pecina, 2023).

Furthermore, this type of approach in the classroom likely promotes the recognition and establishment of invariant basic geometric properties that underlie the continuous construction of curves, and thus the generation of arguments for the development of geometric thinking in high school. Discerning and applying invariant geometric properties, such as Thales' Theorem, to finally arrive at the corresponding conic equation (or a quadratic equation in two variables), adds a different meaning than the commonly associated with conics from an approach of rote teaching through repetition and exercise of its rigorous definition.

Footnotes

[1] In Descartes' Géometrie, the Pappus problem in the case of 3 or 4 straight lines, leads to obtaining the general equation of the second degree in two unknowns, i.e. is an algebraic description of conics. A version of the Pappus problem from Jullien (1996, p. 72), is the following:

> Given four coplanar lines D_1, D_2, D_3, D_4 given in position, consider a point C, [and] denoted as d_1, d_2, d_3, d_4 the lengths of the segments that join C with $D_{1\ (\text{respectively } 2,\ 3,\ 4)}$, under a given angle. The relationship ($d_1.d_2$) is then fixed compared to (d3.d4) … One of the lines is taken as the 'abscissae axis', one point A as the origin, and a direction determines the ordinates. One point C, solution of the problem, is associated with the two coordinates AB=x & BC=y (BC is the first line involved in the analysis of the problem). Descartes then shows, with the help of simple considerations, mobilizing the given lines and angles (those that join C with one of the given lines, under a given angle) … [that] Pappus's problem, up to five lines, produces an equation [with two unknowns] that does not go beyond the second degree… [and] its solutions can be constructed with a ruler and compass. (Jullien, 1996, pp. 72-74)

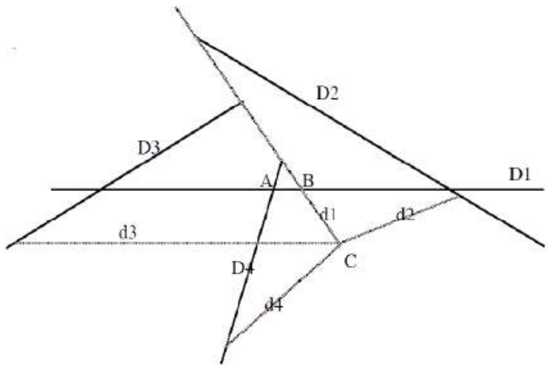

Figure 4: Image of the Pappus' Problem in the case of four straight lines (Jullien, 1996, p.72)

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Appendix 1. Geometry curriculum for Secondary School in Mexico (students of 15-18 years old)

Axis	Components	Central content
From the treatment of space, form, and measurement to the geometric and trigonometric thinking	Structure and transformation: Geometry basic elements	Fundamental concepts of space and form, “the geometric”
		The study of geometric figures and their properties
		Treatment of geometric formulas for areas and volumes
		Visual treatment of geometric properties, the criteria of congruence and similarity of triangles
	Angularity and constructions: Elements of the Plane Trigonometry	Basic concepts of the trigonometry
		Uses and functions of trigonometric relationships in the triangle
		Trigonometric functions and its properties
		Measurements of angles and trigonometric relationships
		From the unit circle to the Cartesian plane. An introduction to ratios of magnitudes to real functions. Visualizing trigonometric formulas and identities
Geometric loci and reference systems. From geometric to analytical thinking	System of reference and location: Elements of Analytic Geometry	Analytical Geometry as an algebraic method for solving geometric tasks. Treatment in various coordinate systems
		Basic concepts of the system rectangular coordinates, orientation and position in the plane. The role of the origin of coordinates in reference systems.
		Recognition and construction of geometric loci: line, circle, ellipse, parabola and hyperbola
		Visual treatment and representations multiples of the geometric loci: rectangular and parametric coordinates, singular points, roots and asymptotic behavior

HOW CAN VIRTUAL GEOMETRY MANIPULATIVES BE USED IN WAYS THAT MITIGATE THEIR ONTOLOGICAL, TECHNOLOGICAL AND PEDAGOGICAL LIMITATIONS?

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The use of virtual geometry manipulatives in mathematics and mathematics education is increasingly commonplace. In this paper we identify three sets of concerns regarding their use. Firstly, ontological ones whereby it is perhaps impossible to faithfully represent the three-dimensional world in two-dimensions. Secondly, technological ones in attempting to represent geometrical objects using digital tools. Finally, pedagogical ones relating to the impact of these two concerns on decisions regarding the use of virtual geometrical manipulatives in educationally appropriate ways. We then suggest the use of trust certificates as a way forward in responding to these three concerns and argue that trust certificates provide students with a procedure for developing confidence in evaluating the accuracy of digital representations. Finally, we propose that this procedure is useful in evaluating problem solutions generated by Artificial Intelligence software. Given the rapid proliferation of digital representations, in the form of virtual geometry manipulatives and artificial intelligence,in our view a process for determining the accuracy of information is a skill that students require.

INTRODUCTION

Manipulatives have long played a critical role in the teaching and learning of geometry (Moyer-Packenham & Bolyard, 2016) and activities using manipulatives form the basis for enhancing a range of geometrical learning processes. Operations using manipulatives, both physical and virtual (Ladel, 2009) are purposeful, and it is essential that students are aware of the relations between the objects and the concepts they are learning. There is a growing body of evidence indicating that quality virtual manipulatives (VM) support mathematical learning (Ladel & Kortenkamp 2016; Larkin, 2016). Given that there has been a rapid increase in the use of virtual geometry manipulatives (VGM) in mathematics education since the last ICMI Geometry Study in 1995, it is opportune to reflect on issues raised by their use.

Moyer-Packenham and Bolyard (2016) define a VM as “an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge” (p. 13). At their core, VM provide students with opportunities to interweave iconic and symbolic representations, with the actions that they perform on them, to emphasize the underlying mathematical concepts. In this paper, VM include both comprehensive geometry packages such as *Cabri, Geometers’ Sketchpad, Cinderella,* and *GeoGebra*, as well as stand-alone geometry apps such as *Coordinate Geometry* or *Cubeling*.

Critiques of VM (Larkin, 2016; Namukasa et al., 2016) have consistently found that most VGM are very unlikely to support the development of geometrical understanding. For example, Larkin (2016) used Dick's (2008) three types of knowledge fidelity (pedagogical, mathematical, and cognitive), to evaluate 53 VGM. His evaluation found that over half of them were considered inappropriate for classroom use and, even for those that were considered as appropriate, teachers were still required to make informed pedagogical decisions regarding their use to ensure their learning potential was maximized.

To combat the paucity of quality VM available to support learning, other mathematics educators have developed theoretical approaches that should underpin the creation of VM. For example, Ladel and Kortenkamp (2013) created Artifact Centric Activity Theory (ACAT) as a model for the development of quality VM and Hirsh-Pasek et al. (2015) use "four pillars of learning" as their conceptual frame in the development of robust VM. Mathematics educators have also been active in building VGM, for example *Cinderella* (Richter-Gebert & Kortenkamp, 2012), *Cubeling* (Etzold), or *Angles* (Etzold). Given the mathematical education background of these designers, teachers can be more confident in using these manipulatives to support student learning.

While these critiques are useful for teachers to use to support their decision-making process, and the frameworks are useful for designers of VM, in this conceptual paper we want to discuss two questions that we think remain unanswered in relation to the use of VGM.

- RQ 1. What issues arise in considering the use of VGM in educational contexts?
- RQ 2. What changes to how geometry is taught are needed to effectively use VGM?

THE PROBLEM

In addressing Question 1, we commence first with an ontological perspective and then discuss concerns from a technological perspective. We finish this section by considering how these perspectives impact the pedagogical decisions that teachers must make regarding the use of VGM.

Ontological concerns

Our ontological concerns are two-fold. Firstly, is it possible to accurately represent 3D mathematical objects in 2D representations (in this case via digital 2D representations)? Any 3D object can only be represented in 2D by projecting it onto a lower-dimensional space and, as a consequence, the projection will not have all the properties of the original object. Thus, we cannot be sure of the physical object without further manipulation of the 2D representation, for example, by change of the projection. As this change will involve time as another dimension, our conclusions must always be supported by argumentation and trust. Secondly, putting aside the ontological problem of 3D objects being represented in 2D, how can students using VGM be confident that the designers of the software are *truthfully* representing *reality* in digital form? It could be the case that designers are, in fact, operating as *bad faith* actors who are either deliberately representing objects incorrectly, or are doing so inadvertently as they do not have the necessary technical skills to represent them accurately. We see this bad faith phenomenon evident in digital environments, where cheaply made apps dominate the market (Larkin, 2016). This is also, in some sense, the dilemma regarding the use of Artificial Intelligence (AI), whereby it is increasingly difficult to place trust in digital information, as we can no longer easily identify the veracity of representations of reality presented to us. We take up this point briefly again in the conclusion of this paper.

Let us illustrate this bad faith phenomenon: Imagine a VGM for 2D shapes that can be moved and rotated on screen, similar to a (real) manipulative of polygons made out of paper, wood, or plastic. The real manipulative supports important invariants of Euclidean Geometry: moving or rotating a shape will not change its inherent properties such as area or circumference. The VGM, however, could change the shapes slightly when they are moved or rotated, and we could experience a proof that the Pythagorean Theorem is true for *any* triangle, not only right-angled ones. There is no sense in creating and using such a VGM; however, the sheer possibility to create such a VGM raises the question why students should trust manipulatives, given that such manipulatives may represent mathematics incorrectly and thus generate misconceptions?

Technological concerns

At first, it should be noted that ontological and technological concerns can be considered as intertwined, for example, the technological problem of presenting a circle in a digital (discrete) manner also relates to ontological concern about the nature of circles. In terms of this paper, we distinguish the two by considering concerns that are based on bad faith and concerns that are based on technological impossibility. Therefore, even if the developers of VGM are aware of the ontological difficulties, and try to truthfully represent geometric content in the best possible way, there are technological problems that cannot be overcome, but at best only be hidden.

We provide two specific examples: (1) The representation on screen is always based on a digital representation of either real world objects or mathematical objects. Usually, this representation consists of a computer screen with high, but finite, resolution. Imagine representing a circle around a point P with radius r. Some pixels on the screen need to be painted. Apart from the fact that these points will have a (small) area, they are not infinitely small, so their distance to the center will not exactly be r, but vary within a (small) ε-range. Consequently, they do not represent a circle in the mathematical sense. What looks like a circle on screen is actually a polygonal shape. (2) The side and the diagonal of a square are not commensurable – so we cannot use rational numbers to express them both: therefore, a triangle created by cutting a square at one diagonal cannot be represented accurately using a digital screen.

Both examples show the difficulties in accurate representation of geometric content on a screen. The introduction of very high-resolution displays hides those fundamental problems from the eye, but they still exist. We are not suggesting that VGM shouldn't be used because of these difficulties, as this problem exists in the real world as well. One could easily ask, on the (sub-)atomic level, "Is it possible to create a true square from wood?" We are just indicating the need for teachers and students to be aware of these issues.

Pedagogical concerns

Presuming that the ontological and technological concerns are resolved, or at least minimized, a third concern remains in relation to how VGM should be used for best effect in the learning process. A common approach to the use of manipulatives is based on Bruner's (1966) suggestion that students learn through three different experiential means:

- *Enactive* (direct sensory) experience where students take an active part in their learning through the manipulation of their learning environment;

- *Iconic* representation of experience where enacted experiences are represented via diagrams, film clips, charts etc.; and
- *Symbolic* representation including written language symbols such as words and mathematical symbols.

What needs to be kept in mind, however, is that these are not staged levels; rather, they should be considered as equivalent in importance. The critical pedagogical element is that there should be a special focus on students transitioning between the three different ways of learning. For example, students should be able to describe symbolically their enactive experiences or find an iconic representation of a symbolic one. What is vital, in a pedagogical sense, is that teachers thoughtfully consider the use of VGM in student learning as, for example, it is problematic if teachers rely exclusively on the use of largely symbolic geometry software packages such as *Cinderella* or *GeoGebra*.

BUILDING TRUST IN GEOMETRICAL SOLUTIONS

In this section we address Question 2 and suggest a pedagogical approach, based on the notion of trust, that we think supports students in learning geometry (and indeed mathematics more broadly). We then provide two practical examples of this approach in high school.

We commence this discussion about trust, as we did with Question 1, from an ontological position, and ask "What is trust?". Although there are many different definitions of trust (see Viljanen, 2005), we use the definition by Amaral et al. (2019) who suggest that "trust is a central component of social life and is frequently referred to as the 'glue of society', vital in economics, social cooperation, organizations, groups, etc. (p. 1)". These authors indicate that the term has been used to refer to different types of relationships, including "between software systems operating in a network" (p. 1) and that trust is "generally the basis for decision making closely related to achieving a goal" (p. 2). In this paper, when we are discussing trust, we mean the trust between a user (student) and a software system (VGM) and we suggest that this notion of trust, and how it can be established, provides a possible mechanism for resolving the ontological and technological concerns we outlined earlier. The establishment of trust in a virtual environment is based on what we will call *trust certificates*.

Trust certificates as a mathematical and pedagogical approach

In computing contexts, the issue of trust between users and software is largely resolved using digital certificates. According to Wikipedia, a "digital certificate certifies the ownership of a public key by the named subject of the certificate, and this allows others to rely upon signatures or on assertions made about the private key that corresponds to the certified public key." In this way, a trust relationship can be established between the subject and the party relying upon the certificate. We suggest that this approach to trust, whereby users build trust in the software via the use of trust certificates, can also be utilized in a modified way, by students as they learn about geometry.

It is important to distinguish here between simply checking an answer and using trust certificates. Students are often taught by teachers to "check their answers" – and will often complete a process to do so. We agree that this is an appropriate pedagogical approach, if implemented correctly. However, often the student's checking is based on the inverse operation of the initial operation they performed, which might be easier or just more familiar to the students. They will check if 15 minus 7 is 8 by adding 7 to 8 to get 15. Of course, this works for correct answers; however, if children are merely following a routine process, then this checking may result in correct logic but an incorrect outcome – **if** 15 minus 6 is 8 **then** adding 6 to 8 gives 15. The relationship between the statements is logically true but the answer is incorrect. By contrast, as seen in the example below, a trust certificate approach builds *confidence* in the correctness of a result, instead of "proving" that it is correct directly. Trust certificates for 15 - 7 = 8 *might* include:

- I "know" (from prior learning also established following a similar trust approach) that an odd number minus an odd number results in an even number; 8 is odd (✓)
- I "know" that 7 is larger than 5, so the answer will be less than 10; 8 is < 10 (✓)
- I "know" double 7 is 14, so subtracting 7 from a larger number will result in a number larger than 7, 8 is larger than 7 (✓)
- Given the three trust certificates above, I can be confident that 15 - 7 = 8.

This approach resembles unit testing (Beck, 2002), a software development method where (small) software units are tested automatically for correctness, increasing the confidence that a function or method of a computer program is indeed giving the correct output. The three trust certificates above could also be formulated by students as mathematical unit tests: (1) Any solution x of $z - y$ should be even, if both z and y are odd. (2) Subtracting a single digit number y from any number z should result in a number in the decade one less than the one of z. (3) If $2y < z$, then $z - y > y$.

The notion of unit testing applies equally in geometrical contexts. As one example, consider finding the intersection of two lines, for example given by two linear equations $y = a_1x + b_1$ and $y = a_2x + b_2$. An easy way to check whether a suggested solution (x_0, y_0) is the intersection of two lines, we can plug the values in each of the equations and see whether they hold. Pedagogically, we can trigger this kind of thought by asking for evidence that a certain proposed solution is *not* the intersection. For example, for the lines $y = 2x + 3$ and $y = -\frac{1}{2}x + \frac{1}{2}$ we can ask students to show t and can ask students to give evidence that $(-2, -1)$ is not a solution, and that $(3, -1)$ is not a solution, which they can do by choosing the correct equation as a (counter-) certificate. A similar approach is possible when searching for the perpendicular bisector of a segment using two circles with the same radius.

Certificates in both real world and virtual contexts

Although our focus in this paper is on VGM, we argue that the use of trust certificates is equally valid in both real world and virtual contexts, and that students should be concerned about the validity of geometric representations in textbooks as well as in VGM. However, given that the ontological and technological concerns noted earlier in this paper relate primarily to VGM, we recommend that students first encounter the use of trust certificates in a real-world context, and then later encounter them in a virtual context that relates to the real world one. We feel this step is necessary for students to build trust in the representations of the same concepts presented later in VGM. However, in later encounters this direction can be reversed. We now provide two examples of the use of trust certificates in contexts related to the teaching of geometry.

How might this approach work in practice?

We first discuss an implementation of the trust approach that took place in a teaching episode, which occurred after a professional development course, where one of the authors presented a VGM that could be used to model the Golden Gate Bridge. Here, using a photo of the bridge, a physical simulation of forces on the bridge can be projected onto the image, enabling the students to experiment with the shape of the bridge created by the forces acting on it [See Kortenkamp and Richter-Gebert (2008) for further details about the content]. It should be noted that the task reflects all the problems we raised in our first research question: The object in question is a 3D object, given as a 2D photograph, which again will be projected on a 2D screen. Students have no opportunity to interact in any way with the real object. As they only have a single photo, they cannot even choose a different projection. Also, as we are using digital imagery, we do not have a proper representation of the real bridge, and the resulting simulation is a polygonal line, instead of a parabola. It is due to the pedagogical intervention by an experienced teacher of physics and mathematics that students were helped to overcome these issues and develop confidence in the trustworthiness of the VGM. He describes this in an email (translation from German by the authors):

> […] I immediately started to work with my group to experiment and built a [physical] model using springs and masses, took a photo and simulated the same configuration with the software. As a "*confidence-building measure"* I took a picture from the side and showed the students that using the right transformation the simulated curve can be projected onto the oblique picture. The culmination was your example with the Golden Gate bridge. […]

He also sent the images (Figure 1) with a description of the technical details.

Figure 8: The simple model was built to create confidence in the simulation

By systematically building up confidence through trust certificates (the touchable instance of a bridge model that is used with the simulation), the teacher could successfully transform the virtual manipulative from a "black box" into a trusted device that students could rightfully accept as experimental evidence.

The second example also involves the connection between the real and virtual world but reverses the direction of how trust certificates can be used. For use in acourse for pre-service teachers (PSTs), a visual proof of the convergence of the geometric series was created by the lead author (Figure 2, left), and then realized as a VGM. Instead of relying on using the digital tool alone, he also produced laser-cut squares that the PSTs could manipulate to determine how these tiles can be arranged to a converging series (Figure 2, right). We claim that the creation of real manipulatives, based on experiences in a VG, builds confidence and note that this is an increasingly affordable option in many schools. As a low-cost alternative, paper printouts from a VGM can often be fabricated to create physical manipulatives that can assist students to trust the digital representations. This notion is also implicit in the theory of Duos of Artefacts (Soury-Lavergne, 2021).

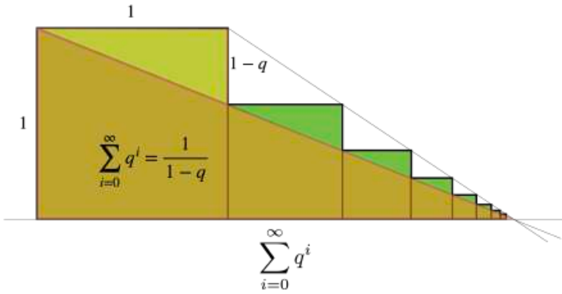

Figure 9: The visual proof in the VM and the laser-cut realization that can provide confidence

Although the thrust of this article thus far has been the use of trust certificates in regard to VGM, we finish with the proposition that it is possible to use trust certificates as a tool for validating information provided by AI. In many cases AI suggests answers to mathematical questions, which appear to be correct, but are often incorrect either in terms of the overall solution, or in terms of the mathematical logic that is used to derive the solution. One of the authors has already begun to use trust certificates with PSTs to help them acquire skills they need to cope with AI generated answers submitted by students. In a similar vein, Kortenkamp and Dohrmann (2023) outline how PSTs can use AI outputs, which answer geometrical problems, as a training tool for helping their future school students generate questions that narrow the problem to understandable solution steps that can be verified independently from the AI output. Overall, the main contribution of trust certificates is that they enable students to build trust, based on verifiable mathematical understanding of the component building blocks, which they can use locally to support argumentation, and which build towards trust in the more complex solutions generated by VGM or AI.

CONCLUSION AND WAY FORWARD

In this paper we identified three sets of concerns regarding the use of VGM. Firstly, ontological ones whereby it is perhaps impossible to faithfully represent the 3D world in 2D and the associated difficulties that can occur with bad faith designers. We then discussed the technological difficulties of creating such representations, which largely relate to discrepancies between formal mathematics and how formal understandings can be represented in digital form. Finally, both these ontological and technological concerns impinge upon the pedagogical decisions required to use VGM. We then suggested the use of trust certificates as a way forward in responding to ontological and technological concerns which, whilst not totally alleviating them, at least provide students with a procedure for developing confidence in evaluating the accuracy of representations in VGM. This is achieved by building trust in the smaller building blocks that comprise the complex solutions, often generated by 'black boxes', where the workings of the technology remain hidden from students, and by providing students with opportunities to replicate digital outputs in more concrete representations (albeit that these in turn are still only models of the real situation). Given the rapid proliferation of digital representations, in the form of VGM and AI, we argue that a process for determining the veracity of information is an important mathematical skill for students to learn.

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SUPPORTING GEOMETRIC REASONING FOR UNDERSERVED STUDENTS IN INDIA THROUGH CONNECTED LEARNING INITIATIVE

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This study explores the impact of module-based interventions on geometric reasoning in underserved Indian students, as part of the large-scale action research project -Connected Learning Initiative (CLIx). The interventions aligned with active learning pedagogy, emphasizing exploration, discussion, and digital resource utilization. Teacher capacity was enhanced through workshops and online courses. The focus was on developing understanding of quadrilaterals. A quasi-experimental approach adopted with the sub-sample of the large-scale implementation revealed significant geometric reasoning improvements in intervention school students compared to non-intervention peers. Female students in the intervention group exhibited notable gains. Schools with higher module fidelity achieved greater success. Observations showed that intervention schools promoted discussions, while non-intervention schools leaned towards lecture-based approaches. Students in intervention schools engaged more actively. Teachers not only facilitated discussions but adapted to contingencies, fostering reflection. Besides cognitive gains, students displayed increased confidence, motivation, collaborative skills, and autonomy. The study underscores the significance of teacher capacity and resource use in under-resourced contexts. This research informs in-service teacher education and resource development for similar settings. The findings bear implications for equitable mathematics education, particularly in resource-constrained contexts.

INTRODUCTION

Students learning, teachers' beliefs, knowledge, skills and practice and their use of resources to support students' learning are intricately connected. However, these connections are often studied in high-resource contexts and there is a dearth of studies that focus on under-resourced contexts. Adler (2000) posits that resources can be considered as an "extension of the teacher in school mathematics practice" (p. 205) which includes hybridized content and pedagogy and mediated use of the resources depending on their relative transparency to support meaning-making by the students. Brodie (2020) highlights the role of knowledge, material/logistical, affective and human resources that support teacher collaboration for professional learning in a community. The process of instrumental genesis (Béguin & Rabardel, 2000) involving instrumentalization and instrumentation is thus facilitated by both how a teacher visualizes and adapts a resource for the classroom and by the function of participation of the teacher in a community that supports this visualization. However, there is a challenge of studying the use of resources at a large scale and characterizing how the teachers' use and adaptation of the resource and school mathematics practice constitute each other.

Resources used for teaching geometry and for geometry teacher education, seem to have a disproportionate focus on the use of software, of which many are proprietary and thus inaccessible in under-resourced contexts (except GeoGebra which is an Open Educational Resource (OER)). Though there are several quality hands-on resources available for teaching geometry, there is scant literature in mathematics education research about the use of such resources for underserved populations or under-resourced contexts. Classroom discourse though focused in research, needs to be considered as a resource in supporting students' learning and resource mediation. However, instead of focusing on only hands-on and classroom discourse for geometry learning in under-resourced classroom, it is imperative to equip the underserved population with the digital skills and the use of digital resources to address equity in mathematics education. Therefore, under-resourced classroom teachers need to learn the skill of managing the use of hands-on, digital and facilitation of classroom discourse in a coordinated manner. It highlights the significance of the selection of tools that meet the learners where they are while providing opportunities to build their skills. Additionally, the tools should support the active exploration and meaning-making of the mathematical content rather than using the digital medium to 'sugarcoat' the mathematics drill and practice or worse use the medium to 'transfer' the content to students replacing the teacher.

In this paper, we discuss some of the findings from a large-scale action research project implemented in India called as Connected Learning Initiative (CLIx). This project involved the design and implementation of blended modules (having both hands-on and digital content) to empower them to develop 21st century skills in Mathematics, Science and English and have been released as Open Educational Resources (https://demo-clix.tiss.edu/softwares/). The intervention involved developing an ecosystem to support the implementation of the modules through engaging teachers through in-person workshops and online courses. Between the period of 2016 and 2023, CLIx has been implemented in 1418 schools, 105000 students and 3200 math teachers across India approximately.

In this paper, we share findings using a subsample from the large-scale implementation from a rural region in India using a quasi-experimental approach for 10 intervention and 10 control (no CLIx intervention, teaching as usual) schools. Implications for in-service teacher education about how resources can be used for addressing and developing teachers' knowledge and skills through the development of a professional learning community are discussed.

THEORY OF CHANGE

The theoretical assumptions behind CLIx math interventions were to provide resources (Digital and hands on) that embed and illustrate active learning pedagogy by encouraging students to explore and develop meaning of mathematics. The resources for students were compiled in the form of an open educational resource – A geometric reasoning module for ninth-grade students and an online course for teachers. Through this course. teachers were involved in providing opportunities to engage in hands-on activities, collaborative discussion with peers, opportunities to learn through mistakes and exploring digital resources. The capacity of teachers to use such resources in the classroom and to use active learning pedagogy was enhanced through topic-specific workshops, engaging in online courses and interaction with peers and experts through mobile-based chat groups supporting the development of the community. The teachers were supported in the school through follow-ups by the team to provide technical assistance for the use of hardware and software and allocating time approved by the administration for exploring these resources. The workshops, online courses and interactions on chat groups focused on developing understanding of theoretical ideas for geometry learning like Van Hiele theory, discussion on types of mistakes and misconceptions of the students, activities that can aid in addressing misconceptions, sharing artefacts from the classrooms in form of students' responses and constructions to delve deeper into students' understanding.

GEOMETRIC REASONING MODULE

The geometric reasoning module was designed to focus on developing a foundational understanding of Quadrilaterals and their properties while the textbook chapter on Quadrilaterals focused on proofs of theorems and application of the theorems. The objective of the module was to focus on foundational concepts of geometry, providing opportunities for students to engage in problem-solving and reasoning, peer discussion and creating a learning environment that is a safe space to make mistakes. The geometric module unit is composed of 5 units designed based on van Hiele theory (Fuys, Geddes & Tischler, 1988). The focus of the unit and the types of resources created in each unit are given in Table 1 below.

Unit	Type of activities	Resources
Concept of Shape	Creating shapes, what changes when rotation, translation, deformation	Origami, Turtle logo, Matchstick Mecano, student handbook activities
Analysing and Describing shapes	Identifying and Describing shapes using properties and qualifiers	PoliceQuad Game Mission 1 and 2, student handbook activities
Classifying and defining shapes	Classification of shapes based on similarities and differences	PoliceQuad Game Mission 3, student handbook activities
Property-based reasoning	Hierarchical relation between shapes, if-then reasoning and need for definitions and alternative definitions	PoliceQuad Game Mission 4, Case of multiple definitions, student handbook activities
Need for proof	Making conjectures, Are verifications proofs? Inductive and deductive reasoning	Geogebra activity, student handbook activities

Table 1: Units, activities and resources in the CLIx Geometric Reasoning Module

The hands-on activities were implemented in the classroom while the digital activities were implemented in the school computer lab. The field staff helped in getting the school computer lab ready and going. Due to the large class size (60-80), the students were divided into batches for accessing the digital activities collaboratively in the computer lab with ten computers. The hands-on activities included paper folding, matchsticks Mecano and a student workbook including several tasks for reflection and discussion. The digital activities involved the use of open software like Turtle logo (Sugarlabs), GeoGebra and a digital game 'PoliceQuad' with 4 missions and 12 levels. The game PoliceQuad was designed as an OER taking the context of playing the role of police and releasing the innocent based on clues provided. The successive missions required the students to identify the 'suspect' shape based on clues describing the properties of shapes (Mission1). Mission 2 required creating the clues as questions using the properties of shapes at display and releasing the innocent based on a 'yes/no' response from the computer. The third mission focused on identifying similarities and differences across two groups of shapes to create a shape belonging to one of the groups approved by the computer. The fourth mission involved guessing the shape on the grid based on the special quadrilateral selected and discovering edges and vertices by clicking on various points of the enemy grid (similar to the Battleships game). The students were expected to engage in the activities in small groups and pairs and give responses after the discussion. The module is accessible in three languages (English, Hindi and Tamil) by toggling on the student platform. The quasi-experimental study reported in this paper involved the implementation of some activities from the first 3 units by intervention school teachers due to constraints of time.

REFLECTIVE MATHEMATICS TEACHING COURSE FOR TEACHERS

The three-day in-person workshop and the 'Reflective Mathematics Teaching- Geometric Reasoning' online course (run for 3 months) for the teachers focused on the development of the teachers' beliefs, knowledge and practice for focusing on developing an understanding of geometric concepts through active exploration, collaborative math talk, learning through mistakes and classroom discussion facilitated by the teacher. The teachers engaged with the module first as learner and then thought through the perspective of the teacher for planning to teach the module and reflecting on the classroom implementation. The course provided opportunities for understanding theoretical ideas like van Hiele's Theory, types of misconceptions that students have, activities that can help students engage with geometric reasoning, pedagogy embedded in activities and resources in the module, reflection on school mathematics practice and reflection on implementation of the activities of the module. In addition to this, the teachers participating in the quasi-experimental study were given two-day refresher workshop on the module and planning of implementation in the classrooms.

METHODOLOGY

A mixed method approach involving a quasi-experimental design was adopted for the study in the year 2018 for three months. The study was conducted in government high schools in semi-urban and rural areas of the Dhamtari district in Chhattisgarh. Ten intervention and ten control schools were selected from the larger pool of schools. Intervention schools were selected based on the condition of the computer lab. Comparability of the intervention and control schools was established using the pass percentage of the tenth-grade official certificate exam using independent samples t-test and no significant difference was found among the two groups. A pre and post-test was constructed for the students adapted from Shaughnessy and Burger (1985) with 8 multiple choice questions (MCQ) and 6 constructed response items. In this paper, we present findings from the MCQs only. Classroom and lab observations were documented using tallies for teachers and student actions and free write documenting the interactions along with the time devoted to lecture mode, individual work, group work and others. For analysis, two observations each from schools that performed best on MCQs for both intervention and control schools have been summarized. Descriptive and inferential analysis was done using the 7 MCQs data from pre and post-test. The frequency of the behaviors and actions from the observation tools was collated. 507 students from 10 intervention schools and 565 students from 9 non-intervention schools participated in the study.

In the control schools, teachers taught the chapter on quadrilaterals from the textbook which focused on the properties of the quadrilaterals, special quadrilaterals, theorems based on parallelogram and mid-point theorem. The teachers mostly explained the content of the textbook and solved the questions given in the textbook on the board. Students were then asked to solve rest of the questions from the textbook exercise for practice. In contrast the teachers in the intervention schools focused on more foundational concepts of understanding angles, parallel lines, analysis of shapes of special quadrilaterals using activities from the modules.

FINDINGS

The analysis of students' responses to seven geometry-related multiple-choice questions (Q1-7) revealed notable differences between the intervention and non-intervention groups. Initially, the intervention group had a lower average pre-test score (29.77%, SD = 20.03) compared to the non-intervention group (33.92%, SD = 20.31). However, after the intervention, the intervention group's post-test scores significantly improved to 43.19% (SD = 22.69), while the non-intervention group showed a more modest gain, reaching 36.73% (SD = 20.49). The pre- to post-test gain for the intervention group was substantial, reflecting a significant increase ($p < .0001$) in comparison to the non-intervention group. Further analysis demonstrated a significant discrepancy in the gains between the intervention and non-intervention groups ($p < .001$), underscoring the efficacy of the module's implementation in enhancing geometric reasoning skills among students. It is also pertinent to mention in this report that a gender-wise analysis of student performance revealed a significantly higher gain in scores for female students in the intervention group than for those in the non-intervention group ($p <0.01$). The schools showing a higher extent of implementation of the geometric reasoning module and with higher fidelity to pedagogical principles underlined in the module, showed higher gains in students' learning as compared to schools with low levels of implementation or low fidelity. This outcome suggests that the module had a positive impact on students' geometric reasoning abilities, as evidenced by the substantial improvement in test scores among the intervention group compared to their non-intervention counterparts.

While 60% of the students in the intervention schools were not able to recognize the shape as square when orientation was rotated, 88% of the students were able to recognize it in the post-test. There was a 17% increase in students being able to identify a common property given a group of shapes and a 7% increase in the number of students being able to identify the class relationship that a square can also be considered as a rectangle.

Observation data indicated that Intervention schools allocated approximately 60% of class time to teacher-facilitated discussions, promoting meaningful 'math talk' between students or between the teacher and students. In contrast, non-intervention schools primarily adhered to the 'Teacher Lecture (TL)' mode, with nearly 70% of class time dedicated to passive student listening and occasional collective responses. Additionally, students in intervention schools engaged in more individual work, predominantly focused on workbook tasks designed to enhance their ability to articulate ideas and foster reasoning. This marked a substantial contrast from the practices observed in non-intervention schools.

Student behavior data from 6 best-performing intervention and control schools (on post-test) depicted in Table 2 shows a higher level of engagement and interaction of students in the classroom discussion in the intervention schools as compared to control. This is corroborated by the data about teachers' behavior in intervention schools depicted in Table 3 as the teachers in intervention schools asked for reasons and explanations and built on students' incorrect/partially correct responses. However, they also tend to give the answers or explanations themselves rather than waiting for students to respond to the question.

Student behavior	Boys/Girls	Average number of instances observed	
		Intervention	Control school
Students gave reasons in support for his/her response	Boys	38	5
	Girls	39	3
Students who gave short answers/ response	Boys	57	38
	Girls	54	41
Students who gave extended responses/ answers (with Reasoning)	Boys	22	3
	Girls	25	2

Table 2: Comparison of student behavior in 6 of the best-performing intervention and control schools

Teacher behavior		Average number of instances observed	
		Intervention	Control school
Teacher asking for reasons/explanations for students' response	For correct response	60	1
	For incorrect response	16	0
Teacher him/herself provided the answer or explanation of the question		33	28
Teacher builds on incorrect/ partially correct response		17	2

Table 3: Comparison of Teacher behavior in 6 of the top performing intervention and control schools

Tables 2 and 3 depict that there is a marked difference in the classroom environment and the discourse as the intervention schools seem to focus more on giving opportunities for students to articulate and reason. These opportunities though embedded in the tasks and activities of the geometric reasoning module, were taken as school mathematics practices by intervention school teachers.

Besides cognitive gains, there were certain non-cognitive and affective changes were also visible amongst students like increased confidence and motivation to learn especially among students who were repeating ninth grades and some girls who were earlier not motivated. Opportunities to discuss with peers and articulation in the classroom also helped in building collaborative skills and autonomy amongst students. The exposure to digital activities helped in developing skills and confidence to engage with computers and some of the students were seen to take charge of managing the maintenance of the computer lab and supporting other students to engage with digital activities by helping them to start the computer, log in details and showing the features which can be used in different software.

DISCUSSION

The study demonstrated a significant improvement in students' geometric reasoning abilities in intervention schools compared to non-intervention schools. This improvement was evident in various cognitive gains, including the development of shape concepts, properties of plane shapes, and hierarchical class relationships, as reflected in higher scores on post-test multiple-choice questions in schools where the intervention was reasonably implemented and maintained. Beyond conceptual understanding, the study highlighted that both the module activities and teachers' actions supported the behaviors necessary for engaging in mathematical reasoning processes. The practices adapted by the teachers in the intervention school were the result of the instrumentation process through which the teachers developed the utilization schemes for activities and tasks in geometric reasoning module. The geometric reasoning module for students and the reflective mathematics teaching online course and workshops for the teachers provided impetus for the instrumentalization that influenced the adaptation of tasks taken up in the classroom using hands-on and digital activities. Both digital and hands-on activities enabled students to articulate ideas that might remain concealed in conventional classrooms focused on repetitive procedures and definitions.

The members of the community belonging to the chat groups as well as in-person interactions with teacher educators and peer teachers provided opportunity for instrumentation by bringing together ideas from module and the teacher course and supporting teachers in identifying misconceptions and difficulties in understanding key geometric concepts. Teachers not only facilitated discussions for tasks within the module but also created tasks, examples, arguments and counterexamples depending on the contingent situations that arose as a result of students' responses. Some of the responses were unexpected for students and teachers struggled to address that appropriately. For example, one of the teachers found it difficult to address the students' question of why there are no right angles in Kite (See Thakur et al., 2020). The increase in opportunities for students to articulate and give reasons also increased the contingent situations in the classrooms, pushing the teachers to make in-the-moment decisions using subject matter knowledge and activating the pedagogical content knowledge. Some of these instances provided opportunities for teachers to re-learn and reflect more deeply on their knowledge. These included teachers coming up with counterexamples to challenge students' naïve or identifications of properties based on the prototypical figure or themselves becoming aware of their own predisposition to use prototypical figures for identifying the necessary property for defining a figure (See Kumar et al, 2019). The contingent situations acted as the catalyst for in-the-moment instrumentation. Thus, the use of artefacts from the modules and teacher course and classroom teaching practice were supported by the dialectic between instrumentalization and instrumentation process.

The instrumental genesis of the teachers in the control school is relatively rigid in comparison to the intervention schools as textbook serve as only artefact for instrumental genesis. The instrumentalization of the questions in the textbook is followed rigidly along with the instrumentation of the practices that expect student to listen and follow the procedure. The impoverished environment with lack of access to other resources and artefacts for supporting teaching, lack of interactions with peers and experts to support the instrumental genesis process has an impact on student learning outcomes as the practices do not factor in contingencies arising from students' difficulties. On the other hand, the presence of alternative tasks and mediums in intervention classroom supported the instrumentalization of adaption of tasks for the classroom to elicit and support student reasoning and responding to contingencies by the dynamic instrumentation process.

The findings have implications for the design of in-service teacher education initiatives and the design of resources for under-resourced contexts which can be implemented at scale. It is essential to note that rather than resources, the development of knowledge and skills of the teachers through adaptation of tasks and resources (instrumentalization) and using discursive practices that supported reasoning (instrumentation) contributed to significant difference in students' understanding in intervention schools. The number of factors that support the implementation of the resources in the under-resourced classroom needs to be studied further to understand how they support or constrain students' and teachers' learning.

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MATERIAL AND DIGITAL TOOLS FOR GEOMETRY IN MATHEMATICS LABORATORY

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In this paper, we present two examples of activities carried out with material and digital tools following the methodology of mathematics laboratory. The aim is to show how activities with the two kinds of tools can enrich students' experience and allow them to construct mathematical meanings. These two examples contain the same artifact used with different didactical functionalities at different school grades.

INTRODUCTION

As reported in the Discussion Document of this ICMI study, the use of material manipulatives and digital tools is widespread in teaching and learning geometry. Our perspective is the use of both material manipulatives and digital tools within the same educational context, thinking about their intertwined use and basing on their respective potential (Maschietto, 2018). Two examples are discussed in this paper. The first example concerns the content of measurement of 2D figures (area and perimeter) at primary and low secondary school; it deals with Tangram puzzles and other artifacts, in their material and digital versions (including GeoGebra); the second example focuses on conic sections at high secondary school through the use of particular artifacts for geometry, the mathematical machines (Bartolini Bussi & Maschietto, 2011). All the activities are carried out within the methodology of the mathematics laboratory (Maschietto & Trouche, 2010).

With these two examples, this paper aims to contribute to the discussion on resources (topic C) and to provide elements for answering the following questions (Discussion document, p.12):

- What is the role of visual tools and manipulatives on geometrical processes?
- How can bridges be created between manipulation and visualization and mathematical thinking in geometry?
- How can students be helped to abstract from perceptive and concrete properties of physical objects?
- What is the place of language, signs, gestures in activities based on manipulatives?

USE OF ARTIFACTS IN MATHEMATICS LABORATORY

The history of mathematics laboratory in the European culture is long and interesting (Maschietto, 2015). In Italy, at the beginning of the 21st century, the UMI-CIIM (Italian Mathematical Union, Italian Commission on Mathematical Instruction) prepared a Curriculum for Mathematics for primary and secondary schools (Anichini et al., 2004), which gave a special emphasis to the mathematics laboratory in its methodological part. This emphasis is maintained in the Italian National Guidelines for Primary and Secondary Instruction, approved respectively in 2012 and 2010. Therefore, in the mathematics laboratory, relevant components are the use of tools and the interactions among people working together for the construction of mathematical meanings. These elements are central in the

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 415-422). ICMI.

Theory of Semiotic Mediation (TSM, Bartolini Bussi & Mariotti, 2008), within which the mathematics laboratory activities reported in this paper are designed and analyzed. The main components of this theoretical framework are:

- The teacher uses an artifact to mediate mathematical meanings.
- An artifact is analyzed in terms of semiotic potential, defined as the double link between the artifact and students' personal senses and the artifact and mathematical knowledge; the conception of tasks for the students is based on this analysis.
- The didactical cycle of activities (working with artifacts, individual work, and collective discussion) always begins with the exploration of the artifact, according to the questions *How is the machine made?* and *What does the machine make?* (they support students' instrumental genesis; Vérillon & Rabardel, 1995); then, it focuses on the emergence of mathematical meanings embedded in the artifact by questions such as *Why does it make it?* and *What could happen if...?* (Bartolini et al., 2011).

The artifacts present in laboratory activities, as already written, can be both material and digital.

REFERENCES FOR LEARNING GEOMETRY

The Discussion Document highlights several components that characterize the teaching and learning of geometry. In this section, we refer to some of those components to support the analysis of the examples that will be presented in the following sections.

The first reference concerns the visualization of geometric figures. Duval (2005) distinguishes between iconic visualization and non-iconic visualization. The first is the most spontaneous approach to a drawing: a subject recognizes an object because its shape is similar to an already known object. In this sense, the shape resembles the typical shape of the real object represented and the contour of the object is mainly considered. Duval claims that this is the usual way of visualizing and pupils tend to deal with drawings through an iconic visualization. Under this visualization, the shapes appear stable: it is difficult to transform them into other similar shapes (for example, a square seen as a rhombus) or different ones (for example, non-congruent figures with the same area). According to a non-iconic visualization, a drawing is one of the representations of a geometrical object; the subject passes from the visual similarity of an object to a familiar shape to its geometric properties. This kind of visualization develops against the spontaneous way of seeing the shape by iconic visualization; passing from iconic visualization to non-iconic visualization is neither easy nor natural, but this transition is necessary to the learning of mathematical proof in geometry. Duval explains that using drawings to solve geometry problems involves three distinct operations: mereological deconstruction, instrumental deconstruction, and dimensional deconstruction. Mereological deconstruction consists of seeing the drawing as a union of superpositions or juxtapositions of figural units of the same dimension. Under an instrumental deconstruction, a drawing is considered the result of a construction process associated with a geometrical object, with the use of tools. Dimensional deconstruction is linked to a mathematical way of seeing drawings: a geometrical figure is considered as a set of figural units related to each other by geometric properties, and the drawing is analyzed accordingly. Non-iconic visualization is a necessary condition for dimensional deconstruction to be operational.

In addition, we consider three further aspects. First, we suppose that an iconic visualization supports a global viewpoint on a drawing, while a non-iconic visualization supports a local/punctual viewpoint. In this sense, relationships between global/local viewpoints should be considered in teaching and learning geometry. Then, another aspect linked to visualization is the relationship between static and dynamic aspects, well highlighted by research in dynamic geometry environment. The variation in geometric drawings allows the identification of invariant properties (the "invariants") and relations between the properties of the represented figures; for example, they are useful in the formulation of theorem statements (Baccaglini-Frank et al., 2009). Finally, the way of seeing the figures and the language to describe them are closely linked. In constructing a geometric vocabulary, the definitions and relationships between the shapes recognized by the students come into play. For example, the type of visualization and the consequent interpretation of the drawings can support exclusive or inclusive classifications of geometric figures (for example, they concern triangles and quadrilaterals, mostly studied in primary and lower secondary schools). Several research studies have been interested in the role of definitions and the implications of these definitions in the conceptualization of geometric figures (2D and 3D).

THE FIRST EXAMPLE: PERIMETER AND AREA AT GRADES 4-5 AND 7

The first example is taken from a collaborative project on mathematics laboratory (https://sites.google.com/view/diffusioneculturamodena2020/home-page) between teachers and researchers. Two teaching experiments were designed and carried out on the topic of area and perimeter, one in primary school (grade 4-5, 4 classes) and one in low secondary school (grade 7, 10 classes). The choice of the topic is based on the results of research (Fandiño Pinilla & D'Amore, 2006) which highlight how these two concepts are kept separate in teaching practice (at least, in Italy) and how consequently students (but also teachers) have difficulty relating them. In particular, the tasks for students often deal with the conservation of areas and perimeter or contextual increase/decrease of measurements without discussing the variation of the area of isoperimetric figures or the variation of the perimeter of equivalent figures. The manipulatives (in their material and digital versions) used in these experiments were the Tangram puzzle, the geoboard (grade 4-5 only), and the mathematical machine for tracing the ellipse by the gardener's way (grade 7 only).

The didactical intervention for grade 7 is composed of three phases. In the first phase, the Tangram puzzle was initially explored in its material version (Figure 1, left), with the objectives of identifying the geometrical figures that constitute it (isosceles right triangles, square, rhombus) and the property (the invariance of the area) common to all the combinations of the seven pieces of the puzzle. The square configuration was represented on paper and pencil and/or by paper folding, favoring the transition from 3D to 2D representations and implementing instrumental and dimensional deconstructions. To support these two processes, the students were asked to write the procedure for drawing the square in small groups (Figure 1) and to exchange the texts with other groups for reviewing and obtaining a correct final version. A Geogebra book with incorrect Tangram squares (Figure 1, right) was assigned to consolidate the construction procedure and the properties of the figures. The digital tool is used to propose tasks that cannot performed with the material Tangram (GeoGebra figures offer strategy feedback by dragging some vertices) and to collect and assess qualitatively students' answers (they had to upload their answers). In this way, it is possible to assign an individual work, as set by the TSM. After this task, the students played the puzzle game in small groups; there the reproduction of the Tangram combinations is based on mereological deconstruction by juxtaposition accompanied by the gestures of moving pieces for composing new combinations.

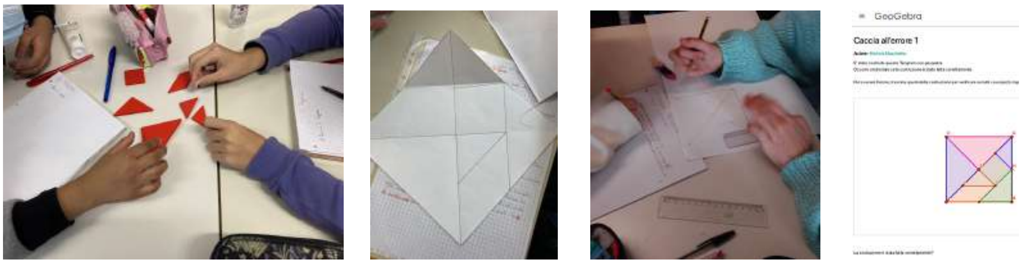
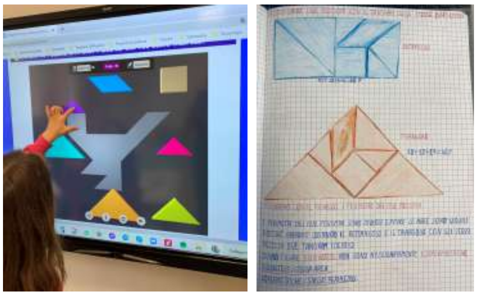


Figure 1: group work; construction of the tangram in paper and pencil; GeoGebra book

The activities with the Tangram puzzle allow students to pay attention to the invariance of the area for each combination of the 7 pieces, and to the measurement of this area and the area of the components using the smallest triangle taken as the unit. Furthermore, this game represents a good context to ask questions about the perimeter of the combinations and to look for combinations with minimum/maximum perimeter, already in primary school (Fig. 2). By the comparison of different combinations, the non-preservation of the perimeter emerges (Fig. 2, center and right).



The perimeters of the two polygons are different, even if the areas are the same since we have built the rectangle and the triangle with the same pieces of two identical tangram. Therefore, equivalent figures are not necessarily isoperimetric.

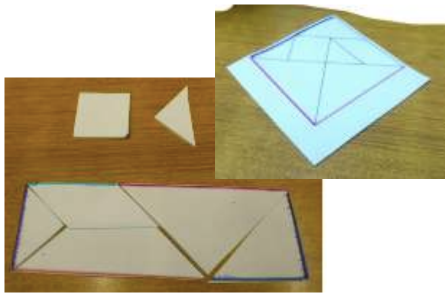
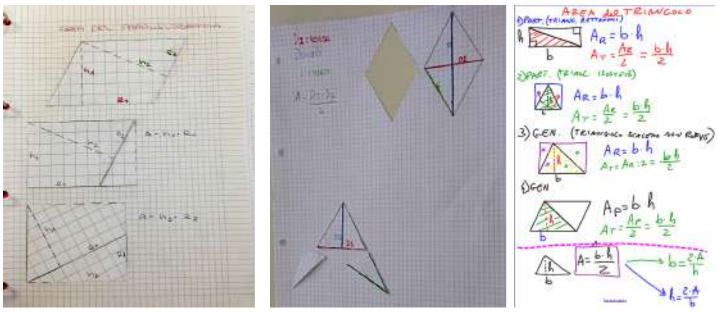


Figure 2: Tangram at IWB; notebook from grade 4-5; comparison of perimeter

The gestures performed by the Tangram puzzle related to the juxtaposition of pieces and the manipulation of equivalent combinations by moving pieces were recalled in studying the areas of quadrilaterals and obtaining their corresponding formulas (Figures 3, left and center). In this case, a digital tool is used to support collective discussion and mereological deconstruction. This work allows students to understand those formulas (Figure 3, right), which are often studied by heart.



S1: We obtained the formulas ourselves and I memorized the formulas better.
S2: Finding the formulas was an extra stimulus and then we remember them better.

Figure 3. Puzzle of quadrilaterals and students' opinions on these activities

In the second phase, the construction of the mathematical machine tracing the ellipse ("ellipsograph") following the gardener's way was proposed to grade 7 classes for asking questions on non-equivalent isoperimetric figures. It was built by students with recycled material (cardboard, pins, and string; Figure 4, left and center). The exploration followed the structure of the questions referring to the TSM and it was guided by a worksheet (Figure 4, center). The ellipse was also traced in the GeoGebra environment (Figure 4, right); students were invited to explore it by varying the parameters (length of the string and focal distance) as homework. The construction in GeoGebra allows students to go beyond the physical constraints of the mathematical machine: by varying the parameters, the students can see different drawings of the ellipse which cannot all be drawn with the ellipsograph. This task aims to support iconic visualization for those students who meet the curve for the first time.

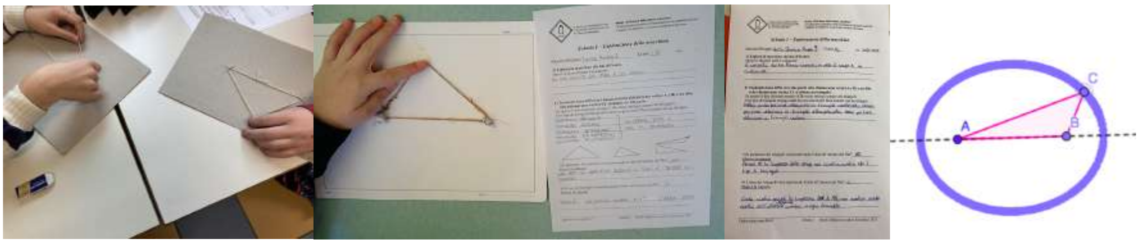


Figure 4. Ellipsograph, worksheets, and GeoGebra figure

The teaching experiment ends with the proposal of a challenge (third phase): “Build a machine that does the opposite of the ellipsograph: equivalent non-isoperimetric triangles”. This assignment requires identifying a change in perimeter and ensuring the invariance of the area; that is another variation to study, and it is a real problem to solve for the students (referring to the question *What could happen if...*). Different machines were proposed (Figure 5, left and center): some of them worked, and others did not. During the classwork, the teacher discussed the solutions, also within the GeoGebra environment, and proposed other problems on equivalent figures. In this case, too, new tasks are proposed by GeoGebra books (Figure 4, right), aiming to support a non-iconic visualization.

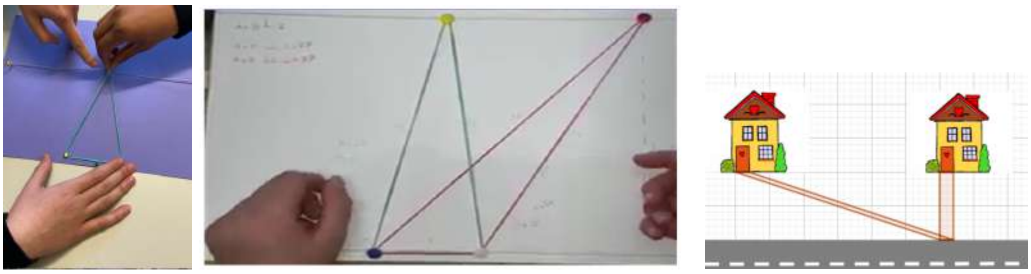


Figure 5. The new machine and new problems within GeoGebra books

THE SECOND EXAMPLE: CONIC SECTIONS AT GRADE 11

The study of conic sections is a classic mathematical content in Italian high secondary schools for grade 11 students. The National Guidelines for High Secondary School contain a precise reference to the approach to be followed, in which the conic sections must be studied from both a synthetic and analytical geometric point of view. The mathematical machines (they belong to the Museum System and Botanical Garden of the University of Modena e Reggio Emilia, www.mmlab.unimore.it) are suitable for proposing a synthetic approach (i.e., Maschietto & Bartolini Bussi, 2011). Two types of conicographs were considered in the didactical cycles (Dondi & Maschietto, 2022): conicographs for ellipse, parabola, and hyperbola functioning by taut string, and crossed-parallelogram conicographs for ellipse and hyperbola. They all embed the metric definitions (i.e., with foci and directrix).

The activities are structured in three phases: in the initial phase, all those elements considered necessary for the exploration of the mathematical machines are presented to students; the central phase is dedicated to the exploration of the conicographs; in the final phase, the theory of conic sections is presented through historical references. The mathematical machines are proposed with two different didactical functionalities (Maschietto, 2015): the taut string conicographs are used to introduce the new mathematical curves, their definitions, and properties; while the crossed-parallelogram conicographs are used to propose a unitary vision of conic sections and to approach the meanings of tangent and normal straight lines to a curve at a point of the curve itself. In particular, for taut string conicographs, the definition of the geometric locus generated by the machine is required, while for crossed-parallelogram machines the justification of the type of curve drawn is solicited (question *Why does it make it?*). Students' work is guided by worksheets, constructed consistently with the framework of the TSM.

We report an excerpt from the exploration of the gardener's ellipsograph by a group of grade 11 students (called A1, A2, A3, and A4). They generally recognize the drawn curve according to an iconic and global visualization related to the curve seen at low secondary school (above all in astronomy), but the analysis of the trace and the identification of parameters, variables and invariants for the definition of the geometric locus is not trivial for them, because it requires a non-iconic visualization and a local viewpoint on the curve. In this excerpt, first, variables and fixed lengths are identified (#3, #4), then a variable triangle (#8) as well as its isoperimetry property (#9, #14) even if it is not explicitly expressed in these terms and does not concern the vertex P (until #15). Nevertheless, the definition of the geometric locus (local viewpoint) needs a formulation involving the point P. We note the use of the terms "the same" (#9, #12, #14) and "equal" (#15) instead of "constant".

2	A2:	So, what does it vary [], [*he forms a triangle with the thread, Figure 6 left*]?
3	A4:	the distance between the two pins […]
4	A4:	this does not vary [*he points to the focal distance*] and this varies [*he points to the other sides of the triangle while A2 moves the thread*] […]
5	A3:	then it varies the distance [*he tightens the string and moves it around the pins*]
6	A*:	the distance of P from F1 and F2 varies [*the students write on their worksheet*]

The students then read the notes and wondered if there were other variables. […]

7	A2:	Beyond the point [*P*], the length of the two sides also varies. […]
8	A2:	Let's look at it as a triangle.
9	A3:	This [*side of the triangle*] plus this [*other side of the triangle*] is always the same.
10	A2:	Let's see more triangles. The two angles vary, and the lengths of the sides vary... up to approximately this limit... [*Figure 6, center*].
11	A4:	They always vary.

A2 moves the string.

13	A3:	That is, if we measure this [*he points to the distance between the foci and P*]
15	A2:	The sum of these two [*sides of the triangle between the focus and point P*] is equal.

Question 3 of the worksheet asked to draw three different configurations of the machine. The objective was to stimulate students to provide a graphical representation of the parameters and the invariant (i.e., by measuring the length of the string and the sides of triangles). However, all answers contain iconic and global representations (Figure 6, right), even if sometimes the invariant is written.

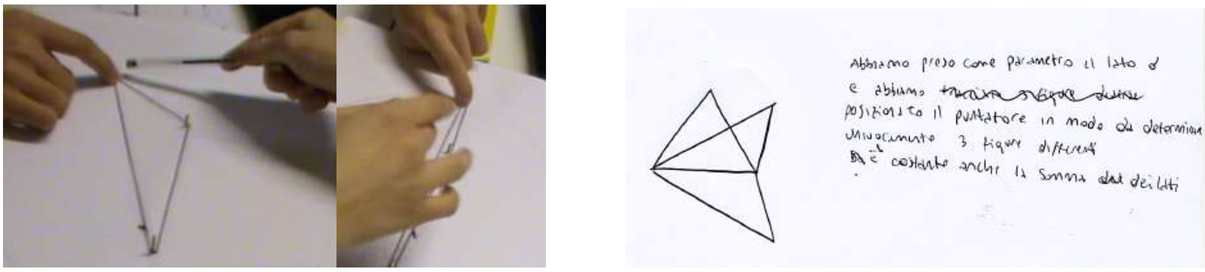


Figure 6. Triangle configuration and representation of these configurations (Question 3)

Contrary to a classical lesson in which the definitions of conic sections are stated (and the students often learn them by heart), working with the mathematical machines allows them to discover and build the definitions and to identify properties, by exploring and exploiting the machines. In the identification of parameters, variables and invariants, the analysis shows some difficulties in referring to the meanings of terms they have already encountered in another context before the study of conic sections. The use of different machines allows students' conceptualization to be tested: for example, each conicograph offers different material constraints and different representations for the parameters of the curves. The construction of the ellipsograph in the GeoGebra environment (as in Figure 4, right), asked after the work with the material machine, highlights the need to work more on parameters, variables, and invariants: for instance, in that construction, the possibility of choosing foci as free points hides the choose of that parameter of the curve. Compared with the first example discussed in this paper, GeoGebra is used with different didactical functionalities: there the students were supported to observe the shape of the ellipse depends on the variation of its parameters, while here the students were asked to make explicit their conceptualization of variables and parameters by constructing the curve.

CONCLUDING REMARKS

This paper presents two examples of didactical activities to contribute to the discussion about the use of material and digital tools in learning and teaching geometry. Through them, we aim to show how a bridge between the manipulation of artifacts and mathematical thinking in geometry can be created, proposing tasks by which the students describe procedures and drawings, represent the mathematical objects in different environments, and look for regularities and invariants. Concerning the use of only material artifacts, the use of digital tools allows students to perform tasks and problems that can not be proposed through the formesr; for teachers, it can be possible to structure sub-tasks and implement evaluation and strategy feedback, supporting in this way students learning by collective and individual tasks. On the other hand, the use of material artifacts encourages collaborative group work and students' exploration processes within a rich semiotic activity. In our perspective, each kind of tool offer complementary representations and approaches to mathematical objects. In the two examples, the activities carried out by the students through the digital tool often corresponded to individual activities, completing the didactical cycles of the TSM.

The artifacts are chosen following their preliminary analysis (semiotic potential in the TSM), which suggests which characteristics can be exploited with students at a certain grade and how to construct appropriate tasks. For instance, the use of the gardeners' ellipsograph at different grades is based on this kind of analysis. Furthermore, several tasks can involve the same artifact, because the emergence and the construction of mathematical meanings requires time (also for instrumental genesis), and it is not always possible to activate and end these processes in only one or two lessons. Another common element between the two examples is that the gestures made during a task can be performed in other tasks or with other artifacts. For instance, the gesture of moving pieces in the Tangram puzzle for composing new configurations is used then for quadrilaterals, and in general for transforming figures into equivalent ones that are not congruent. For conic section tracers, the gesture of usage of tightening a string is common to all three conicographs; in addition, it corresponds to creating a segment (cognitive root for the straight line) and gives a strong experience of a straight line. The analyses show that students very often have an iconic visualization, which can be questioned by tasks involving mereological deconstruction and the comparison between the variation of geometrical quantities and the invariant properties. Thus, the transition to a non-iconic visualization is supported.

In our classes, the manipulation of material artifacts, intertwined with digital tools, supported students' engagement in geometry activities. And this is not a secondary aspect for student' learning.

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DEVELOPMENT OF RESOURCES TO STUDY GEOMETRY REUSING MATERIALS

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The Professorship in Mathematics of the National University of Rosario (Argentina) supports both educational research projects linked to teacher training, as well as university extension projects to articulate the work with schools in disadvantaged or vulnerable contexts through the use of physical didactic resources, in particular, for the teaching and learning of geometry. Access to these resources is not always direct, and limitations of various kinds are usually encountered. In this instance, some attempts to address such obstacles are documented, highlighting the intrinsic relationship between university extension and educational research for the improvement of educational proposals in geometry when teaching resources made by teachers and students themselves are used. As a peculiarity, it should be noted that for its elaboration it is sought to put into action the ecological 3Rs as a transversal contribution according to the Sustainable Development Goals.

INTRODUCTION

The Professorship in Mathematics (PM) of the National University of Rosario (province of Santa Fe, Argentina) is one of the 30 Professorships of this type in the country. It has existed for 35 years at the University and is going through its third curriculum since 2018. In it, the Professional Teaching Practice (PPD) is conceived as an articulating project through training, where the link with the environment is fundamental for the delimitation of the professional profile with which the institution contributes to society. It is sustained from research projects that take as their object the training in, with, from and for PPD. For example, the current one is called “The processes of knowledge construction about the PPD in the PM” and is framed in a Research Program with other careers.

Within this framework and as a plausible form of university social commitment, for more than a decade university extension and volunteer activities have been promoted, sustained and strengthened, which, in this case, include articulated work with schools in a variety of contexts, especially disadvantaged or with some type of vulnerability. In particular, these activities are part of the mathematical work with educational resources, initially digital (within the framework of the Connect Equality Program, University Volunteering 2011), with the incorporation of analog manuals (from a University Extension Project 2013) and in turn with attention to environmental care in its design (from the Integrating Program 2018). All this gave rise to the ReMatEd team (acronym for Resources + Mathematics + Education) composed of students, graduates and teachers of the PM.

On this occasion we share the resources to teach and learn geometry that have been designed, as well as some aspects related to their elaboration that contribute to the task of the Mathematics teacher who wishes to produce them. We have been implementing these resources in schools in the city of Rosario and surroundings. About the educational levels, we are responsible for both secondary and primary, and teacher training for these levels, being the secondary level (students 12-18 years old) that we will center. Briefly, it is intended to illustrate possible routes with accessible resources to study geometry, which can be designed by multiple actors (students, teachers, parents, researchers), with the elements that are available according to the possibilities. This with the purpose of taking advantage of inputs that people discard in their usual consumption, to activate the 3Rs of ecology (reduce, reuse, recycle) that encourage to cushion the human impact on the environment through a more efficient use.

BACKGROUND

There are numerous definitions of "teaching resources". Alsina et al. (1988) consider it as any object, device or means of communication that can help describe, understand and consolidate concepts in different phases of learning. Without differentiate resource and didactic material, we consider the didactic resource as a material that although it may have specific intentions from its design, often transcends the original intention of use and admits varied applications (Alegre et al., 2018).

On the other hand, although geometry is recognized as one of the most important components of the school curriculum of mathematics (Atiyah, 2001; Jones et al., 2006), in secondary school it is poorly developed and often even forgotten how to treat in teacher training (Gutiérrez, 2010). For Corica & Marin (2014), its exclusion or superficial study obstruct different ways of reasoning, inferring and producing properties and relationships. We consider then that the use of didactic resources in the geometry area can be an engine of promotion for its approach in the school.

It is also necessary to address the particularities of educating in vulnerable contexts. Promoting participatory methodologies linked to the game allows overcoming master classes, capturing the attention of students in non-intimidating environments and more fun ones, as reported by Hierro & Seller (2020). From our perspective, we believe that the use of concrete manipulative resources can be especially motivating in these contexts, but we should not forget to consider the cost of them.

For more than a decade in a sustained manner, ReMatEd participated in science dissemination events in public spaces such as squares or public university premises, reaching a diversity of public and socioeconomic sectors. In addition, game kits (many of those mentioned in this paper) were developed and shared with public primary and secondary schools in the region.

CURRICULUM FRAMEWORK

As presented above, in this work a cut is made in terms of educational level. In reference to this, in Argentina, the Secondary School corresponds to the last stage of compulsory education. In particular, in the province of Santa Fe, it consists of five years (students are 12 to 18 years old). Likewise, of these five years, the first two ones make up the Basic Cycle, and the last three, the Oriented Cycle.

At the national level, there is a curricular document called Priority Learning Cores (NAP), which reflects the knowledge considered key and relevant for adolescents. In turn, based on the NAP, the province of Santa Fe proposes the Curricular Guidelines at the jurisdictional level. In both cases, only work with resources of a technological nature, referring to dynamic geometry software, is mentioned, and no reference is made to resources of another type, such as manipulatives.

As for the Geometry area, the NAP promotes teaching situations that allow the use of properties of figures and geometric bodies to solve problems, as well as the production and analysis of geometric constructions, and the production and validation of conjectures on geometric relationships. In the

Curricular Orientations, the importance of validating the constructions through processes that pass the empirical test and favor deductive argumentation is added. They also indicate that a dynamic geometry software would enhance formulation of conjecture, argumentation and modeling, and make explicit the integration with the axis of Algebra and Functions, where certain geometric objects are treated algebraically.

As for the priority established contents, a large part of Secondary Education is devoted to those related to synthetic geometry, the last year being the only one totally dedicated to analytical geometry. The Basic Cycle deals with the study of triangles (properties of their angles, notable points, congruence and similarity, Pythagorean theorem), angles between lines, quadrilaterals, polyhedral bodies and isometric transformations, promoting constructions with ruler and compass, demonstrations and relationships between concepts (such as, between area, perimeter and volume). In the Oriented Cycle, the study of polygons in general (similarity) and triangles, specifically, with trigonometry, is deepened. In the last year, in particular, analytical geometry is approached from the work with Cartesian plane and equations of the line, circumference and parabola.

RESOURCES MADE WITH YOUR OWN HANDS

Below we share four manipulative resources and five adapted board games, that we have produced in greater quantity and with which we have worked frequently in secondary level institutions. These types of manipulative resources offer students the possibility of linking mathematical concepts not only through the auditory sense, but also through the visual and tactile sense. In this way, abstract interpretations are combined with concrete interpretations of the geometric definitions and relationships being worked on. Some examples of activities can be found in Sgreccia et al. (2019).

The peculiarity of our contribution lies, on the one hand, in the reused materials that we employ to produce the resources, with an ecological touch. On the other hand, we prioritize that they can be carried out by teachers and/or students, in turn activating geometric knowledge in this elaboration process. This "learning by doing" is similar to movements such as New Scholl and Maker Space, particularized here in geometry and with affordable materials.

Geoplane

The geoplane has a great versatility that, as Villarroel & Sgreccia (2012) highlight, allows it to address varied contents linked to the plane and, in some cases, to space. It consists of a plate with fixed points (nails) in which segments, angles and geometric figures can be formed using elastic bands, which are possible to stretch achieving the desired length.

It is easy to manufacture depending on the type of material used according to the purposes for which it is intended and the resources available. It can be made of wood with nails or screws (fixed geoplane), or failing that with some firm, but perforable material, such as cork sheets or EVA foam sheet pricking with push pins (mobile geoplane).

Different working templates can be used on each geoplane depending on the location of the nails: quadrangular, isometric or circular. The mobile geoplane, although it is more unstable or less durable than the fixed one, allows to adapt the template according to the work carried out in the classroom by changing the printed paper that is placed on the cork sheet. This is one of the advantages of creating the own geoplanes, with the desired templates.

We show as an example in Figure 1, the process of making geoplanes with reused wood and screws, carried out by students who are part of the ReMatEd project.

Figure 1: Elaboration of circular and quadrangular geoplanes

Although in vulnerable contexts there is not always an adequate device (it should be noted that for more than a decade the Connect Equality Program has been providing equipment to institutions, teachers and students), another option to use this resource is through virtual geoplanes. These, in terms of digital simulation, offer the same possibilities as manipulative geoplanes by simply accessing from mobile devices, tablets or computers; for example, Geoboard by The Math Learning Center.

Among the contents of secondary school that we have addressed with the geoplane are: construction of flat figures, similarity of triangles, analysis of formulas for the calculation of surface measurements, area, perimeter and their relationship, angles, notion of parallelism, obliquity and/or perpendicularity, Pythagorean Theorem, Sine and Cosine Theorem, trigonometric ratios.

At the same time, on certain occasions we were able to detect some limitations of the resource. For example, grids do not show explicit links with Cartesian planes, a difficulty that could be overcome by combining with some other resource of a technological nature such as GeoGebra (as we have done, for example, in a sequence to introduce the sine trigonometric ratio). Likewise, we can argue that proposals that require the use of geoplanes strongly promote the motor development and creativity of students linked to specific contents of geometry. For example, the construction of a segment, closed polygonal or open polygonal in direct relation to the placement of elastic bands on the geoplane.

Polyforms

Polyforms are recessed pieces with flaps that allow the creation of polyhedron bodies. These constructions strongly use the notion of a polyhedron's face together with that of flat development. For its manufacture, cardboard or firm plastics are enough, which can be used from product packaging such as tea or cereal boxes, disused X-ray plates, among others. Due to this characteristic, it is easy to prepare and very low cost, allowing it to be considered an object of promotion of the activity of recycling and conservation of the environment, in the key of sustainable development (Corbetta, 2022). We show in Figure 2 the process of developing this resource.

Figure 2: Elaboration of polyforms with radiographic plates and cartons in disused containers

The pieces we have designed at the moment are regular figures, from three to five sides, although they can be made of more types (not regular, more sides). With a kit of twenty regular triangles, six regular squares and twelve regular pentagons, it is possible to construct (assembling and disassembling) the five Platonic solids. Although these regular bodies constitute an entity in themselves, the work with polyforms also allows the construction of special bodies (for example, a stellated dodecahedron) according to the free production of each student. Action that makes it possible to make geometric work an artistic and creative activity, strongly promoting special reasoning and the determination of conjectures about spatial relationships of objects. Likewise, if the development of the resource is carried out by the students, it allows them to work on the construction of geometric figures with a ruler and compass and all their properties.

As a possible didactic limitation of this resource can be considered the association of the notion of figure (two-dimensional) with that of the piece (three-dimensional) of the polyforms, because each piece takes the concept of figure to be considered face of the constructed body. Working in parallel with two-dimensional representations on paper can help overcome this type of obstacle. Polyforms, as a manipulative resource, give rise to a special concrete conception of the three-dimensional, promoting construction skills, which is difficult only with pencil and paper.

Among the contents that we have addressed with polyforms in different courses of secondary school are: construction of regular polyhedra bodies, recognition of faces, vertices and edges of polyhedron bodies, Euler's equation, plane developments, notion of area and volume of polyhedron bodies.

Circular sectors

The kit of circular sectors that we have assembled is formed by six discs of the same diameter: five of them divided with the same amplitude -1) two circular sectors of 180°; 2) three of 120°; 3) four of 90°; 4) six of 60°; 5) eight of 45°- and one whole (without cuts). This resource has a total of 24 pieces.

For its elaboration, the materials have to allow an easy cut, because having to round the discs becomes very expensive the task. To avoid this cumbersome procedure and avoid messiness or imprecision, we make them with disused compact discs and on these we simply make straight cuts to create the circular sectors. Then they are covered with EVA foam sheets, also preferably reused, of different colors. As mentioned for other resources, its processing cost is almost zero and enables awareness spaces about the ecological 3Rs. We share in Figure 3 the process of developing these circular sectors.

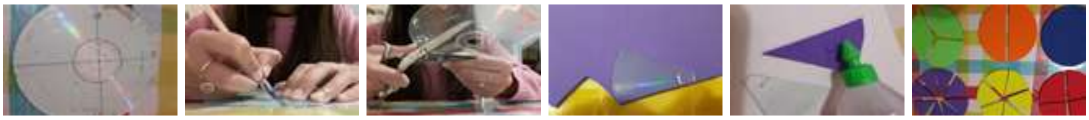

Figure 3: Construction process of circular sectors with disused CD and EVA foam sheets

As for the mathematical contents promoted by the work with this resource, one can clearly allude to the study of angles, but also to the notion of circular sectors, central angles, circumference and circle, part-whole relationship, equivalent fractions, trigonometric ratios, among others. Working with this type of materials allows one to interpret and identify geometric concepts through the action of assembling-disassembling, relating and comparing amplitudes, a much less intuitive activity that is carried out only with pencil and paper. On the other hand, constructing the circular sectors themselves allows choosing the number of pieces or measurement of the angles according to the objectives of the teaching activity, as well as, if created by the students, encouraging the measuring of angles and the construction of figures with ruler and compass.

Tangram

Tangram is a very popular educational resource as a puzzle game for all ages. Its objective is to create certain geometric shapes or specific silhouettes with small figures (regular and irregular). There is an immense variety of types of tangram according to the figures that make up each one. Among them:

Chinese (the most popular), Ovoid, Cardiotangram, Brügner, Fletcher, Lloyd, Stomachion, Triangular, Pentagonal, Hexagonal, Pythagorean, Spatial, as reported in Villarroel & Sgreccia (2012) and exemplified with activities in teacher training in Dominguez (2021).

For its elaboration plates of a material that is easy to cut but resistant to deformations are required. We usually use EVA foam sheets to make the different pieces of each tangram, as they are accessible from diverse discarded inputs in good condition or, failing that, in bookstores at affordable costs.

The richness of this manipulative in terms of learning geometry is well recognized. It allows to develop visual, drawing and construction skills when reproducing models and given data, communication when interpreting geometric information, and reasoning when exploring concepts and regularities. It promotes geometric learning not only during the same game of construction of the desired figure but also can take advantage of the same process of elaboration of the material by the students. For this construction, various geometric reasoning is put into play combining concepts such as: angle measurement, bisector, mediatrix, perpendicularity, midpoint, rational numbers.

As with the geoplane, the tangram also has digital versions that promote geometric skills; for example, Tangram Online for Kids in Cokitos. There are numerous options, varying in complexity, so they can be used for different levels of schooling and intentionalities.

Adapted board games

Board games are those that, as the name implies, can be played on the table, although any space where a group of people can meet can also be used where the participants are arranged surrounding the game. They are also versatile in terms of their elaboration, that is, anyone from some recycled materials can recreate and customize them according to their preferences. This represents a benefit in contexts of vulnerability and has allowed us to create copies for various schools without making a significant expense. Even, on more than one occasion, the same institutional actors were the ones who elaborated them from collecting disused materials for reuse. The elaboration of the resources themselves determines a moment of reflection on the concepts addressed, allowing different mathematical notions to be put into criticism, relationship and contrast.

Board games adapted to mathematical content are a tool that can be used from the initial level to the secondary level and even higher. These games allow us to address a wide range of mathematical concepts since they are easily adapted and customized according to the specific learning objectives. Some of the geometric contents that can be addressed are area and perimeter of plane figures, geometric bodies, angles, classification of triangles, similarity of triangles, among others. Particularly, the board games that we have adapted are Domino, Bingo, Ludo, Memotest and Game of the Goose (Figure 4a, b, c, d & e); for all ones were designed the same criteria as the original game.

Figure 4: Adapted board games: a) domino; b) bingo; c) ludo; d) memotest; e) goose game

Adapted board games have the advantage of being versatile; that is, they can be adapted to any content, in particular geometric ones. They offer an interactive and relaxed way to learn mathematical concepts and allow students to develop mathematical skills in a playful way. They are likely to encourage the participation of students favoring their progression and the consolidation of content.

CONCLUSIONS

Experience has shown that the teaching and learning of geometry in classrooms with didactic resources can be limited by different aspects. On the one hand, the resource itself may constitute an obstacle by presenting mathematical or didactic restrictions. An example, from our experience, is the case of the introduction of trigonometric ratios through the use of fixed geoplanes. To overcome this physical limitation, we have combined it with another of a technological nature. Likewise, it merits accompanying the material with an adequate didactic analysis, planning and reflection by the teacher.

It is essential to remember the exploratory need for certain resources, such as geoplanes, polyforms, circular sectors and tangrams. This exploratory power lies in "letting touch" and "letting do". Although, as has been indicated, it is convenient to implement oriented instructions, it is important that the teacher offers each student a previous time of personal manipulation of the resource. On the other hand, an aspect that also highlights the work with manipulatives in the teaching and learning of geometry is the potential to put geometric concepts into practice not only during the implementation of a didactic sequence or during the game but also during the previous process of elaboration. This shows the usefulness of giving students the opportunity to develop their own learning materials. During elaboration, students make guesses (correct and incorrect) through trial and error, interrelate concepts, take measurements, and pose new questions.

In addition, users may constitute another limitation. On the one hand, students may have difficulties for manipulation, either due to motor issues or prior knowledge. But it is also important to note that access to resources starts with teachers. Knowing them, manipulating them, analyzing them... These are aspects to be taken into account also in the initial and continuous training of teachers. The mere fact of presenting a manipulative resource to the class does not disoblige the teacher to be present with all his senses and to have an objective or intentionality with the activity. Precisely, as the experience of working in various concrete schools has marked us, with the mere fact of manipulating a resource you do not learn. The exchange and sharing of productions between students collaborates a lot. It is not only about using the resource, but requires a specialized look from whoever is in charge, with awareness of how and for what. Indeed, we have been noticing in the territory different levels of specificity of use of the resource: absent, dependent, intermittent and critical. Hence the need to consider the resource as a scaffold that does not generate dependence every time one have to work with the concept, becoming an obstacle. In critical use, to which we aspire as extension teachers and researchers, it also comes into play to know when to dispense with the resource.

Specifically, in the PM, the content "didactic resources" is present in the four years of the program, in each of the annual curricular spaces of PPD (I to IV). They are integrated into didactic sequences that the students themselves develop, as well as in microclasses in which future teachers play the roles of teachers and students. The referenced ReMatEd projects are integrated, year after year, by a minimum of 10 future teachers in Mathematics (a requirement of the University, according to the relevance of living this type of experiences) and the majority of the teachers of PPD also integrates ReMatEd. There is, in this sense, a community established in this regard with an approach aimed at documentary genesis processes (Gueudet & Trouche, 2009).

From ReMatEd we have gone through different contexts: from closets full of abandoned resources and without knowing their existence by the members of the institution, to schools with the intention of using them but with ignorance of how or the economic inability to have copies. The proposal of workshops located with teachers, either for the construction or for the incorporation of the resources to their classes, the implementations with our accompaniment, and the resource kits, as well as teaching how to assemble them, were some of the strategies carried out to overcome these difficulties.

Thus, in the present work we seek to document some of these attempts to address possible ways to achieve the objectives in terms of learning geometry in classrooms and contexts with limitations, both socioeconomic and training. Finally, we highlight as a key the amalgam between the extension and research work located for the formation of the team. The extension has allowed a close link with the environment and the investigative gaze has empowered to take these experiences as an input for reflections, for their subsequent critical analysis in order to make contributions to the community.

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HOW DO TEACHERS COORDINATE STUDENTS' EMPIRICAL PERCEPTION AND LOGICAL REASONING IN DYNAMIC GEOMETRY ENVIRONMENTS: CHINESE AND FRENCH CASES

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This paper presents a part of a French-Sino PhD project. It investigates mathematics teachers' practices, while using 3D dynamic geometry environments (DGE), for coordinating students' empirical perception and logical reasoning, and the teachers' justification for their coordination behavior. Adopting a qualitative approach in a multiple-case study, we select two teachers both in France and in China and develop a framework to analyze teachers' coordination behavior in lessons with 3D DGEs. The analysis reveals several patterns of coordination behavior, which are unevenly distributed among the four teachers' cases. Subsequently, the teachers' justifications are identified and contrasted. The findings illustrate how teachers' coordination patterns could be influenced by institutional, pedagogical, material and temporal factors. Finally, some implications for teacher education are drawn, along with a cross-cultural perspective for future research on 3D geometry teaching with technology.

INTRODUCTION

This paper addresses teachers' didactical challenge when they use 3D DGEs in upper secondary school. 3D DGEs refer to digital files or websites constructed with a dynamic geometry software like Cabri 3D or GeoGebra. They enable users to create and manipulate 3D dynamic models, giving access to different viewpoints and to feedback (Hohenwarter & Jones, 2007). While 3D DGE bring new opportunities for teaching 3D geometry, they create a challenge for teachers for balancing students' empirical perception (e.g., visual and sensory motor) and logical reasoning (e.g., inductive and theoretical deductive reasoning). At the beginning, researchers always highlighted a progression from empirical perception and inductive reasoning to theoretical deductive reasoning in 3D DGEs (Accascina & Rogora, 2006). But later researchers argued the rationality of perceptual and inductive approaches to geometric knowledge, considering cultural, institutional, temporal and material contexts of teaching (Nardi et al., 2012). We call *coordination behavior* teachers' instructional behavior considering both aspects in students' activity – empirical perception and logical reasoning – in mathematics lessons with 3D DGE, and *coordination patterns* the regularities in their coordination behavior. This paper reports representative *coordination patterns* of French and Chinese teachers unveiled in Shao's PhD study (2022), situated in a Sino-French Cooperation Project which offers two advantages: a defamiliarization mechanism helping researchers re-see teaching practices in their original cultural context; an analytic process evidencing variations in factors like didactical traditions, institutional contexts and available materials, likely to impact teachers' practice.

In this paper, we do not preset criteria for what constitutes an appropriate coordination pattern, but rather try to understand teachers' context and difficulties leading them to that pattern. After unveiling several coordination patterns, we identify influencing factors from the justifications that teachers proposed for their behavior or decision-making, to answer the research question: "how do factors influencing teachers' coordination patterns vary from one country to another?"

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Coordination behavior, a notion specifically proposed in our study, is described by Shao (2022) as by adapting Toulmin (2003)'s diagram of argumentation (Fig. 1). This diagram is now widely used to model informal argumentation processes in mathematics instructional settings. Our adapted framework of coordination behavior encompasses several specifications to some of its components, aiming to reflect a mode of interaction between students' perception and logical reasoning. We also integrate, from research literature (Conner et al., 2014; Accascina & Rogora, 2006), types of teacher support, including questions (e.g., *requesting an idea*, *requesting elaboration*), supportive actions (e.g., *directing*, *promoting*, *evaluating*), and typical usages of 3D DGEs (e.g., *dragging for a better view*, *dragging for contrast*). These reflect teachers' strategies to foster students' empirical perception and logical reasoning. To save space, we do not elaborate on them, but directly present, in the Results section, the coordination patterns obtained by applying this diagram to data analysis.

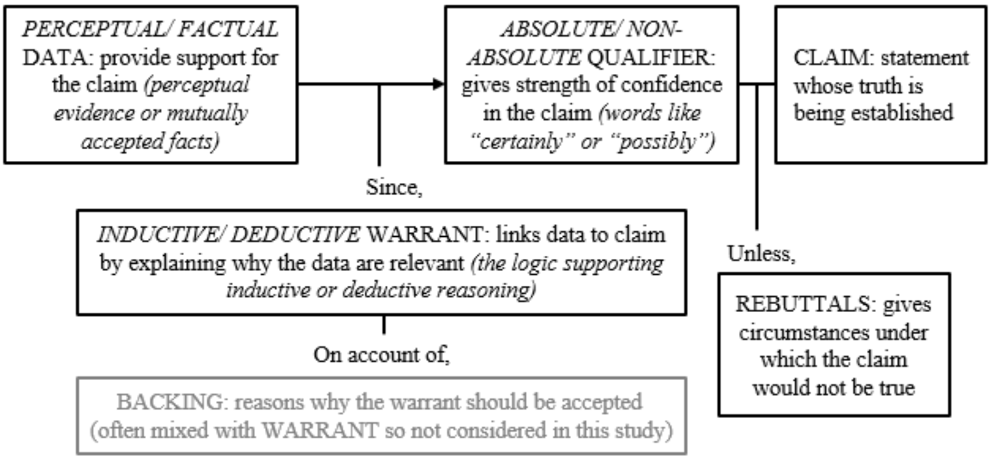


Figure 1: Components in Toulmin's diagram (normal) and our specifications (italics)

Besides unveiling teachers' coordination patterns, this study attempts to identify influences on the patterns. To do this, we identify *a priori* a range of such influencing factors: the epistemological factors (a mathematical theorem or definition), pedagogical ones (a pedagogical principle), institutional ones (recommendation in a curriculum or textbook), empirical ones (teacher's personal teaching experience), or evaluative ones (personally held view or belief) from Nardi et al. (2012); the temporal, material, and cultural contexts from Herbst & Chazan (2011). In our analysis, we refer to these factors but remain open to other influencing factors emerging from data.

METHODS

We adopt a qualitative approach to a multiple-case study, involving two French and two Chinese teachers who use 3D DGEs in upper secondary 3D geometry lessons (3 lessons of French teachers and 6 lessons of Chinese teachers). The four teachers are experienced in, or at least keen on, using 3D DGEs; thus their cases are supposed to evidence fruitful coordination patterns and allow for a maximum exploration of the focal question within the time limit. For each teacher, the data collected include lesson videos – to faithfully record teachers' coordination behavior – and a follow-up interview – to elicit teacher's justification and influencing factors. The interview is conducted after an initial analysis of the empirical and logical reasoning activities in the lesson videos, and we question teachers' decision-making related to the activities and their justifications. A qualitative content analysis is performed on the lesson videos with the adapted diagram of argumentation. The videos are divided into episodes by the tasks or subtasks in the lesson. Each episode is modelled by a diagram of argumentation. Then the episodes are grouped by the basic structure of their diagrams – combination of argumentation components. As a result, each group of episodes shares the same basic structure in the diagrams and corresponds to one coordination pattern. Then, the interview scripts related to each group are gathered, from which teachers' justifications and influencing factors on their coordination patterns are identified.

Since a range of influencing factors have been identified in literature, we focus on those relevant to teachers' coordination and briefly describe their states in France and China (Table 1).

Factors	France	China
Epistemological	Modern mathematics system, including axiomatic Euclidean geometry, established early in France but late in China	
Institutional	Both curricula take mathematical proof at the heart, and highlight development of spatial visualization, deductive reasoning and task-solving abilities; Exam requirements vary from one task to another	
Pedagogical	Didactical tradition of highlighting rigorous proof and pure mathematics system (Gueudet et al., 2017); Theory of didactical situations (Brousseau, 2002)	Didactical tradition of highlighting classroom efficiency and "two basis" (Xu et al., 2013); Theory of teaching with variations (Gu et al., 2004)
Empirical and evaluative	Teacher's personal teaching and learning experience, views, values or beliefs, different among individuals	
Temporal	Tight teaching schedule; Flexible way of managing time for one lesson	Tight teaching schedule; Well-structured time for one lesson
Material	A set of well-structured 3D DGEs offering a progression of 3D geometry tasks, GeoGebra tools, and feedback	3D DGEs mostly need to be designed by teachers themselves

Table 1: Factors likely to influence teachers' actions in coordination

DATA ANALYSIS AND FINDINGS

This section presents representative coordination patterns emerging from data, and the corresponding teachers' justifications. The italics in the following text are elements in the new behavior framework.

Coordination pattern 1: From empirical perception to theoretical deductive reasoning

This pattern is characterized by a chain of *perceptual data – non-absolute qualifier – claim* and a chain of *factual data – deductive warrant – absolute qualifier – claim*. That means teachers move students from a perceptual to a theoretical approach to confirm a conclusion. When the first claim is incorrect, teachers propose a *rebuttal* (i.e., rebut students' biased perception) by *dragging for a better view* or *evoking feedback* in 3D DGE. This pattern is identified when Dora helps students construct intersection of planes (EID) and (ABC) in a 3D DGE (Fig. 2). At first, student S1 claims that lines (EI) and (AD) intersect to produce one point in the target intersection, drawing on perceptual effects on the screen. Dora lets S1 construct the intersection, evoking the 3D DGE internal feedback of "nothing happens". Then Dora drags for a better view in the 3D DGE (Fig. 3), which makes S1 uncertain about her claim and propose a new one: (EI) and (AB) intersect. Dora constantly asks for S1's explanation (*request elaboration)* and *evaluates* her response only when she says the two lines are "both in plane (EAB)". Here, S1 probably refers to the geometric theorem she learned earlier – two lines in space are either secant or parallel when they are coplanar, and it is clear that Dora prefers students to utilize theorems rather than empirical perception.

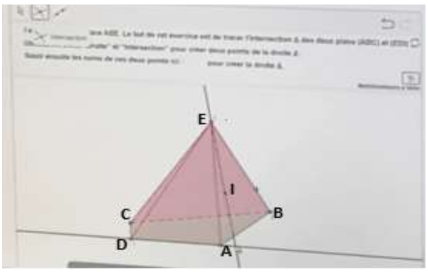


Figure 2: S2's screen at the beginning

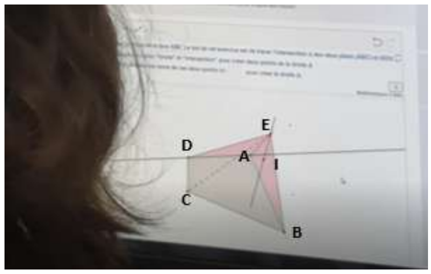


Figure 3: Dora drags for a better view in 3D DGE

Coordination pattern 2: Combine directed empirical perception with geometric theorem

This pattern is characterized by a chain of *perceptual data – deductive warrant – absolute qualifier – claim*. That means teachers lead students to combine empirical evidence in 3D DGE with theorems and make them certain of the conclusion derived. They always *direct* students to explore perceptual data that can satisfy the precondition of a theorem. Then they *promote* students to get a claim by combining the perceptual data with the theorem (deductive warrant).

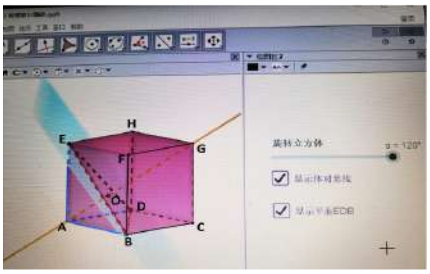


Figure 4: The cube overlaps with its initial position

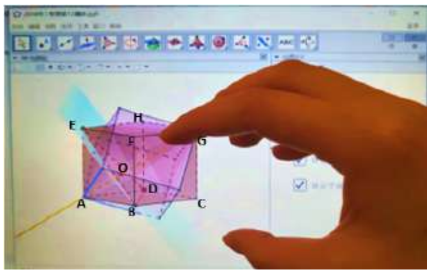
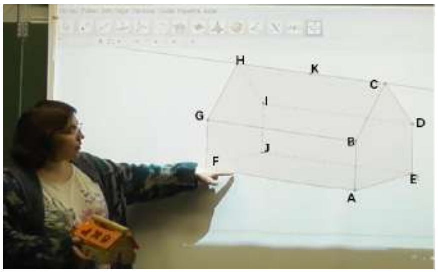


Figure 5: Huang simulates the rotation of the cube

This pattern is identified when Huang helps students determine a section plane that forms equal angles with all edges of a cube (part of a multiple-choice task in his lessons, Fig. 4). Huang shows the condiguration in Fig.4 and reminds students of a theorem: "a figure always keeps its shape and size while being rotated". Then he rotates the cube around the diagonal (AG) with a slider, showing that [AE], [AB] and [AD] would respectively coincide with [AB], [AD] and [AE] (Fig. 5). Combining the "coinciding effect" and the theorem, Huang leads students to, and makes them certain of, a conclusion that the three triangles ABO, ADO and AEO are congruent, and thus the angles between plane (EDB) and [AE], [AB], [AD] are all equal.

Coordination pattern 3: Focus on empirical perception

This pattern is characterized by a chain of *perceptual data – absolute qualifier – claim*. That means teachers make students certain of a conclusion only by fostering their empirical perception. Perceptual data concern multiple observable examples, which are sometimes evoked by teachers with the modality of *dragging for contrast*, sometimes provided by students under teachers' questions of *requesting elaboration*. With the perceptual data, teachers directly propose a conclusion about which students generally hold no doubt. This pattern is identified when teachers help students understand in which condition the 2D representation can faithfully represent properties in space (Fig. 6), or when they teach axioms about relative positions in space.

a) 

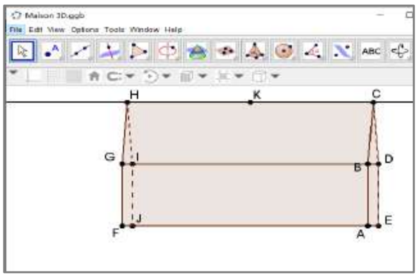
b) 

Figure 6: Dora drags to contrast a bottom angle (∠JFA) (a) and a front angle (∠BGF) in a 2D representation (b)

Coordination pattern 4: Combine empirical perception with inductive reasoning

This pattern is characterized by a chain of *perceptual data – inductive warrant – absolute qualifier – claim*. That means teachers lead students to combine inductive reasoning with perceptual evidence and make them certain of the generalized conclusion. Perceptual data mainly include evidence that teachers evoked *dragging for separation* and *dragging for a better view* in 3D DGE.

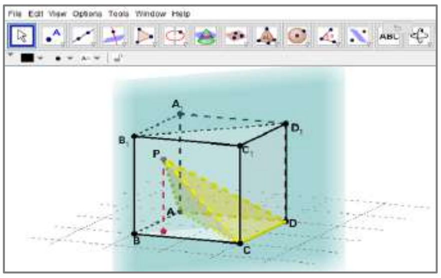


Figure 7: Huang drags to help students to evidence the invariant aspect – P always projected onto the bisector of ∠ADC

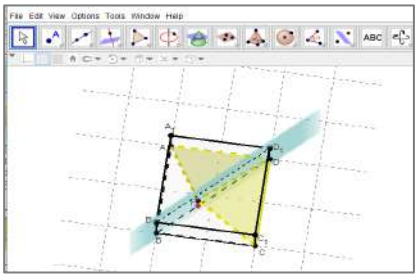


Figure 8: Huang drags for a "better view" to observe the position of projection of P

The pattern is identified when Huang teaches students a conclusion about a figural pattern: for any pyramid P-ADC situated in a cube, with P being in the diagonal face of the cube, the orthogonal projection of P to the base plane always falls on the bisector of ∠ADC (Fig. 8). Huang uses the two dragging modalities (Fig. 7-8) and guides students to generalize the conclusion by inductive reasoning.

Coordination pattern 5: Empirical perception as auxiliary support for deductive reasoning

In this pattern, characterized by a chain of *factual data – deductive warrant – absolute qualifier–claim*, the teacher support aiming to elicit chain components involves perceptual effects serving as auxiliary support for theoretical deductive reasoning. This pattern is observed when teachers use 3D DGEs, gestures or real objects to illustrate the structure of a 3D task figure or a geometric theorem.

In summary, five coordination patterns are identified, which differ in the interaction mode between theoretical and perceptual activities, and in the role played by 3D DGE. Table 2 shows frequencies of the coordination patterns in each teacher's case. Next we contrast patterns and related justifications of the French and Chinese teachers.

Episodes involving coordination pattern	Dora	Sonia	Huang	Xia
1. From perception to theoretical deductive reasoning	32	16	0	2
2. Combine directed perception with geometric theorem	12	2	42	0
3. Focus on perception	4	10	0	18
4. Combine perception with inductive reasoning	0	0	4	0
5. Perception as auxiliary support for theoretical deductive reasoning	5	0	28	14
Total Episodes	53	28	74	34

Table 2: Coordination pattern frequencies by teachers

Patterns 1 and 5, fostering students' theoretical deductive reasoning, occur frequently in all teachers. French teachers most frequently move students from empirical perception to theoretical deductive reasoning (pattern 1). This corresponds to the fact that in two of their three lessons, every student has access to a computer. In the interviews, both Dora and Sonia express their wish to let students learn by autonomy in the 3D DGEs, designed by a researcher-teacher team in France and aiming to create a "milieu" for students to interact with and learn from feedback. Here we identify pedagogical influence on Dora and Sonia's coordination patterns (Theory of didactical situations). Dora and Sonia also express the pedagogical influence of the French didactic tradition of highlighting rigorous proof:

Dora: I want them to reason all the time, or more precisely, to appropriate the theorem.

Sonia: There are two things to make an intersection, 'construct' and 'justify'… they should know how to do, and meanwhile know why to do this (Sonia).

For the two Chinese teachers, pattern 5 is more frequently observed than pattern 1. The 3D DGEs they use are all designed by the teachers themselves, and in pattern 5, they choose to control them throughout their lessons, providing perceptual support only when students struggle with theoretical deductive reasoning. Both Huang and Xia think that in this way, the lessons can go "more efficiently". Furthermore, the corresponding tasks are all short-essay tasks in exams requiring students to rigorously demonstrate their reasoning. Huang and Xia express concern for the exam requirements:

Huang: They won't have any aid except for the paper, pens and drawings in the exam. If I present them the dynamic models at very first, they still cannot manage it when back to paper.

Therefore, we identify here institutional influence (exam requirements for particular 3D geometry tasks) and pedagogical influence (didactical tradition of highlighting efficiency in the classroom).

Notably, Dora and Sonia mention that the availability of well-prepared 3D DGEs leads them to allow students exploring tasks in autonomy in a computer room. In Dora's lesson in an ordinary classroom, only pattern 5 is observed: Dora controls 3D DGEs and guides students' work on paper. Here the material factor (3D DGE availability) emerges as another influence on teachers' coordination patterns.

The other three patterns either focus on fostering students' empirical perception (patterns 3 and 4) or combine students' perceptual activities with incomplete theoretical deductive reasoning (pattern 2). These patterns are unevenly spread over the four teachers, among which Huang's case is very specific: although pattern 3 is not observed, he is the only teacher mobilizing pattern 4 and he shows preference for pattern 2 much more than the other three teachers. The geometry tasks corresponding to the two patterns are small exam tasks, i.e., multiple-choice or fill-in-blank tasks that only require a final answer. In pattern 4, Huang keeps students at the level of rational conjecture derived from empirical perception and inductive reasoning, which is enough to correctly answer the corresponding task. When evoking this pattern, Huang attaches importance to developing students' visualization in space and the ability of formulating rational conjectures, and expresses his concern for exam requirements:

Huang: Students should not spend too much time on such a task, it takes up a small portion of the total marks of the exam. If they have a good vision in space and can "guess" the correct answer on that basis, I will say OK, it's enough.

Similarly, pattern 2 is identified in Huang's case, when he explains small exam tasks that involve complex 3D figural structures. Huang encourages students to take perceptual evidence in 3D DGEs as a fact to replace a part of theoretical deductive reasoning, because he believes that the empirical perceptual support in 3D DGEs could allow them to "accumulate images that can be remobilized back to the paper" and to "improve their task-solving efficiency." Here, we identify institutional (exam requirements for particular 3D geometry tasks) and evaluative (Huang's value in 3D geometry teaching and views on advantages of 3D DGEs) influences on coordination patterns.

Pattern 2 is also identified in Dora and Sonia, but their justification is more influenced by temporal factors and the content of the tasks at stake. They always value students' theoretical activities but compromise with perceptual support from 3D DGEs due to complex task figures and time constraints:

Dora: The students have a good vision, but they did not dare to go. It's my role to reassure them that it's the right direction… if I don't help them out, they would be stuck and not learn much from this lesson…if we had three hours, I would leave them to try by themselves.

Pattern 3 is identified in both French teachers and Xia. For these teachers, students only need to "understand" but not "prove" a geometric axiom or rule, that is why they focus on fostering students' empirical perception when helping students to understand a rule. As can be seen, the three teachers have similar personal views on advantages of 3D DGEs in illustrating geometric axioms or rules. From that we identify evaluative influences on their coordination patterns.

DISCUSSION AND CONCLUSION

To conclude, the teachers, while using 3D DGEs, show different coordination patterns and they refer to different factors to justify them. Firstly, pedagogical (e.g., didactical tradition in a country) and evaluative factors (e.g., teachers' views on the educational value of 3D DGEs) have significant influence on the four teachers' behavior. The evaluative factors differ among individual teachers, and we observe more differences within groups than between groups of teachers. Secondly, the two

French teachers refer more to material and temporal factors in their justifications, whereas the two Chinese teachers refer more to institutional ones, especially exam requirements. Finally, the content of 3D geometry tasks is a new factor emerging from data. This study can be related to Herbst & Chazan's (2011) study that confronts teachers with others' teaching episodes in which some norms are breached to elicit teachers' practical rationality, i.e., their dispositions towards norms of customary teaching. Here, we adopt an alternative technique – confronting teachers with their own teaching episodes – to elicit their practical rationality. Furthermore, this study attests the interest of a cross-cultural approach for deepening our understanding of what constitute a "rational" teaching practice (coordination pattern) and what factors lead teachers to that pattern. The diverse rational coordination patterns form a repertoire of practice resources that teachers can mobilize according to their didactic objectives. The intricate web of factors that have different influences on different group of teachers are supposed to help teacher educators to design more targeted training programs.

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WHAT MAKES A SHAPE 2D OR 3D? – USE OF TEACHING AND LEARNING RESOURCES IN GEOMETRY YEAR 5/6 CLASSROOM

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Two-dimensional (2D) and three-dimensional (3D) shapes are crucial geometric concepts that children work with during their primary school years. Building an understanding of 2D and 3D shapes requires them to learn about the mathematical construct of dimension. Dimension is often described as length, breadth, and height for measurement purposes, and not necessarily as a geometric construct that allows children to see shapes as a plane or solid. This paper draws attention to how Year 5/6 children (9 to 11-year-old) use their language and gestures to communicate their understanding of dimension as they engage with physical and digital teaching and learning resources used during geometry lessons. Audiovisual data of six lessons was collected from a New Zealand primary classroom. The findings suggest physical teaching and learning resources such as play-dough and sticks with adhesive may lead to quality differences in classroom interactions where some resources may allow opportunities for children to use their language and gestures to express their thinking about dimension and another set limit children's responses to one or two words.

INTRODUCTION AND BACKGROUND

In primary school geometry, two-dimensional (2D) and three-dimensional (3D) shapes are often described as plane shapes with two dimensions (that are length and breadth) and solid shapes with three dimensions (length, breadth, and width), respectively. These definitions may suggest dimension as a measurement concept where characteristics of an object such as length, breadth, and/or width are measured for 2D and 3D shapes. Or, typologically speaking these definitions may also suggest 2D shapes as surfaces such as sphere region, circular region, plane, polygonal shapes etc., and 3D shapes as solid objects like spherical region, cylindrical region, etc., (Manin, 2006; Ural, 2014). Both of these understandings of 2D and 3D shapes may contribute to children's understanding of dimensions. Yet, this mathematical construct of dimension is rarely explored in mathematics education with few exceptional studies (e.g., Morgan, 2005; Panorkou, 2011). Limited research exploring children's conception of dimension informs us that children construe dimension in many ways. Lehrer et al. (1998), after a 3-year longitudinal study with primary school children, argued that children view dimension as a malleable property of shapes or objects where a shape could be morphed into 3D or 2D shapes by "pulling or pushing" (p. 142). In a phenomenographic study, Panorkou (2011) explored the experiences of twelve 10-year-old children about dimension using three digital tools (which are Elica applications, Flatland the film, and Google SketchUp) and found that children construed dimension in different ways as (i) a material property of an object as a thick or thin, (ii) as a vector expressing ideas of position, direction and orientation, and (iii) as capacity, where lower dimension objects can be contained in objects with higher dimensions (e.g., a cube contains a square). Interestingly, Morgan (2005) also identified the multi-faceted description of dimension in her study of Year 5 classroom discussion (see extract presented in Appendix in Barwell, 2005) where children and the teacher seem to describe dimension in a moment-specific manner with multiple layers of meanings attached to it. Barwell (2005) added to Morgan's analysis of the discussion by suggesting that the teacher introduced the ambiguity of stating a physical/material resource such as plastic shapes as 2D shapes by mentioning "coz they look like three dimensional don't they. They're thick but they're not meant to be, they're meant to be two dimensional" (Barwell, 2005, p. 123). Barwell argued that ambiguities introduced in classroom discussions such as one presented in Morgan's study (2005) provide opportunities for children to use their language to do mathematics. Yet the affordances of the teaching and learning resources in this case- the physical manipulative – plastic shape to become the object of mathematical discussion were not discussed.

A variety of teaching and learning resources are used in geometry classrooms. These resources include physical manipulatives, digital tools such as videos, virtual manipulatives, and worksheets. The use of physical manipulatives in geometry education is not new and is often regarded as part of effective teaching practice. However, research suggests the use of manipulatives in a classroom is not effective in itself, rather there are instructional characteristics such as teacher guidance, length of instructional time, and the perceptual richness of the manipulative (Carbonneau et al., 2013; Sarama & Clements, 2016). In their meta-analysis of 55 studies, Carbonneau et al. (2013) found that higher levels of instructional guidance while using manipulatives are associated with higher levels of student learning. Daher (2014) published a semiotic analysis of the group work of four Grade 5 children in Palestine. The group was exploring how many triangles could be built using numbered sticks of varying lengths. In his analysis, he explored children's interactions and "how these interactions were reflected in their actions, productions, and communications" (p. 424) as worked on finding triangles as a group. He argued that the complexity of interactions was found in instances where children reflected on their actions and productions while working with manipulatives, thus leading children to develop a coherent understanding of the geometric relationship of sides of a triangle. Sarama and Clements (2016) also argued that to build an understanding of mathematical ideas, children "must *reflect on and talk about their actions* with manipulatives to do so" (p. 75). In a recent study, Arvanitaki and Zaranis (2020) conducted a mixed method study to investigate the impact of Information and communications technology (ICT) on geometry learning of nets in Grade 4 class. They designed activities incorporating ICT and Augmented reality (AR). They found that ICT and AR activities had positive impact on students' learning because they provided provided interactive opportunities. Moyer-Packenham and Bolyard (2016) too suggested that the interactive features of the virtual tools help users to explore the concepts beyond static representations.

In this paper, I explore how Year 5/6 children use their language and gestures to express their understanding of dimensions as they work with different teaching and learning resources using the discursive approach informed by ethnomethodology and Bakhtin's dialogic theory (1986). The discursive approach (including both ethnomethodology and Bakhtin's dialogic theory) takes language as the primary mode of social activity and takes interactions as discursive practices where the focus is on the social actions performed by participants' utterances instead of just the content (Edwards & Potter, 2005). These interactions take place in a dialogic space that is shared by all participants to generate meanings through engaging in dialogue (Bakhtin, 1986) The dialogic space implies dynamic space, where all possible meanings are taken into account in a continuum of meaning. The dialogic space allows for a dialogic gap where meaning is realized in the process of active, responsive understanding –an understanding that accounts for the ways one looks for meanings in everyday life.

Thus, the dialogic space allows the possibility of multiple meanings by reducing but not resolving differences in diverse speakers' perspectives. The aim of dialogic understanding is not to reach one true meaning but to appreciate the diversity of multiple meanings (Wegerif, 2013). The ethnomethodological focus in this study allows detailed description of how children make sense of dimension as a mathematical construct as it unfolds during classroom interactions in everyday setting. The analysis in this paper focuses on the diversity of meanings that are given about dimensions through their utterances and gestures as children engage with different teaching and learning resources during geometry lessons.

CONTEXT OF STUDY AND DATA

The study took place in a Year 5/6 class in an English-medium state-run school in New Zealand with a multilingual student population. 15 children (9 multilingual: 1 Somali, 2 Tongan, 4 Māori, 1 Chinese, and 1 Filipino) and a teacher with 7 years of teaching experience participated in the study. Informed and voluntary consent to participate in the study was sought from the participants following ethical approval from the University of Waikato Division of Education Ethics Committee. Audio-visual data from six lessons on shapes and their properties were gathered using two uni-directional cameras and five audio recorders. Fieldnotes, semi-structured interviews with the teacher, semi-structured focus group interviews with children, and relevant documents including New Zealand Curriculum and resources, teacher's unit plan, and students' work samples were also collected. For in-depth analysis of the classroom interactions, some Conversation Analysis (CA) techniques were used (details of analysis are provided in Sharma, 2023). Children's and teacher's utterances were considered as the unit of analysis. The in-depth analysis using CA techniques with Bakhtian dialogic theory allowed explication of intended taken-as-shared meanings about dimensions during classroom interactions.

ANALYSIS AND FINDINGS

Thematic analysis of the audiovisual data, field notes, and semi-structured group interviews with children revealed that children may construct their understanding of dimension in several ways. Some of these constructions are visible in the following children's utterances:

a) D is dimension. Like they are at different place. (Zara, Focus Group Interview 4)

b) [for 3D] Side to side, in and out. (Ethan, Fieldnote, Lesson 4)

c) 3D is three ways to go and 2D is two ways to go. (Matiu, Focus Group Interview 2)

d) Dis dimension. 2D is flat and 3D is fat. 3D has a lot of stuff. Like a 3D has some stuff in it. 2D is like flat and it has nothing. It's like his, his body was like he just, it's like squished over from the car. (Ozan, Focus Group Interview 1)

The first utterance (a) is an example of children's construction of dimension as as "another world". Utterances (b) and (c) show dimension as "different ways to go"; and the last utterance (d) talks about dimension in terms of "flat or fat". In-depth analysis of classroom interactions exploring how children interactionally construct shapes as 2D or 3D differently in different moments is presented elsewhere (see Sharma, 2023). The focus of analysis in this paper is on exploring language and gestures used in interactions where different teaching and learning resources are used for building children's understanding of dimension as property of 2D and 3D shapes. Table 1 provides a summary of teaching and learning resources used during two of the six geometry lessons, that focused specifically on dimensions. The table provides details of activities and resources. I, here, present three interactions from Year 5/6 class where children talked about whether the shape would be 2D or 3D and tried to give their explanations during whole class or group interactions. The first two interactions are from lesson 2 and the third interaction is from lesson 3.

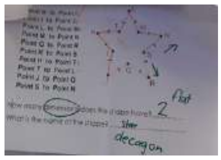
Lesson	Activity	Resources/ Materials	
2	Making shapes with play dough or sticks and adhesive they already know	 Figure 10	 Figure 11
3	Watch YouTube video, Join the dots activity	YouTube video: Math Antics video on Points, Lines, and Plane	 Figure 12

Table 2: Details of lessons, activities, and resources used in geometry lessons

Interaction 1/ Lesson 2

The first interaction is taken from the lesson 2. During this activity, children were asked to make shapes that they already knew using play dough. Figure 1 shows the shapes that Matiu made using play-dough. In terms of language use, it appears Matiu tried to initiate a conversation about the possibility that it is not possible to make a shape flat using play-dough. .

#	Speaker	Text
12	Matiu	I am gonna..trying to. It's like they are all just
13		three d you can't like. make. not make a fat
15	Garry	yes you can

During this interaction, Matiu was working with a physical material – play-dough when he suggested that it was not possible to make a flat shape using playdough because all the shapes that he would make using playdough would be 3D. It seems Matiu was reflecting on his actions and properties of the physical material, in this case, the play-dough, that does not easily lend itself to making 2D shapes.

Interaction 2/ Lesson 2

This interaction is also from the same lesson, lesson 2. During this interaction, children were engaged in whole class discussion. This interaction took place when the teacher asked a child, Elie, to describe the hexagonal shape she had made using sticks and an adhesive (see Figure 2).

#	Speaker	Text
341	Teacher	anyone else got some right. Um.. Elie with your sticks
343	Elie	um… I forgot what this shapes called
344	Teacher	very good. So..how many ..so... so umm describe it

346 Elie its got one two its got one two three four five six.. its got six..corners

349 Teacher got six

350 Elie and its got

351 Teacher Elie just hang on a minute.. is it three d or two d

353 Elie umm I think its three d because its not. a two d (she was holding the shape and rolling it around her finger))

355 Teacher put it down on a on the ground is it. flat or fat

357 Elie its fat

358 Teacher its fat. is it coming out towards you

359 Elie (looks at the shape holding it near her eye level)

360 Teacher okay lay it on the ground

361 Kimi no its flat

362 Teacher its its okay. so its not actually coming out of the ground or going through the ground..so we call so we call..we call that a two d. okay. So

366 Elie uhm

The teacher asked Elie (line 341) to describe the shape that she had made using sticks and adhesives. Elie used "um" as a hedging device, paused, and stated that she did not remember the name of the shape (line 343). The teacher then (line 344) asked Elie to describe the shape that Elie had made. Elie counted to six and stated that the shape had six corners. The teacher (line 344) then asked Elie if the shape was 2D or 3D (line 351). Elie responded that the shape that she had made was 3D (lines 353-354). At this moment, Elie also used her gestures of holding the shape and spinning it around her fingers. This action of spinning the hexagonal shape is of interest here. It seems that Elie used her action to indicate that the shape was 3D instead of 2D as she could hold it and spin it. As the conversation proceeded, the teacher asked her if the shape was fat or flat (lines 355, 358). In line 359, Elie again used her gesture of bringing the same to her eye level to perhaps indicate that the shape is 3D as she could see its thickness. Again, it is Elie's action here that seems to convey Elie's argument that the shape is indeed 3D and not 2D as she could see the shape. It seems that the teacher noticed Elie's gestures and; therefore, attempted (line 362) to develop a mutual understanding with Elie by using a persuasive way of speaking by using the phrase "so we call" phrase (line 364) twice in her utterance. At this point, Elie responds with "uhm", which could be interpreted as Elie's way of withdrawing from the interaction instead of agreeing with the teacher. During this interaction, Elie's actions of using her gestures of spinning the shape around her fingers and bringing the shape to her eye level appears to show the affordances that physical material allows in case where the teacher and the child may not have an agreement. The affordances of the hexagonal shape allowed Elie to show her thinking when the teacher argued that the shape is flat via gestures. Moreover, the use of language here focused on providing justification for Elie's thinking where Elie used flat pitch that English speakers use to indicate confidence (Ward, 2019).

Interaction 3/ Lesson 3

During lesson 3, children watched a YouTube video – Math Antics– Points, Lines, & Planes. The teacher used this video resource to support children's learning of D in 2D and 3D shapes. The teacher consciously highlighted the key points about what a plane is and what makes a shape 2D and 3D with children while showing the video. This following interaction takes place as a whole class interaction. The teacher asked the children to take turns drawing lines between the points and then answer the two questions about the shape that they see in the worksheet (see Figure 4).

#	Speaker	Text
585	Teacher	how many dimensions does this shape have
586	Zara	(puts her hand up for answering the question)
587	Teacher	Zara
588	Zara	ten?
589	Teacher	no. Ethan
590	Ethan	nine
591	Matiu	(laughing)
592	Teacher	no..think about what I am asking. I am saying how many dimensions. does this star have..Nikau
595	Nikau	I dont know like
596	Elie	o my god (rolled eyes)
597	Ethan	Oh (loud voice)
598	Teacher	(circling the word ‘dimension on task sheet)
599	Ethan	Two d (loud voice)
600	Teacher	how many dimensions in the space does this star have
602	Zara	Matiu whats dimension
603	Teacher	Elie (as Elie raised her hand to answer)
604	Elie	five
605	Olivia	ay?
606	Teacher	no..Alyssa
607	Alyssa	ten
608	Teacher	no. think about what the word we talking. with dimension..
610	Zara	dimension
611	Teacher	do we go upto ten dimensions WE WOULD (loud voice) be on the movies if we went into ten dimensions..would be on the sci.fi movie.Matiu
617	Matiu	two
618	Teacher	thank you very much (exhalation of breath, closed her eyes and slightly tilted her head back)
619	Ethan	I said that (loud voice)

After completing the join the dots activity, the teacher asked “how many dimensions does the shape [star] has” (line 585). She selected Zara as the next speaker (line 587). Zara responded “Ten”. As the conversation proceeded, the children stated that the star shape had nine, five, and ten. The teacher at these times repeatedly emphasized the word “dimensions” to draw the children’s attention to what they had just watched in the video (YouTube video) before this task. It seems that Ethan had realized the correct dimensions of the star shape as said “two d” in a loud voice (line 599). However, Ethan’s response was not considered in the classroom interaction, as he was not selected by the teacher as the next speaker. The teacher explained why the answer could not be ten and selected Matiu as the next speaker (lines 611-615). Matiu responded with the correct answer (line 617). The teacher, in her following utterance (line 618), thanked Matiu for the correct answer. Moreover, the teacher’s gestures informed us that she was relieved to get the correct response, as she slightly tilted her head back. In this interaction, the focus was on providing correct number of dimensions for the shape, which limited children’s responses to one to two words. The use of worksheet as a teaching and learning resource did not provide children with opportunities to use their language or gestures extensively. Moreover, though children had watched the YouTube video, it seems not many children could connect the concepts from the video to the worksheet.

The analysis of the three interactions reveal two findings. First, physical materials lend opportunities for children to act and reflect on their actions to build understanding of dimensions. On the other hand, video resources (such as YouTube) may support visualisation yet may not support building understanding of concepts due of lack of interactive opportunities. Second, the children’ use of language and gestures in interactions where they engage with the materials (Interaction 1 and 2) is different from their language and gestures in interactions where they merely watch a video showing different representations (interaction 3).

DISCUSSION AND CONCLUSION

In this paper, I explored the language and gestures that Year 5/6 children use to express their understanding of dimension as they work with different teaching and learning resources. Thematic analysis revealed that children construe dimensions as “flat or fat” or as “different ways to go”. This finding is re consistent with findings from Leher et al. (1998), Panorkou (2011) and Morgan (2005). Regarding the use of teaching and learning resources, it is evident that physical materials (such as play dough and sticks with adhesive) allow children to interact, act and reflect on their work with materials. Daher (2016) and Sarama and Clements (2016) too suggested that children reflect on their actions as they work with objects that could be manipulated. Lack of opportunities for interacting with the resource may not be useful when developing mathematical understanding as evident in interaction 3. Arvanitaki and Zaranis (2020) too has suggested that it is the interactivity with the materials that support children’s learning. Qualitative comparison of the use of language and gestures in Interactions 2 and 3 are of interest. In interaction 2, Elie used arguments in her language and used her gestures to support those arguments. However, in interaction 3, children’s language is limited to one or two words and they only raised their hands to get the turn to speak. From dialogic theory perspective, in the interaction 2, Elie seemed to make use of gestures to convey her argument. This was possible because of the affordance of the physical material with the possibility of spinning it or bringing it to her eye level. Contrastingly, the resources used in interaction 3 was the worksheet and the video showed to them. Both of these resources did not allow children to play or interact with the shape individually which could have resulted in limited understanding of dimension. It is possible that although children could see different representations of planes and shapes in the video, the lack of opportunities to interactively engage with the shapes did not allow students to gain that intended learning (Moyer-Packenham & Bolyard, 2016; Sarama & Clements, 2016). It is not to say that physical materials are more useful for all geometric concepts as we have seen in interaction 2 teacher and Elie both construed the hexagonal shape differently in terms of its dimensions. It is the dialogue between the teacher and the child that contributes to the meaning making process that could be facilitated due to a physical material. Therefore, it could be argued that different teaching and learning resources could lead to difference in the quality of classroom interactions.

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CHALLENGES OF INTEGRATING DGS WITH GEOMETRY EDUCATION

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Mathematics teachers in Iran, have many challenges in integrating DGS with their geometry teaching. To overcome these challenges, we conducted research taking qualitative approach to identify the major constraints that mathematics teachers have in using technology in geometry education. The informants were 18 volunteer mathematics teachers from different cities in Iran and the data collected virtually by a semi-structed interview. All recorded interviews were transcribed and the MAXQDA was used for analysis of the data. The findings consisted of four themes as technological infrastructure, policies, mathematics curriculum, and teachers' resistance. The study exposed that teachers developed innovative strategies to reduce these constraints to facilitate the integration of DGS with geometry education. The study concludes that even in highly centralized education system, teachers are main player in promoting technology in mathematics teaching and thus, their professional development is a necessary enterprise.

INTRODUCTION

Iranian Education system is centralized in which, there is a prescribed curriculum and a single textbook for every school subject at the national level. The tolerance of the highly centralized system to the diversity from different perspectives including culture, language, economic situation and even climate, is limited and instead, the policymakers have more focus on general educational demands (Karimi et al., 2023). However, Gholamazad (2020) criticized this radical look on the degree of centralization prior to that and gave an evidence-based explanation that the highly centralized education system in Iran has undergone unavoidable changes at a slow but overwhelming pace, including technology as becoming one of the infrastructures of educational activities. This forecast was in line with Sinclair et al (2016) report that regarding geometry education, technology has become a global trend. Along with this trend, using "Dynamic Geometry Software" (DGS) for teaching geometry, gained more attention. However, at the end of 2020, the outbreak of Covid-19 pandemic caused the closure of schools around the globe and unexpectedly, all education systems had to switch from physical-face to face- to virtual education and rapidly, technology became the main platform for teaching and learning all school subjects including geometry and the use of DGS. During that era, Iran was not the exception and technology became integrated with all aspects of education rather rapidly and with no well-thought plans as well. Nevertheless, the new situation created a unique opportunity for some young mathematics teachers to use digital technology and DGS for their teaching. We designed a multi-phase study to investigate the role of DGS on geometrical proof and reasoning of senior high school students in Iran. At the first phase, we focused on the teachers' views about using DGS in teaching geometry and found out whether that practice became a legacy to last or just a temporal action during virtual education. So, the first phase of the study focused on teachers' views about the obstacles of using DGS in geometry teaching and this paper is reporting on the preliminary findings.

LITERATURE REVIEW

Straesser (2002) specified that digital tools allow students to construct geometric figures and see relationships that are not limited by the small steps of conventional paper and pencil drawing techniques and so, expand the scope of possible geometric constructions and solutions. On another study, Reyhani et al. (2009) investigated the effect of DGS on improving high school students' reasoning in geometry and found that using DGS enabled them to enhance their problem-solving skills considering geometry. In a project led by Lavicza et al. (2010) in England, a professional development network was created with nine teachers who were enthusiastic in using GeoGebra for their mathematics teaching. They formed a core group to support mathematics teachers via professional development, as well as exploring the ways in which, GeoGebra could be aligned with England national curriculum. The project also involved developing and collecting resources for mathematics teaching and learning with GeoGebra and establishing a local searchable GeoGebra site and an online support structure. However, Stols and Kriek (2011) found that teachers' general technology proficiency is a prerequisite for using DGS effectively in classroom. With the development of digital technology, many mathematics education researchers became interested in attesting the usefulness of DGS in mathematics teaching and learning. Ruthven (2014) alleged that relevant knowledge and expertise of mathematics teachers is the most critical factor for the effective incorporation of digital technologies into the regular teaching practice. To enhance teachers' knowledge, he introduced a framework with five components of "working environment, resource system, activity structure, curriculum script, and time economy of classroom" to support mathematics teachers for integrating digital technologies into their mathematics teaching. On the other hand, Chan and Leung (2014) systematic review of quasi-experimental studies showed that DGS-based teaching, improved students' mathematical achievements with a large and positive effect size.

Later, Sinclair et al. (2016) reported that the impact of "Dynamic Geometry Environments" (DGEs) on geometry education from elementary to tertiary level and its various aspects including proof, definition, and visualization, has been widely investigated. However, they stressed that the nature of communication between geometric objects and concepts in this environment is different from the traditional paper-and-pencil methods and it is important to make necessary modifications when using DGS. Aside, Ottenbreit-Leftwich et al. (2018) performed a study with four mathematics teachers who had received extensive training to use technology for teaching. After completing the training, they had various field experiences as well. In all teaching sessions, these teachers demonstrated high levels of self-efficacy, knowledge, beliefs, and intentions to use technology. However, they encountered some external barriers that sometimes, they could not use technology in their real classroom settings. In a recent study, Hillmayr et al. (2020) employed a meta-analysis approach and examined 92 primary studies that investigated the effects of using digital tools for teaching and learning science and mathematics at secondary school. Their analysis showed the significant positive impact of digital tools on students' attitude and learning outcomes. Furthermore, they concluded that one of the key factors for the successful implementation of digital tools in mathematics and science classes is teacher training in advance. They also suggested that digital tools are more beneficial when are used as complement rather than replacement of other instructional methods. Clark-Wilson et al. (2020) reviewed the current state of research on teaching mathematics with technology at the secondary level. They found that the predominance of studies was on students' learning and barely on teaching. They also highlighted the need for more research on teacher professional development that can support them to use technology in their classrooms. According to Højsted (2020) looked at the historical development of research regarding DGEs, and he classified his findings into three periods; first, the main emphasis was on investigating students/learners' cognitive processes to better understand how they learn geometry in DGEs. Second, the research attention shifted to how to design effective tasks that align with learning objectives and affordance of DGEs. In the third period, research focus has moved to how teachers can integrate DGEs into their pedagogical practices and what kind of professional development is needed to prepare them to teach geometry using DGEs. Højsted (2020) concluded that by the rapid development of DGEs and their accessibility, it is necessary to consider learner-task-teacher and their interplay as three main dimensions of research plans and methods in this field. Ruthven (2014) points to the several factors such as poor resourcing of schools, limited recognition of DGS in school mathematics curricula, and exclusion of digital tools in examinations, have hindered the implementation of these tools in mathematics classes. His findings were in line with Clark-Wilson et al. (2020) results as insufficient time and supportive professional development, limited access to appropriate digital technology, and poor technical support as constraints for teachers in using technology in class. In this line, Pittalis (2020) developed and validated a comprehensive model that could assess secondary school teachers' intention to use DGS in geometry teaching that depends on their evaluation of how well the software aligns with their pedagogical and learning goals. However, teachers expressed that sometimes the external barriers such as structures, policies, culture, and resources of their schools, were quite challenging for them. Further, Thurm & Barzel (2022) stressed the importance of continues projects that their purpose is to provide professional development for teachers to become able to incorporate technology into their teaching and concluded that this is an evolutionary rather than a revolutionary process where teacher change occurs in incremental steps. They also pointed out that teachers' beliefs toward their own self-efficacy, potential benefits of technology, and extra time requirement, have visible consequence on using digital technology for teaching mathematics. Finally, Benning et al. (2023) conducted a study in Ghana to investigate that how a professional development program using GeoGebra, might change the dispositions of mathematics teachers concerning technology. The results proved that at the end of the program, teachers' ability to use technology in their mathematics teaching improved significantly. They also expressed that three contextual factors comprising administrative support, continuous professional training, and availability of adequate technology resources are involving teachers' use of digital technology.

METHOD

The purpose of the present study was to identify the major obstacles for integrating DGS with geometry teaching in Iran. Thus, the research guided by this research question and according to its explanatory nature, a qualitative approach was employed. The researchers formulated several questions according to the research findings that presented in the literature review. After piloting the questions, the cover letter for interview sent to several mathematics teachers' groups that are formed in social media including WhatsApp and Telegram and invited those who are interested in using DGS to voluntarily participate in a semi-structured and online interview. At last, 18 senior high school teachers were interviewed. The informants' diversity in terms of gender, geographic distribution, years of teaching experience and economic status of schools that they were teaching. The interviewer indicated that for the sake of confidentiality, the name and other personal information of participants is not part of interview protocol. All interviews were recorded by permission and transcribed, and for the data analysis, MAXQDA software was used.

RESULTS

The data analysis revealed four main themes related to the challenges of using DGS consisted of “technological infrastructures”, “policies”, “mathematics curriculum”, and “teachers’ resistance”. Based on these themes, teachers’ solutions to overcome these challenges, three main themes of “in-classroom strategies”, “out-classroom strategies”, and “promotional activities” were found as well.

CONSTRAINTS OF USING DGS IN GEOMETRY EDUCATION

Mathematics teachers are faced with various challenges in using DGS in geometry education that are grouped into four main themes as "technological infrastructures", "policies", "mathematics curriculum", and "teachers' resistance" (Figure 1.)

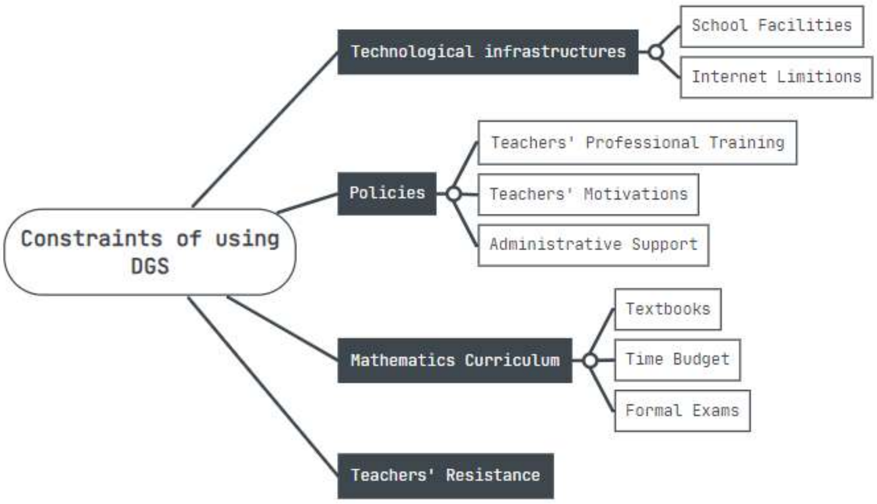

Figure 13: Constraints of using DGS from mathematics teachers’ perspective.

Technological infrastructures

For teachers, lack of adequate technological infrastructure in most schools, is a major constraint in using the DGS in geometry classes. Although in their views, schools’ facilities ranged from technological readiness with a smart board in every classroom and a computer lab in some private schools and special schools for gifted students, to no technological facilities in the classrooms or a single computer and video projector either in every class or entire school. Extremely low internet speed in schools, limits teachers to use online software and tools as well and makes it difficult to share supplementary videos with their students through virtual platforms. Another even more serious challenge is the internet restrictions imposed by external sanctions or internal filtering in Iran that make many online platforms and websites, inaccessible for Iranian users, like some GeoGebra applets and Desmos do not function properly due to filtering is even worse.

Policies

Teachers expressed that in highly centralized educational system in Iran, the Ministry of Education has the authority to form new policies for supporting teachers to use digital technologies. But in contrast, they pointed out that institutionalized procedures and the lack of attention to the preparation of teachers for using DGS is evident. In their view, "many teachers haven't even heard about different software". In this situation, "it is not surprising that they don't feel the need to learn and use DGS". In addition, "some teachers haven't even heard GeoGebra", though "usually, the local education offices offer workshops or in-service courses to introduce GeoGebra to teachers", but "they are not effective and widespread". Most teachers believed that "if mathematics teachers were introduced to DGS and GeoGebra and such, they would definitely use it". Nonetheless, teachers stated that the Ministry of Education is responsible for professional development of teachers and should take this matter seriously by making implementable policies with sufficient monitoring and support. Otherwise, only offering online courses without adequate quality is not responsive to this great need. Some participants tried to promote DGS among other teachers by directing workshops to demonstrate its features. The feedback indicated that "they often had a positive initial impression", but they discontinued its use and did not integrate it into their teaching that might implies that for using DGS, the policy makers should invest on long-term training not only short-term outcomes.

Mathematics curriculum

Mathematics textbooks are a vital component of school mathematics in Iran and in fact, are curriculum itself and are the main source of teaching. However, teachers argued that "the structure of current mathematics textbooks is not acceptable, and there is rarely any demand for integrating geometry teaching with DGS". In addition, they reasoned that "mathematics textbooks are outdated and don't include technology-based activities and mathematical applications". Teachers also explicated that "time allocated" to geometry is not sufficient to incorporate DGS to its teaching for two reasons; the organization of content that is not based on technology and require them to design their own teaching material suitable for this integration and second, "outdated equipment" and "lack of technical support", consume more time and put more constrains on using DGS[21]. However, for few teachers, "DGS saved time", because "it facilitated the drawing of figures", "attracted students' interests", and "avoided unnecessary repetitions in explaining some topics". Another important constrain for integrating DGS with teaching geometry was national "university entrance examination" that is "multiple choice", and students are not allowed to use technology. Thus, "it is not useful for students to spend their time on technology. They need to do more drills and practice" and even "some families don't want to wase time on technology and rather spend more time on practicing". They also "expected to cover the entire textbook for final exams, and their performance is evaluated by the exam results". So, "using DGS is not a high priority" for many of them.

Teachers' resistance

Another theme that emerged from the analysis of the data was "teachers' resistance" to learn or adopt DGS in their teaching. For some teachers, using technology is still conceived as "cheating" and a source of "mental laziness."

TEACHERS' STRATEGIES TO OVERCOME CONSTRAINS

Several active teachers with more positive views towards integrating DGS with geometry teaching, developed some strategies to facilitate this process without expecting any kind of top-down reforms

[21] Private schools and schools for gifted, do not follow this policy and usually allocate more time to various subjects. As well, they have modern technology labs with specialized staff.

and shared them with other teachers via their social groups. Their initiatives are grouped into three themes (Figure 2.)

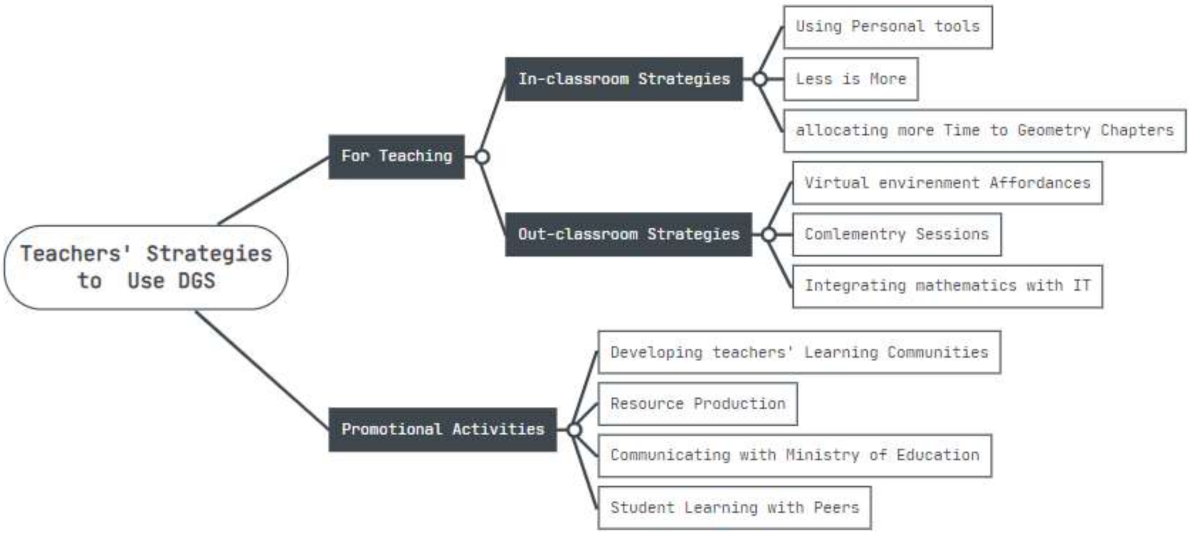

Figure 14: Teachers' strategies to overcome challenges of using DGS in Iran

In-classroom strategies

Teachers initiated different strategies to overcome some of the constrains that introduced above. For instance, challenges of inadequate technological resources in schools, many of them used their own personal devices for teaching as they did during the covid-19 pandemic. In one surprising occasion, one teacher in a deprived school, used his laptop to demonstrate a geometry activity in a school that did not have video projector. In his words, "I turned my laptop screen towards 30 students in class and they crowded around to see the activity". In terms of limited time allocation, some teachers adopt a "less is more" strategy as one of them explained, "instead of covering all the textbook exercises, I did some of them thoroughly and used GeoGebra to demonstrate their different aspects". She then said with confidence that "they usually could do the rest by themselves". Other strategies adopted by teachers included "talk less-do more", "avoid doing repetitive tasks", "give less exams" and instead, "spend more time using DGS". At junior high school, geometry is part of mathematics textbook and some teachers "devoted more time to geometry" comparing to other chapters to save time for using DGS. Finally, a promising strategy for overcoming teachers' resistance was to provide effective professional development programs and offer more support to mathematics teachers.

Out-classroom strategies

To deal with teachers' resistance, those who were enthusiastic to teach geometry using DGS, posted gifs or video clips created by DGS and were ready to use, to encourage and support other teachers to become knowledgeable and confident to utilize technology in their geometry teaching. Further, many willing teachers initiated virtual environments for their classes and posted extra materials such as video clips and gifs for their students. For instance, they used DGS to teach "how to sketch different sides of a solid", and to their experience, "sending several related gifs was sufficient for students to solve such tasks by themselves". As well, in schools that did not have video projector, teachers sent video clips and other teaching material via virtual environments to both overcoming lack of technological equipment and insufficient teaching time. In addition, teachers provided tutorials for students who were instructed to use GeoGebra for solving geometry problems. Some teachers posted interactive activities made by DGS where "students could drag shapes, measure segments and angles, make conjectures, and answer questions". Another teacher described that:

> In my class, I demonstrated the intersection of altitudes of a triangle in different conditions with GeoGebra and then asked students to construct similar dynamic shapes for medians, bisectors, and perpendicular bisectors of a triangle and send them to our virtual group.

In sum, several teachers developed various strategies to face the challenges of integrating technology with geometry education. Even some of them gave supplementary sessions as extra-curricular activity or asked other teachers to give them permission to use their possible spare time. Finally, one of the mathematics teachers asked the school principal to allow her to combine two "Mathematics" and "Information Technology (IT)" subjects to integrate DGS with her teaching.

Promotional activities

Most teachers indicated that their first exposure to DGS happened in mathematics communities including mathematics houses, conferences, and online communications such as various mathematics teachers' virtual groups. They used these platforms to disseminate their successful practices with DGS. Even so, some of them initiated numerous free online workshops for their colleagues to decrease their resistance by enhancing their knowledge and skills in using digital technology. They also shared their ready to use material with others via virtual groups to balance teaching time.

CONCLUSION

The purpose of this study was to identify the main constrains that Iranian mathematics teachers encountered for integrating DGS with teaching geometry. By analysis of the data, the constraints grouped into four themes labeling "technological infrastructure", "policies", "mathematics curriculum" and "teachers' resistance". In addition, this study presented the initiatives of enthusiastic mathematics teachers who were willing to share their successful practices with other teachers and encourage them to adopt DGS with geometry teaching. The study presented examples of how teachers compensate for the lack of school resources with their great effort and beyond school hours by creating virtual environments for their classes as well as for their colleagues to exchange resources. They also used various educational communities such as mathematics houses and conferences, produced ready to use materials for both students and other teachers, and communicated with the Ministry of Education and offered their willingness to collaborate with them to promote technology in school mathematics from different aspects. These initiatives could inspire mathematics teachers around the world to overcome the constraints in using DGS with geometry teaching. The implication of this research could particularly be useful for highly centralized education systems and disadvantaged schools and districts that "where there is a will, there is a way!"

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DEVELOPING ANALYTICAL THINKING IN THE RECOGNITION OF UNUSUAL GEOMETRICAL SHAPES WITH YOUNG PUPILS (FROM 4–8-YEAR-OLD)

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In this research, we present how we conceived a resource for teaching geometrical shapes and their recognition from the age of 4. We show how, through pupils' ways of speaking and acting, in situations, we are able to understand their way of thinking about geometry, in particular by enriching their initial iconic visualization with analytical thinking.

INTRODUCTION

Several research works in geometry education have pointed out a rupture between primary and secondary school: either due to the transition to deductive geometry (Berthelot and Salin, 1992), to a paradigm shift between natural and axiomatic geometry (Houdement & Kuzniak, 2000). Therefore, we postulate that it is possible to act from kindergarten and contribute to reducing this rupture and smoothening the transition. To this end, we have created a specific artefact (Coutat & Vendeira, 2015) and developed teaching-learning situations in geometry around shapes recognition (Coutat and Vendeira, 2015-2018) for pupils aged 4 to 8. In presenting our manipulative and visual tool and the teaching activities we have designed, we want to contribute to the topic C2: "resources for teaching and learning geometry" of ICMI Study 26 Discussion Document. Indeed, our research tackles the issue of manipulation and visualization as a sufficient mean to allow effective learning processes in geometry in early childhood. In particular, we question the role of language and gestures to identify ways of thinking in geometry with pupils aged from 4 to 8. Duval (2005) distinguishes two possible types of visualization when identifying geometric objects and their relationships: the iconic and non-iconic visualization. The first represents a global apprehension of the shape, while the second is more analytical. In the iconic visualization, 2D objects are perceived globally, as pieces of surface of certain shapes arranged in a sort of a catalogue, a bit like in a botanist herbarium (Duval et al., 2004). In a non-iconic visualization, one goes beyond the globally perceived shape to deal more analytically with the properties of the object. Duval underlines the necessity to change the way of looking at figures, from an iconic to a non-iconic visualization, in order to reach a certain level in geometrical expertise. This change though is not a radical immediate change, but rather a progressive "enrichment" of the way of looking at shapes towards an analytical way of looking at geometric figures. We distinguish shapes describable through their characteristics from figures involving properties and requiring a higher level of abstraction. Objectively, it seems to us that talking about "properties" with pupils aged from 4 to 8 is premature. This is why we prefer to use the term "characteristics" meaning more precisely the state of conceptualization of geometric objects at that age. The nuance between characteristics of shapes and properties of figures is based on the vocabulary used, the different ways of looking at geometric objects and the relationships considered to define an object. In our work (Coutat & Vendeira 2017), we talk about characteristics when pupils still need representations of the objects such as diagrams or drawings even if these are partial. This is however no longer necessary for the properties of figures because they exist independently of any representation. At the stage of the characteristics, the pupils have only limited access to the comprehension of geometric shapes: 1) all properties are not considered and 2) these are not necessarily put in relation. Our research began in 2015. We built some equipment made of various shapes cut in wood or synthetic resin with which we have experimented many activities in classes, in the French-speaking part of Switzerland, mainly in Geneva with pupils aged from 4 to 8. Overall, the whole experiment has been very positive. Pupils manage to juggle between the two different types of visualization of geometrical objects, highlighting the enrichment of the way of looking at them. Beyond this enrichment, we also understood that there is a need of a certain flexibility in these changes. Indeed, the visualization must be adapted to the situation: an expert procedure in a certain situation could, for example, no longer be adequate in another, depending on the modifications of the didactic variables (Brousseau, 1982). In this contribution, we propose to first describe the developed equipment and explain our didactic choices to enrich pupils' way of looking at shapes. We then give some elements on our analysis framework. Finally, we pursue by presenting succinctly some examples of concrete classroom analyses before concluding.

THE MATERIAL AND DIDACTIC CHOICES

As in kindergarten we cannot use neither instruments, nor construction tasks we have developed a specific equipment adapted to work around shapes recognition at this level. Our idea was to work on a special collection of shapes, which force to change the way of looking at them and to gradually work around the characteristics of shapes from the age of 4 up to higher levels in primary education. At four already, pupils are quite aware of the characteristic of the number of sides of a shape. Yet, pupils of this age are still learning the concept of number, which therefore remains fragile when it comes to counting sides. In addition, other skills may interfere, such as enumeration (Briand et al., 2004), which further complicates the task. For these different reasons, it is important not to focus exclusively or preferentially on this characteristic. It is also necessary to deal with the presence of straight or curved side, symmetry, parallel opposite sides or even the fact that the shape is convex or not. Correct mathematical terms are, however, not expected. At this early age, it is important to let the identification of such characteristics be made with a personal vocabulary as long as it is understandable and shared by the other pupils. Our pieces are not the usual shapes like squares, rectangles, triangles and circles. We made this choice because of some obstacles we observed in classrooms (Coutat & Vendeira, 2015). Indeed, dealing with usual shapes, the pupils do not refer to their characteristics but only use the name of the shape, which seems to be its only characteristic. Moreover, non-prototypical shapes (such as the rightangled triangle or the rectangle not having lengths twice as long as the widths) are not recognized as belonging to their class (triangle/rectangle), because they are perceptually different from their model. This phenomenon is also confirmed by research in psychology, for example, Gentaz (2013) questions how children during their early development manage to overcome specificities in favor of generality. Based on the above, we developed a collection of unusual shapes that do not have names (Figure 1).

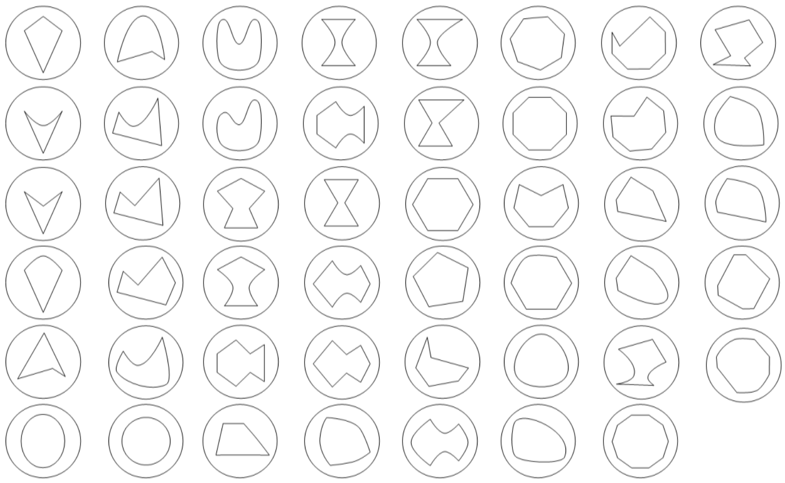


Fig. 1: The developed material consisting of 47 pieces of different unusual and unnamable shapes

The choice of this collection does not prevent the pupils from focusing on the overall aspect of the shapes. A resemblance to a known everyday object is possible, such as a fish, a vase, or a mountain. However, depending on the set of the pieces and their iconic proximity, a treatment by characteristics only will be effective (case of the selection made in fig. 2). In fact, a pupil would not be able to use the resemblance of one of the pieces below to a fish, since they all resemble to a fish. According to the objectives to reach, an adequate selection of shapes by the teachers is therefore essential.

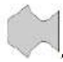
Fig. 2: Example of iconically close shapes

Research in cognitive science (Gentaz et al., 2009) and mathematics education (Duval & Godin, 2005; Duval, 2005), pointed out the importance for young pupils to be able to manipulate objects. This is why we chose to have physical manipulable objects involving haptic exploration rather than images on paper (even if we use these sometimes too) (Coutat & Vendeira, 2019a). Gentaz et al. (2009) showed that the haptic modality enables pupils to represent shapes better than when they are only represented as images on paper. Our equipment (Figure 3) is therefore composed of two interlocking parts (templates and stencils) allowing two distinct types of manipulation (the inside of the stencil and the perimeter of the template). Thus, when touching the pieces, where points are detectable on a template, holes are associated on the corresponding stencil.

Fig. 3: The stencil (hollowed-out part) – The template (full part) – The two-part puzzle

These different configurations give rise to the perception (both visual and haptic) of four distinct "shapes". Those of the stencil (the empty part and the solid part), that of the template and that of the puzzle of the template and the stencil. In order to avoid favoring any orientation, the stencils were cut out in a form of a disc. The embedding of the templates in the stencils provides generally an appropriate and reliable feedback for the pupils (Brousseau, 1997). A priori, if the pupil does not manage to embed her or his template in the stencil, it is that it is not the right one (or possibly that it would have to be rotated (turn the piece in the plane) or returned (in the case of a non-symmetrical shape). Once the material had been produced, an accompanying handbook was developed to support its use. This handbook offers a few explanations of the learning challenges and objectives and present some possible tasks for achieving them. The two main activities are 1) recognizing shapes from a set and embedding the template and stencil for validation, and 2) classifying shapes according to several criteria. We then play with the didactic variables to generate a variety set of tasks (Vendeira, 2019). In the following section we develop the analytical framework, which enabled us to observe pupils in class in order to determine their way of looking at shapes.

ANALYTICAL FRAMEWORK

In order to analyze the classroom sessions, we use the tools from Bernié (2002) such as the ways of acting, speaking and thinking necessary for the notion of a discursive community. We enrich it with the approach of iconic and non-iconic visualizations from Duval (2005). As each of these ways of thinking is associated with distinct ways of speaking and acting, we can use this tool in order to analyze activity in class. When pupils think globally with an iconic visualization, they usually try to name geometric objects based on their resemblance to corresponding familiar objects. On the other hand, a pupil who thinks about objects analytically with a non-iconic visualization, will refer to their characteristics or properties. Depending on the age of the pupils, the geometric vocabulary used is either conventional and formal or more spontaneous and referring to everyday life. For instance, “non-convex figures” can be designated as: “shapes with holes”. However, in both cases it is indeed a property or a characteristic that is referred to. Thus, the difference in language provides information on the pupil's state of knowledge. Below is the summary table of the elements taken into account for our analyses. We take as an example the following pieces of our collection: .

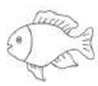
Visualization	Iconic	non-iconic	
		Characteristics	Properties
			
think	Global	Analytical	
talk	Common and spontaneous vocabulary related to the global aspect of the piece	Common and spontaneous vocabulary related to characteristics	Conventional and formal geometric vocabulary
	"It looks like a fish"	“It is a shape with holes and with straight and curved sides etc. »	“parallel lines, an axial symmetry, …”
act	Frequent handling	Precise/focused handling	Almost no manipulation
	Many insets are attempted (all pieces that look like fish). Gestures such as: - put the pieces side by side to compare them; - return the pieces in an orientation that favors comparison; - move around the work table, etc.	Pieces are touched non-randomly. Gestures such as: - make the outline of the piece with the finger - touch a specific part such as a vertex	The pieces are scanned.

Fig. 4: Summary table of the elements taken into account for the analyses

Regarding ways of acting, it is very likely that pupils who think more globally manipulate more than others who think more analytically. Some gestures can be identified with one way of thinking rather than another. For example, a pupil who turns over a piece to orient it in the same direction as another one (to compare them visually) is very likely to mobilize an iconic visualization. On the other hand, a pupil who identifies a specific feature of a part of shape with his finger is more likely in an analytical thought process. Nevertheless, we shall remind that it is not necessarily always the analytical way of thinking that is the most effective, but that it depends on the situation. For instance, if the task given to the pupils is to decide on a way of separating a group of shapes (only the templates without the stencil) into two sub-groups, a pupil can use the properties of convexity (even if he or she talks about shapes with and without holes) to make the two sub-group. However, if in the group all shapes are non-convex, this analytical way of seeing will be totally inefficient and maybe separating the shapes between these which looks like fishes and the others will be very efficient. Therefore, we believe that the two ways of thinking must coexist and that a certain flexibility is necessary in order to adapt to the proposed situations. The class sessions with the developed equipment are thus analyzed according to two levels: the operationalization of the enrichment of ways of thinking and their flexibility according to a given situation. In the following section, we show how we used this analytical framework to analyze pupils' productions.

SOME EXAMPLES OF ANALYSES

In this section, we do not seek to prove the effectiveness of the developed material. Instead, we give examples that highlight, through the ways of speaking, the enrichment of the way of looking at shapes and how a flexible way of looking is developed. The examples are based on data collected over several years with pupils aged between 4 and 8. In this paper, because we focus on ways of talking, which are easier to describe than ways of acting, our examples are with 6–8-year-olds where tasks involving language are mainly proposed. Our data are based on a variety of tasks. In order not to complicate the reader's understanding, we have deliberately chosen to focus our examples only on two of them (classification and recognition of a shape by its written description).

Global or analytical ways of thinking

Here are a few examples of how the pupils we observed think. In a task involving the classification of a group of given shapes into 2 to 4 subgroups, pupils regularly group together figures without being able to justify their choices other than "because they look a bit alike". It's an iconic visualization. We also find more substantiated arguments relating to this same visualization. For example, for figures 1 and 2 below, one pupil justifies his choice by saying "this is the round family". It's the resemblance of the two shapes to a circle that counts.

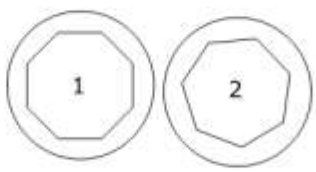


Fig. 5: Iconic visualization "the round family"

In another task when asking to divide a group of six shapes into two subgroups, one pupil justifies putting A, C and F together because they all contain at least one "half-circle". This is also a use of a type of iconic visualization.

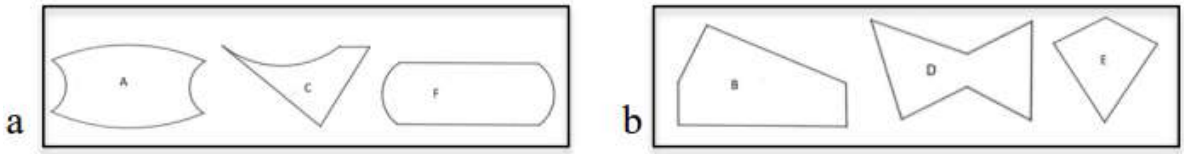


Fig. 6: a Iconic visualization with “half-circle” - b Summary table of the elements taken into account for the analyses

In many cases, however, we see a more analytical classification of these same three shapes. Some pupils refer to waves (for curved sides) or to freehand tracings as opposed to those made with a ruler (fig.6b). In this case, we see a common and spontaneous vocabulary related to the characteristics. Finally, in some rare occasions, we find examples where the vocabulary is more conventional and formal. For shapes B, D and E, pupils say: “they are polygons and have almost the same number of sides”, “they have straight sides” or “they all have right angles”. These few examples show how it is possible to obtain information about pupil's way of thinking.

Flexibility: Visualization must be adapted to the situation

For pupils aged from 6 to 8, it seems difficult to be exhaustive in enumerating the characteristics needed to solve the task described below (Figure 8). In fact, each of these figures can be distinguished from the one marked with the star by a single geometric criterion. These are, in order, the number of sides, convexity, symmetry and the presence of only straight or some curved sides. When only characteristics are used, those are rarely exhaustive and do not allow to succeed. Pupils must therefore be flexible, mobilizing both iconic and non-iconic visualization. This situation is an opportunity to test this skill.

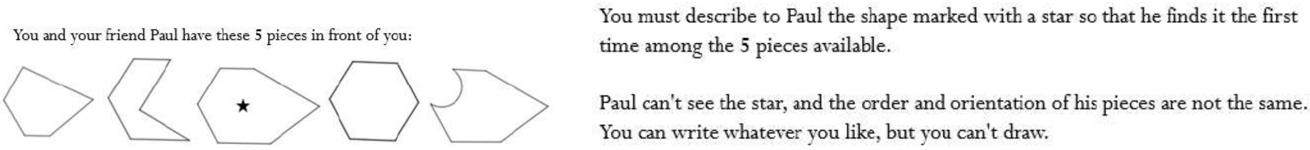


Fig. 7: Task to assess visualization flexibility

Here are two examples where only the global aspect is exploited which is not enough to find the right shape: “the shape I’ve chosen looks like a hexagon”, “I’d like a shape with one piece of a hexagon and one piece of a triangle”. Since many of the shapes are similar, an exclusive use of the iconic visualization is not operational in this situation. Pupils of this age often use the term “hexagon” to designate any polygon with more than 4 sides, which makes it difficult to ascertain its meaning. On the other hand, relying on the characteristics of the shapes only requires a degree of comprehensiveness that is difficult to achieve for pupils of this age. Indeed, between 6 and 8, they generally haven't yet worked on all these characteristics in class. Here are some typical examples: “It has 6 straight sides. One side is bigger than the other”, “It's drawn with a ruler. It has 6 sides and 6 vertices”. In these descriptions, the iconic visualization is not mobilized and the proposed characteristics are insufficient in order to determine the right shape. In addition to the mastery of the characteristics, it is also necessary to know the geometry vocabulary and its signification. From the following example, we can see that the pupils are not yet familiar with the specific vocabulary: “It has no right angle. It has a vertex. It has straight lines. Lots of sides”. We can imagine that this pupil, when he uses the term vertex, refers to something sharp. If the pupil knew the geometric definition of this term, he would probably understand that all the proposed shapes contain several vertices. We might also wonder what the pupil means when using the term “right angle” because this notion has not been yet worked on in class. We now present some examples where pupils jointly mobilize both ways of thinking as intended. However, this doesn't always lead to success, as in this case: “it looks like a pointed flint, the sides are straight”. In this case, there's not enough information to determine a single shape. Finally, here are some examples of successful productions, such as: “The shape has 6 sides, it's almost a hexagon and the part on the right is deformed”. In this production, however, we can see that the pupil uses spatial cues, even though the instruction forbids it. In many productions, pupils use spatial vocabulary such as “on top”, “on the right” and so on. A certain number of the pupils’ productions are also based on other aspects such as measurement, or the decomposition of shapes into known subshapes. Here's an example of what one could expect: “It doesn't have curves sides. It has 6 angles. It has 6 lines. There's one place that looks like a hexagon and another place that looks like a triangle.” These few examples show the diversity of what pupils can produce, depending on whether they use iconic, non-iconic or mixed visualizations.

CONCLUSION

Our research proposes to work on geometry in early school in order to reduce the gaps highlighted by several research studies in mathematics education. To do this, we have developed material based on manipulation and visualization. In order to see how the pupils learn, we observed the ways in which they act and speak, to have tray to determine in their way of thinking about geometrical shapes, whether they use either iconic or non-iconic visualization. Our research shows that the use of the equipment leads pupils from the age of 4 to enrich their initial iconic visualization with some elements of non-iconic visualization. So, they are able to take into account some of the characteristics of shapes. The choice of tasks and the group of shapes proposed by the teachers are essential in training the flexibility of their look. Pupils had to be able to mobilize the most effective visualization according to the environment they were confronted with. However, the results on this point are more nuanced. It is more difficult for the pupils to know when to use a global rather than an analytical way of thinking depending on the situation. Similarly, for teachers, understanding this more cross-curricular objective also seems more complex. At the moment, we're looking at ways of disseminating this material ensuring that teachers make the most of it. We therefore continue conducting our investigations in two directions: looking at issues concerning in-service teacher training and considering evaluation and institutionalization.

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DESCRIPTIVE AUTOMATED ASSESSMENT: FACILITATING INQUIRY IN GEOMETRY

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As designers of technology to support learning, we are interested in supporting feedback processes in the context of guided inquiry instruction of geometry. We explore the potential of automatically associating mathematical descriptions as resources for reflection and inquiry learning. We study the potential of what might be thought of as a mirror that speaks: it reflects student's actions and describes the mathematics it sees in a student's submission constructed using interactive diagrams. We illustrate the inquiry process by a thought experiment around a task related to inscribed triangles. We discuss the design of descriptive assessment where students reflect on and communicate their own learning, utilizing individually-reported multi-dimensional automatic analysis of their submissions in response to example-eliciting tasks. Based on Hershkowitz's theory of fundamental concepts (1990) and on theories of exemplification and variation we outline a research based framework for automated descriptive assessment that is designed to facilitate inquiry in geometry.

INTRODUCTION: INQUIRY LEARNING AND TECH-BASED FEEDBACK PROCESSES

In this paper, we suggest that automated descriptions of student work are a new strategy for feedback that can be designed into technology for providing students with support for learning geometry. We illustrate this strategy with an example-eliciting-task (EET) designed in the STEP platform (Olsher et al., 2016) that is intended to support students in developing conjectures about areas' relations among inscribed triangles. We argue that such tasks have the potential to initiate and support autonomous inquiry situations. We are interested in conceptualizing feedback as an ongoing process and study the use of automated information to support technology-enhanced didactical situations that involve inquiry processes with non-judgmental personal feedback. The tasks include interactive diagrams (applets) in Geogebra (Hohenwarter et al., 2009). The focus of our design effort is providing students with mathematical verbal descriptions of their submitted examples. A main challenge is to design an environment (tasks and tools) that can involve students in exploration of examples and submission of answers to tasks that go beyond the specific examples. Personal feedback provided automatically to students, which allows them to reflect on their work, can serve as a fundamental tool to define pedagogical and mathematical goals amenable to automatic assessment. Our overarching research question is how automatic characterization of students' examples serves as feedback which involves students' reflections on their own understandings of the examples they have created.

THEORETICAL BACKGROUND

Exemplification

We view inquiry learning in mathematics as centrally involving the coordination of examples and of concepts, where the definition of a concept identifies criteria for classifying instances as examples or non-examples of that concept. DGE is a natural tool for designing inquiry tasks and inquiry situations for learning geometry. Studies suggest how dragging allows students to experience multiple examples of multiple representations, visualize mathematical concepts, and explore dependencies among properties of the diagram. This can help students learn to categorize diagrams according to their characteristics, discern what could be true, and experience opportunities to overcome difficulties in conjecturing (Arzarello et al., 2002; Luz & Yerushalmy 2023). Participating in such learning, students need to learn to note and communicate about patterns, structures, or regularities in mathematical examples and to question reasons for these phenomena. Although an example cannot confirm a universal statement they can be "supporting" in the context of universal statements (Buchbinder et al. 2017) and having students construct examples of a concept is an important activity that can be a catalyst for enhancing students' conceptual understanding and expanding students' concept image (Zaslavsky & Zodik, 2014). Given this perspective, example-eliciting-tasks (EETs) are an important component of our approach to guided inquiry with a digital environment.

Interactions with DGEs are known as helping students to generate examples, identify patterns and categories of constructions in order to formulate conjectures. Yet, research has followed learning episodes where the feedback did not lead to concept-formation and the achievement of curricular goals. Thus, a major challenge is to create a feedback process that would support students who are working on EETs and guide conceptualization of mathematical objects and actions that are central to the curricular goals. To overcome the obstacle of the particularity of an example and to gain insight into the breadth of students' concept image, we describe two theoretical views that helped us to design the STEP platform and outlay the principal aspects for automated descriptive assessment.

Variation

To discern a concept or an object of learning, learners must be familiar with a large example space: a collection of examples of the concept, together with the methods that make it possible for them to construct their own examples (Mason & Watson, 2008). Learners can engage with new ideas through making their own examples. Within STEP, students' submissions to EETs consist mainly of examples, which are instances created with the interactive diagram of the task and stored on the platform. EETs prompt students to generate their own examples but also to produce multiple-examples' submissions that are as different as possible from one another. The collection of outcomes, including the personal example spaces (PESs) of individual students (or of the collective example space of the entire group), are analyzed online and filtered according to specified characteristics.

Generating examples is rarely sufficient for meaningful learning in autonomous inquiry situations. Most learners need to have explicit stimuli to form a class they can denote as a collection of similar or related examples. However, what is required for learning through exemplification should exceed observations of patterns; it has to involve developing awareness of the variants and invariants of the PES. And it has to become part of a feedback process that supports discerning and formalizing patterns symbolically or verbally. Usually, such awareness is triggered by external guidance, most often by teacher's probes. The theory of variation (Marton & Tsui, 2004) analyzes the critical aspects of how learners can learn concepts from examples (for a variation-based automated analysis in geometry see Luz & Yerushalmy, 2023). Watson and Mason (2005) relates to variation as the features of an example that learners recognize as eligible for change, without losing examplehood.

Characterizing examples with words

To further enhance conceptualization using mathematical characteristics as stimuli we follow Hershkowitz's theory of fundamental concepts with its focus on definitions and examples (Hershkowitz, 1990). By concept, this theory refers to a combination of critical and non-critical characteristics; by concept definition, to the minimal combination of critical characteristics (necessary and sufficient) to define the concept; and, by concept image, to the collection of examples and the derived properties reflected in students' work. From this perspective, characterizing students' examples in words might be thought of as providing a mirror that speaks, in some ways like the mirror in the Snow White tale. The sort of "talking mirror" we aim to create "speaks" to the student by offering a description of each example and of the generated PES. Yerushalmy et al. (2022) further explain the theoretical foundations and roles of the mirror that speaks. The kind of mirroring that STEP provides is led by the designers of the tasks or the teachers, to enable students to think in terms of categories rather than of an example; compare and distinguish between their examples, suggesting mathematical descriptions or contextual descriptions of mathematical phenomena. As Watson & Shipman (2009) who quote Vigotsky, we appreciate autonomous learning that is "structured by expert others" (1978, p.57) where these experts speak to the student by automatically characterizing PESs.

ILLUSTRATING DESCRIPTIVE AUTOMATED ASSESSMENT IN GEOMETRY

In this section, we build on the terms and theories described above to illustrate the descriptive assessment of a geometry inquiry process: The section starts by presenting a STEP geometry task (Figure 1). Then it presents the characteristics and the mathematical and pedagogical considerations for choosing them. Thirdly we demonstrate an automated personal task report; the section ends with a description of a learning situation with the resources of STEP.

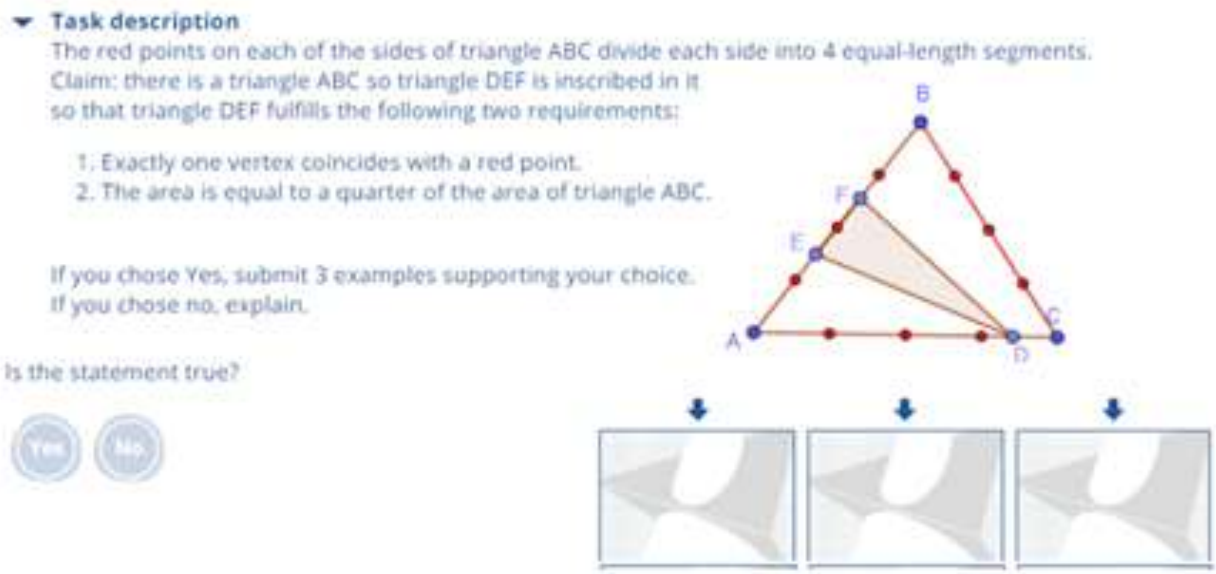

Figure 1: The inscribed triangles task

An example-eliciting-task in geometry:

This task requires that students construct a triangle DEF inside triangle ABC where the area is ¼ of the original ABC and only one vertex is on a red dot. The red dots subdivide each of the three sides of ABC into four equal segments. The shaded triangle is triangle DEF. Points A, B, C, D, E, F are draggable (the positions of points D, E, F are limited to the perimeter of triangle ABC).

The task's aim is to prompt interaction with the area of a triangle formula while using proportional reasoning to define a set of examples that would preserve a final ratio between two results of multiplication: segment * altitude to the segment (or it's extension). The low entry point for this task enables the students to rely on the division of the big triangle into 4 equal-area sub-triangles by dividing one of its sides into 4 equal-length parts. This method enables the students to initially succeed when working on the task while setting the stage for prompting them to further interact with the problem.

Critical characteristics	Non-critical characteristics
• Exactly one vertex of DEF coincides with a red dot • The area of DEF is ¼ of the area of ABC	1. One common vertex to triangles ABC and DEF 2. Two common vertices to triangles ABC and DEF 3. Three common vertices to triangles ABC and DEF 4. Triangles ABC and DEF have no common sides 5. One side of DEF lies on a side of ABC 6. Two sides of DEF lies on a side of ABC 7. ABC has a right angle 8. DEF has a right angle 9. Common altitude to triangles ABC and DEF 10. In triangle DEF one side is a midsegment of triangle ABC 11. In triangle DEF two sides are midsegments of triangle ABC 12. In triangle DEF no sides are midsegments of triangle ABC

Table 1: Critical and non-critical characteristics

Characterizing variations

The key for the variation of the answers lies within the understanding that the result of the multiplication of the segment and its corresponding altitude of triangle DEF is a quarter of the multiplication result of the altitude and segment of triangle ABC. Below we describe ways in which the non-critical characteristics can offer students ways to either reflect on their submitted work, or plan for different examples that might fit or not fit the marked characteristics. We group the different characteristics to categories and explain the rationale of the choice:

Reducing the complexity of the problem (7,8,9): ABC has a right angle; DEF has a right angle; Common altitude to triangles ABC and DEF.

Students might find it easier to drag ABC to be a right triangle. In which the altitude coincides with a side of the triangle. The same could be said for the construction of triangle DEF. Another way to deal with only one side's ratio is to use a common altitude. While this might not be the way that students perceive their examples, reflecting this characteristic in their answers could help them communicate their strategy in a more mathematically robust fashion.

Limitations of the example space (1,2,3): One common vertex to triangles ABC and DEF; Two common vertices to triangles ABC and DEF; Three common vertices to triangles ABC and DEF.

Students might look into these characteristics in pursuit to meet all of the listed criteria. This would eventually lead them to see that some characteristics contradict critical characteristics. In this case, there cannot be a correct answer with three common vertices, while one and two are possible.

Variation of non-critical characteristics (4,5,6): Triangles ABC and DEF have no common sides; One side of DEF lies on a side of ABC; Two sides of DEF lies on a side of ABC.

All of these non-critical characteristics can be demonstrated in correct answers.

Contradicting conditions (10,11,12): In triangle DEF one side is a midsegment of triangle ABC; In triangle DEF two sides are midsegments of triangle ABC; In triangle DEF no sides are midsegments of triangle ABC.

As the midsegments connect two red dots, 10 and 11 are impossible to achieve. This could also be hinted by the fact that any examples meeting the critical characteristics also meet 12.

Personal descriptive assessment

In this section, we describe a thought experiment based on our experience with student's feedback processes using STEP. We demonstrate how interactions with the personal characterization of each example are manifested in the feedback process. Below we mark these actions as (a) discerning the PES: Students analyze the examples, observing critical characteristics described by STEP, thus distinguishing between examples and non-examples. To comply with the diversity's requirement, they would (b) use the characterizations for categorizing examples according to non-critical characteristics - turning the non-critical to "critical" within a special case of the solution. This, in turn, leads to (c) formulating conjectures through generalizing distinctions between examples and non-examples using descriptive statements or symbolic expressions. Sometimes they will generate new examples to argue for truth through supporting examples, refutation by new examples and by following relations, contradictions, and dependencies among non-critical characteristics. It could provide the initial stepping stone for proof construction. Often STEP tasks ask for textual or symbolic explanations that are manually assessed.

In Figure 2, we see that the student submitted three different examples; each example is an instance describing construction of an inscribed triangle DEF that its three vertices are on the sides or vertices of ABC and its area is ¼ of ABC. Example 1 demonstrates a triangle DEF having altitude that is ¼ of the altitude to a common side AC (which is also FD); Example 2 demonstrates an DEF that its altitude to FD is ½ of the altitude to AC and FD is ½ of AC (AC and ED are overlapping); In Example 3 triangle DEF complements the area of the triangle constructed in Example 2 to ½ of ABC, probably imagining folding ABC upon the median BD to get triangle ABD which is half area of ABC and recognizing DF to be a median, thus DEF is ½ of ½ of ABC.

Following the submission of the answer and the examples, the student starts analyzing the information presented online in the personal report. If the student answered "No" the report would present the student's explanation. Otherwise, the personal report consists of the list of characteristics that the algorithm assess and highlights automatically per submission. Firstly, it identified critical characteristics (action (a)): The report identifies example 1 as fulfilling both requirements while Examples 2 & 3 are identified by only one requirement. While both - example 1 and 2 - could have been a result of a similar strategy for producing the required ¼ area, DEF in example 2 is located on two red dots. Example 3 as well has two red dots as vertices. At this point the student considers more general inscribed triangle than the limiting requirement of the given task.

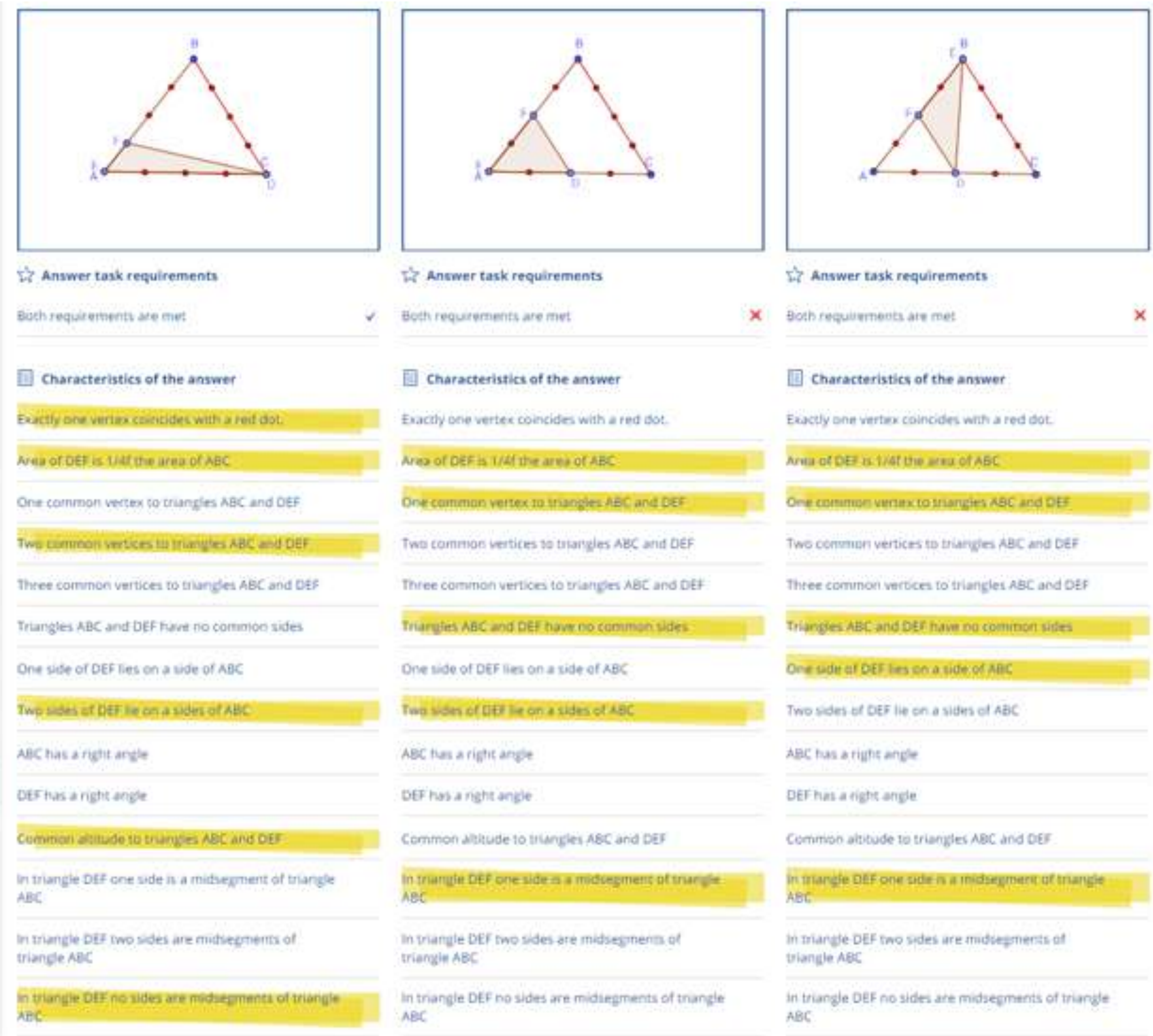

Figure 2: An online personal post-task report, listing and highlighting characteristics for three submitted examples (From left to right)

The inquiry continues by action (b): Comparatively reflecting upon the non-critical characteristics of an example and a non-example.

> *"I notice that characteristics 3,4,5 identify the number of common vertices. In Example1 ABC and DEF share two vertices and in Example2 they share a single vertex. Mmm! so it must be that the other two vertices are located on the sides of ABC. And, Example3 also demonstrates ABC and DEF that share one vertex.*"

The comparative analysis of the first three noncritical characteristics then led to a question:

> *"Should I always construct DEF to share two vertices with ABC?"*

Seeking further information that can support an answer to this question the student continues to scan to non-critical characteristics list (action (c)). The next characteristic on the list (6) relates to common sides. Both non-examples are marked as not having a common side. Are the two aspects of sharing sides and sharing one common vertex provide consistent feedback?

A positive answer helps to conclude that: *Two triangles share a side if and only if they share two vertices*. But this quite trivial conclusion does not provide further explanation and the student returns to comparatively analyze the remaining list of the non-critical characteristics of Example1 and Example2:

> *"…there must be another non-critical characteristic that distinguishes between an example and a non-example of triangles that share a side and ¼ area."*

Idyllically, one can come closer to an answer by returning to the interactive diagram and generating new constructions such as in Figure 3.

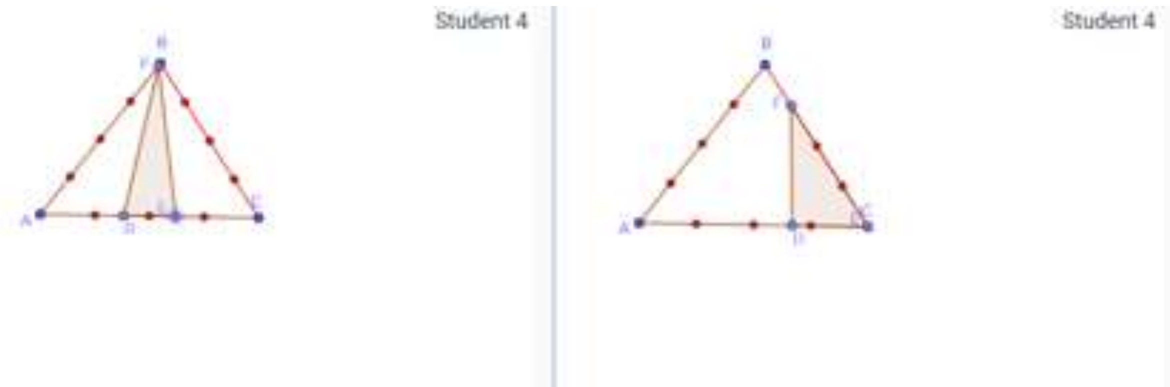

Figure 3: Possible correct constructions for the inscribed triangle task

> *"Ah! Blue dots are not necessarily vertices! the vertices of DEF might be any point on the ABC sides which isn't red."*

In the case of this task, developing symbolic arguments most probably triggered by external teacher prompts. Generalizing the answers, for $\frac{DF \cdot h_{DF}}{2} = \frac{1}{4} \cdot \frac{AC \cdot h_{AC}}{2}$ (DF and AC can be replaced by any segment of the triangle, or in terms of the original triangle: $DF = aAC, h_{DF} = bh_{AC} \Rightarrow \frac{aAC \cdot bh_{AC}}{2} = \frac{1}{4} \cdot \frac{AC \cdot h_{AC}}{2}$, we just need to keep this relation: $a \cdot b = \frac{1}{4}$.

While this generalization is a goal within itself in terms of student's understanding of ratios between areas in geometry, there are many meaningful interactions with different mathematical characteristics of this specific setting that could promote inquiry that can be supported by an interactive feedback process that suggests different perspectives, as described in the list of non-critical characteristics.

TAKING DESCRIPTIVE ASSESSMENT IN GEOMETRY FURTHER

As this illustration suggests, the strategy of characterizing student submissions with mathematical descriptions points to less common directions in the field of learning analytics (LA): Mostly used LA uses measures like time on task, correctness, misconceptions, and more such information that could suggest important aspects of student's learning in general. We consider automatic characterization of student examples in words as a way for students to interact with points of view that may be different from their point of view while engaging in inquiry. We argue that through reports constructed for individual learners, such communication could be useful, enabling students to further their learning by reflecting on their understanding.

Furthermore, the digital nature of these tasks, and the rich descriptions in a digitally analyzable format, enables us to envision mechanisms that would benefit and use this information as a "training ground" for artificial intelligence agents that could then automatically suggest relevant characterizations of mathematical tasks, as well as sequencing of tasks to create a relevant learning trajectory. In addition, we end with a thought about the possible interactions with large language models, to be trained with the relevant mathematical content, as part of personal learning with a "speaking mirror".

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TOPIC D

Multidisciplinary perspectives

EXPLORING CREATIVE PROBLEM-SOLVING WITH EYE-TRACKING METHODOLOGY

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We present an empirical investigation of creativity-directed problem-solving linked to Visual Geometric Multiple Solution Tasks (VG-MST). The problem-solving strategies used by the subjects were analysed for fluency, flexibility, and originality of the problem-solving process, using a thinking-aloud procedure. The corresponding eye-tracking patterns reordered along the solution process were examined using scan paths, heat maps and entropy. Differences between the solution strategies employed by two university students who are both expert in mathematics suggest that (a) producing similar problem-solving strategies can be linked to different cognitive processes, and (b) mathematical insight is linked to more efficient eye-movement patterns.

BACKGROUND

Studies have indicated that students with creative thinking abilities are more adept at tackling the intricate problems of the modern era (Pellegrino & Hilton, 2012). *Creativity in mathematics* promotes the development of mathematical proficiency and, in turn, depends on the level of mathematical proficiency (Elgrably & Leikin, 202; Schoevers et al., 2022; Leikin & Lev, 2013; Levav-Waynberg & Leikin, 2012). The present study is aimed to shed light on creative problem solving in geometry problems, with special attention to creative mathematical processing, using both behavioral (problem-solving ways used and their accuracy) and neurophysiological (eye tracking) measures.

We explore creativity using Multiple Solution Tasks (MSTs), which explicitly require solving a problem using multiple strategies (Leikin, 2009, 2013). Over the past decade, MSTs have been found among the primary tools for assessing mathematical creativity (Leikin & Sriraman, 2022), with creativity assessed based on the analysis of the solver's fluency, flexibility and originality (Leikin, 2009). In addition, due to the centrality of insight to creative endeavors (Haavold & Sriraman, 2022; Leikin et al, 2016; Leikin & Guberman, 2023), we use insight-allowing MSTs for which one solution is insight-based.

This paper presents a study that focuses on solving Visual Geometric Multiple Solution Tasks (VG-MSTs), with a visual insightful solution. These tasks require implementation of visual skills, deductive, abstract and logical reasoning (Hanna, 1996; Herbst & Brach, 2006; Clements & Battista, 1992). VG-MSTs provide a rich basis for inquiry, implementation of proving skills and generalization. While similarly to most geometry problems, solutions of VG-MSTs are not algorithmic and, therefore, involve heuristic reasoning (Polya, 1981; Schoenfeld, 1985), insightful solutions additionally require seeing the underlying structure of the problem.

Over the past decade, *neuroscience* has been applied to enhance understanding of mathematical processing (e.g., Grabner & De Smedt, 2016). Among other neuroscience methodologies, the *eye-tracking methodology* has also entered the terrain of mathematics education research and is shown to

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 473-480). ICMI.

be beneficial. It was demonstrated that eye fixations and eye movements correspond to mental operations (Grant & Spivey, 2003). Learning about mental processing during problem-solving using eye trackers is increasingly key to educational research (e.g., Scheiter & van Gog, 2009), particularly in mathematics education (e.g., Andrá et al., 2015). For example, Dewolf, Van Dooren, Hermens, and Verschaffel (2013) used eye movements to validate students' strategies related to solving mathematical word problems and found that students scarcely looked at the representational illustrations and that the illustrations did not affect the realistic nature of their solutions.

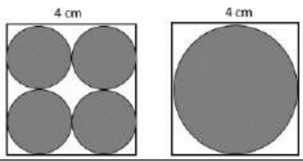
Strohmaier (2020) found that eye tracking is useful for studying processes, revealing mental representations, and evaluating subconscious aspects of mathematical thinking (Strohmaier et al., 2020). For example, Norqvist (2019) utilized eye tracking to examine algorithmic and creative reasoning strategies in mathematics tasks and discovered that students who practiced creative tasks performed better than those who practiced algorithmic tasks in subsequent evaluations (Norqvist et al., 2019). While eye-tracking technology shows potential when it comes to understanding students' mathematical strategies, its application in graph interpretation is still limited. In this study we employed eye-tracking methodology, keeping in mind the challenges outlined above, with the aim of bridging the gap between behavioral data and learners' cognitive processes.

THE RESEARCH GOAL

The study's main goal was to characterize creativity-directed problem-solving processing in order to deepen our understanding of the flexibility associated with the production of multiple solution strategies and the originality associated with mathematical insight. We evaluated the creativity of the problem-solving process using Leikin's model (2009, 2013). The research methodology integrated a thinking-aloud procedure with eye-tracking to obtain a deeper understanding of the behavioral and neuro-physiological patterns associated with solving MSTs.

PARTICIPANTS AND DATA COLLECTION

As mentioned before, we used a VG-MST, depicted in Figure 1. The problem allowed algorithmic computational solution strategies and insightful solution strategies without computations (see Results section).



Given two congruent squares with side 4.
One square subscribes 4 small tangent circles while the other subscribes one big circle.
Compare the area of the big circle and the area of 4 small circles?

Figure 1: The Visual Geometric Multiple Solution Task (VG-MST) used in the study

We report on the problem-solving performance on the VG-MST of two BSc students studying Computer Sciences (S1, S2). These students both have a high level of mathematical competencies. The problem chosen for the study was easily approachable for these students; the requirement to find multiple solutions allowed us to examine the participants' creativity rather than their knowledge.

We employed individual interviews with a thinking-aloud procedure accompanied by Camtasia software recording and the Pupil Labs GmbH software. The subjects solved the task while wearing a pupil eye-tracker [Pupil Labs GmbH] and using a DELL 24" touch screen on which the tasks were displayed and on which the participants could write while tackling the tasks. Camtasia allowed synchronized recording of the thinking-aloud problem-solving process with solution writing on the computer screen, the shifts between different solution strategies, and eye movements.

The study was conducted in a small and quiet room, while the participant was sitting approximately 35 cm away from the computer screen. The eye tracking data was translated into eye movement characteristics, including the number, frequency, and length of the saccades, their directions, and the location of fixations.

DATA ANALYSES

The solution strategies employed by the participants were categorized into distinct categories, and the evaluation of the creativity linked to the problem-solving strategies was aligned with these categories (Leikin, 2009). Subsequently, the student's performance was evaluated based on creativity components - fluency, flexibility, and originality - using the framework and scoring system developed by Leikin (2009). This model for evaluating creativity has been consistently utilized in prior research investigations (Leikin, 2009, 2013; Levav-Waynberg, 2014).

The eye tracker reordered eye movements and screen coordinates at approximately 200Hz. This generated a continuous stream of (x, y) coordinates and screen images every 5 milliseconds. The correspondence between the solution strategies and eye movement patterns was classified based on this analysis using techniques such as heatmaps, scan paths, and entropy. The strategies were then classified into distinct qualitative categories, each characterized by distinct movement patterns, including scan paths and the duration of fixations. The final phase encompassed a validation and refinement of these identified categories.

Our calibration procedure for the Pupil Eye tracker embraced both internal and external methods, as outlined by Schweizer et al. (2021). The external calibration technique utilized barcodes at the screen periphery to establish the screen as a two-dimensional surface within a three-dimensional context (Elmadjian et al., 2018). This calibration allowed us to generate a gaze direction graph during task completion. The screen was divided into a 9x9 grid of bins, wherein fixation points were quantified for each bin and depicted as varying-sized green circles. Saccades, or rapid eye movements between fixations, were visually represented by connecting lines, as illustrated in Figure 3. In our analytical framework, we harnessed an array of quantitative metrics derived from eye-tracking data. Beyond basic statistical parameters like mean and variance, we gauged the dispersion of fixation points across the screen using Shannon's Entropy (Shannon, 1948). To achieve this, we assigned P(i) to signify the proportion of fixations occurring within bin i relative to the entire set of fixations. High entropy indicated significant dispersion, whereas low entropy corresponded to clustered fixations. The entropy calculation followed this formula$-\sum_i p_i \log(p_i)$.. Furthermore, our analysis encompassed the assessment of saccades based on their length, variability, and angles between successive saccades. Larger angles denoted smoother transitions between fixations, while smaller angles indicated more erratic movements between different locations.

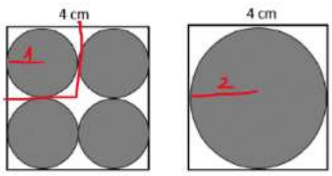
The problem-solving process was meticulously recorded, including participants' handwritten notations and their concurrent verbalized thoughts through the think-aloud procedure. Each instance of a solution strategy was temporally bounded, stretching from the initiation of the solution process to its culmination.

FINDINGS

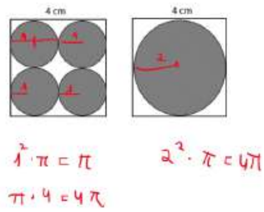
Problem-solving strategies used by the participants

Figure 2 presents different solution strategies for the VG-MST used in this study.

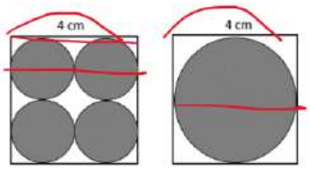
Solution A1: Calculate the area of a large circle and one small circle



Solution A2: Calculate the area of a large circle and 4 small circles



Solution B: Insight-based – similarity of the small and big squares



Solution C: Calculate: Check and multiply by 4

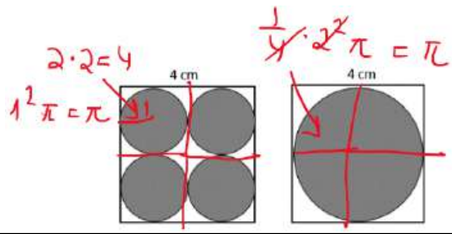


Figure 2: The task and solution strategies produced by the participants.

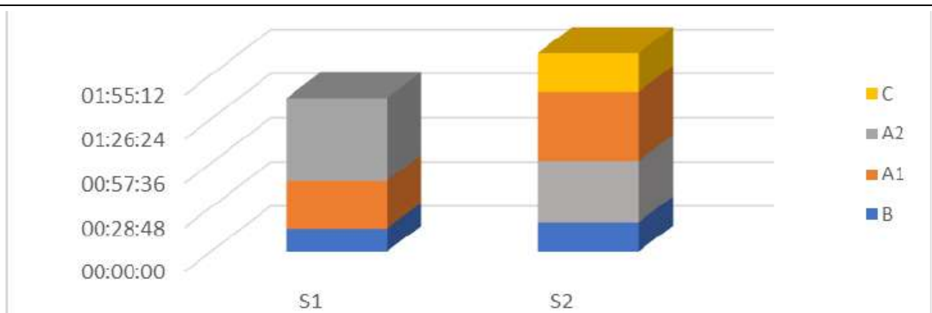


Figure 3: Appearance of the different strategies when solving the tasks

Figures 2 and 3 depict the different solution strategies and their sequences and durations as employed by the two study participants. Four solution strategies were identified (see Figure 2). Figure 3 shows that S1 implemented four solution strategies, and S2 implemented three solutions. Both S1 and S2 started by solving the task with an insight-based solution. They each solved the problem using calculations. The two additional solutions performed by S1 and S2 were similar (denoted as A1 and A2 in Figure 3). S2 provided an additional solution using the area of a circle formula.

Eye movements related to the creativity of the problem-solving performance

In this section, we present an analysis of the eye movements linked to the creativity–emerging problem-solving process for the task. We focus on the differences in the thinking processes of the two subjects when performing similar solutions, and on the different ways of solving performed by the same subjects. We examine the differences displayed in the scan paths of specific solutions for the task performed by S1 and S2 (Figure 4).

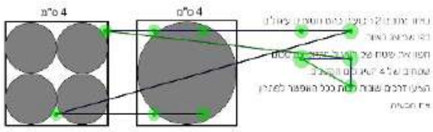
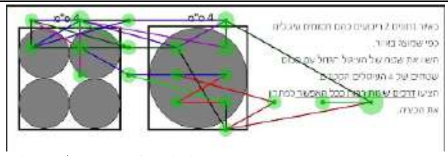
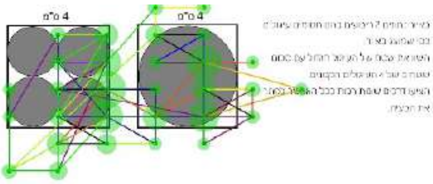
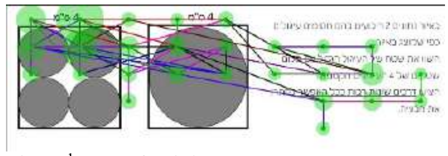
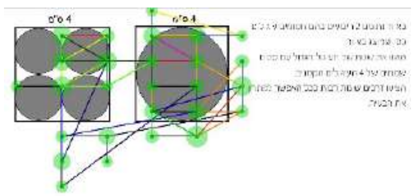
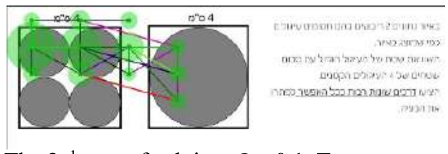
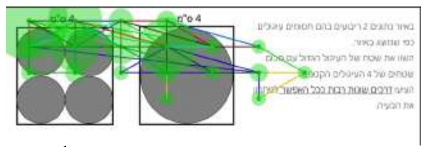
	S1	S2
Solution strategy B	 The 1st way of solving, Or=10, T=15:630 sec.	 The 1st way of solving, Or=10, T=19:665 sec.
Solution strategy A1	 The 2nd way of solving, Or=0.1, T=54:175sec.	 The 3rd way of solving, Or=0.1, T=45:444 sec.
Solution strategy A2	 The 3rd way of solving, Or=0.1, T=31:13sec.	 The 2nd way of solving, Or=0.1, T=40:813sec.
Solution strategy C		 The 4th way of solving, Or=1, T=65:175 sec.

Figure 4: Eye tracking analysis of specific solutions for the task by S1 and S2

Figure 4 demonstrates that different solution strategies are (naturally) associated with different thinking processes reflected in different loci of attention and different scan paths**.** This difference can be seen in the difference between the scan paths corresponding to strategies A1, A2, and B for each one of the participants. Moreover, we find that similar solution strategies in different participants are associated with different thinking processes and foci of attentions. For example, solution B, performed both by S1 and S2 as the first solution strategy, was a similar insight-based strategy:

S1-B: The same diameter here, i.e., the ratio between them is equal, i.e., the radius of 2 circles equals the radius of one circle.

S2-B: It is possible to refer to the ratios between the radii. They are equal.

The corresponding scan paths demonstrate that S1 and S2 focused on different locations on the figure as well as following different connections between the figure and the text. While S1's utterance related to this solution is longer, the scan path demonstrates that he was more efficient, with less time spent on the solution, fewer fixations and a smaller number of saccades. At the same time, while S2 was less verbal he spent more time producing the solution and his scan path differed in location, number of fixations and saccades. S1 primarily focused on the square's sides, whereas S2 focused on radiuses. These differences may indicate different thinking processes that led to the use of similar solution strategies.

Similarly, the scan paths of S1 and S2 corresponding to solution strategy A1 differ. The analysis shows that they focused on different parts of the figure, but these different approaches led to similar calculation strategies. In solution A1, S1 focused on the circles and writing below the illustration, and S2 focused mainly on the top side of the rectangle and returned to reading the text.

S1-A1: [...] The diameter is 4, the radius is 2 cm, and the area of the circle is 4π [...] It is half a square, 2 cm minus 1, is a radius, π in a square, one square is a π and a big circle 4π.

S1-A2: Suppose it is four and divide by that to calculate the area of a tiny circle and then double four and one big circle. Moreover, if there are four here, then 4π.

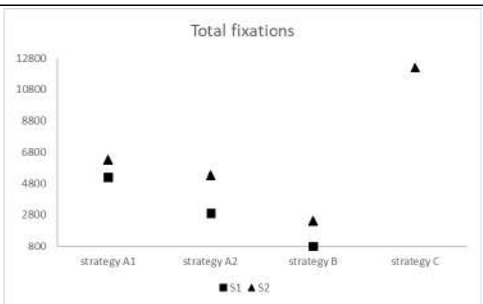
S2-A1: Check the area of the large circle and each area of the small circles.

S2-A2: It is also possible to take the area of a tiny circle multiplied by four circles, and it is the same area, check and multiply by 4.

As mentioned above, comparison between the scan paths of different solutions - for example, A1 and A2 - demonstrates the differences in corresponding eye movements of each participant in the same strategy: when performing solution A2, S1 focuses on the radius and inner areas, compared to S2, who concentrates on the top left side of the square, and thinks of division into four small circles. As seen in the scan path, eye movements differed for S1 and S2, each of whom focused on specific regions of interest.

This analysis assumes that the scan path identified areas of interest during mental processing, and identified both *subject-dependency* and *strategy-dependency* aspects of problem-solving processes. In this way, eye-tracking analysis deepens our understanding of the different thinking processes involved in different subjects' performance of similar solutions.

An additional eye-tracking variable that allows analysis of the thinking process associated with problem solving is *entropy*. Entropy refers to the degree of dispersion of fixation points on a display. High entropy indicates a large dispersion, while low entropy indicates a tight clustering of fixations. Figure 5 presents the analyses of each solution's entropy and total fixations.



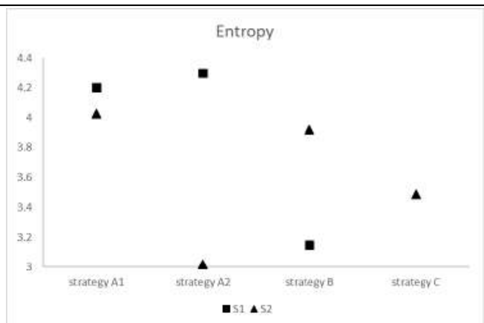


Figure 5: Eye tracking analysis of specific solutions for the task by S1 and S2

From the entropy analysis, which demonstrates the dispersion of the fixations, we can understand the differences between S1 and S2 in solving the tasks. In solution B, S1 was lower entropy than S2, which means the fixations are tightly clustered for S1 and more dispersed for S2. Another exciting example was in solution A1 (similar for both subjects), which indicated that S1 and S2 had high entropy and large dispersion. The last example for the same solution is demonstrated in solution A2; S1 had high entropy, his dispersion was large, and S2 had very low entropy, which means the dispersion of the fixations was tightly clustered, implying the same strategy may still involve a different thinking process. The total fixations showed that S2 performed more eye movement in each solution than S1. For example, in solution A1, S2 solved more quickly than S1, but used more eye movement in this strategy.

Based on the above, we argue that eye-tracking analysis is adequate for understanding differences in mathematical thinking that lead to similar solution strategies and outcomes. Such understanding could not be attained from the interview transcript analysis alone.

DISCUSSION

This paper presents an empirical investigation of creativity-emerging problem-solving by employing a VG-MST that explicitly requires participants to solve a mathematical problem in multiple ways. Our goal was to characterize creativity-emerging problem-solving processing by participants with the same level of creativity through integrating behavioral and neuro-physiological perspectives that focus on problem-solving strategies and corresponding eye-tracking patterns. We call the process of solving VG-MST creativity-directed since we do not require participants to perform creative solutions but fosters creative thinking.

We integrated eye-tracking methodologies with individual clinical interviews to characterize patterns of processing associated with flexible mathematical thinking and insight-based processing in solvers with different strategies. Due to the small sample size (the major limitation of our study), we performed qualitative data analysis. The data analysis included behavioral measures – fluency, flexibility, and originality and the corresponding eye-tracking tools: scan paths and entropy. The findings described in this paper shed light on several processes and phenomena.

Eye tracking analysis further reveals differences in thinking processes associated with employing seemingly similar solution strategies. This finding is based on observing differences in scan paths of identical solution strategies. The differences are related to regions of interest, entropy values, number of saccades, and total fixations. The number of saccades and the longer duration of fixations, combined with a higher level of cognitive load, can indicate either a higher level of problem-solving creativity or problem complexity. Uniquely, our study demonstrated that insight-based solutions have different and identifiable eye-tracking patterns related to more focused regions of interest with smaller numbers of saccades and fixations. In summary, based on the findings of our study, we argue that eye-tracking methodology can make a meaningful contribution to research in mathematics education since it reveals underlying problem-solving processes that cannot be discovered by behavior analysis alone.

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REIMS (France), 23-26 April 2024

TEACHING GEOMETRY TO ADVANCE DESIGN JUSTICE

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The study of geometry provides unique opportunities for students to appreciate geometric patterns in the world and apply geometry to solve authentic problems. These opportunities are well aligned with design thinking approaches to problem solving. The geometry curriculum can incorporate design challenges that require geometric thinking. We apply a design justice perspective to geometry instruction and provide examples of lessons that can allow students to advance social justice in their communities. This perspective can help to rehumanize mathematics by engaging students in creating new math, broadening what constitutes math, and developing students' sense of ownership. We posit that by embracing a multidisciplinary approach to the teaching and learning of geometry, we can position students as designers, thus preparing them for future jobs where design thinking is at the core. At the same time, by drawing on a design justice perspective, students can develop empathy and empower their communities through the study of geometry, thus making their math instruction more meaningful. Overall, design thinking can model the historical development of geometry knowledge.

INTRODUCTION

The U.S. Geometry curriculum has tried to attend to various goals which are represented by four arguments justifying the high school geometry course (González & Herbst, 2006). Proponents of the *intuitive argument* established that geometry provides unique opportunities to read the world. Consequently, students' ability to appreciate geometry in their surrounding such as seeing patterns in geometric designs is important. Proponents of the *mathematical argument* stressed that the study of geometry offers opportunities for students to think like mathematicians. From this perspective, students' development of conjectures as they explore geometric problems is a key activity. Proponents of the *formal argument* consider that geometry can allow students to apply logical reasoning that they can transfer to other situations. In the U.S., the use of the two-column proof format is an example of how school geometry incorporated resources for teaching students how to make logical arguments (Herbst, 2002). Proponents of the *utilitarian argument* considered that geometry is especially important in preparing students for the workforce. Geometry problems that position students as professionals such as navigators or architects are helpful to achieve the goals of the utilitarian arguments. While these four arguments stem from the study of the U.S. geometry curriculum in the 20th century, they continue to co-exist in textbooks such as in problems pertaining to geometry and visual arts (González, 2020). Initial evidence from U.S. geometry teachers suggests that the intuitive and utilitarian arguments are most valued in contrast with the formal argument, which seems to be less valued (González & Rinkenberger, 2023). Additionally, research suggests that the emphasis on proofs characteristic of the formal argument is less relevant in the geometry curriculum and geometry instruction (Otten et al., 2014; Sears & Chávez, 2014).

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Pre-Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 481-488). ICMI.

At the same time, questions about the role of school mathematics in supporting students to adopt a critical stance towards society have become increasingly relevant. The pandemic has made explicit societal inequities for which mathematics can be an important tool to examine the world (Bakker et al., 2021). Mathematics educators' discussions about whether geometry instruction will meet the current needs to teach students mathematics to read the world are important as we consider the justifications for teaching geometry in schools. Moreover, the study of geometry can prompt students to dream of a better future and conceive of new ways of living in the world. Gutiérrez's (2018) "Rehumanizing Mathematics Framework" exemplifies the call for new ways of engaging students in mathematical activity by centering mathematics around social justice goals.

FOCUS

We are investigating how to use design-based pedagogy for geometry instruction. Design-based pedagogy is a learning environment that uses instructional scaffolds to support students as they solve a problem by implementing design practices (Royalty, 2018). We are particularly interested in applying a design justice approach so that students can inquire into problems that are relevant to them and their communities (Costanza-Chock, 2020). In doing so, we align the justification for why students need to learn geometry with the goals of the intuitive and utilitarian arguments. A design-based perspective can allow students to appreciate geometry in their surroundings and address challenges affecting them, their communities, or people in the world. We strive for students to develop empathy by understanding the needs of those affected by the challenges (including themselves). At the same time, we are in alignment with the utilitarian argument by positioning students as designers. The opportunity to learn skills and procedures that designers use can strengthen their desire to study geometry and find connections with the ways in which designers apply geometry in their profession. Moreover, with a design justice approach, we seek that students will be able to examine their agency in proposing solutions that are just and equitable.

Main questions

The main questions guiding our study are:

1) In what ways does geometry allow for students to study the world by using design?
2) How does a design-based approach to geometry instruction enable students to investigate authentic problems as designers?
3) What are examples of geometry lessons that use a design justice approach?

With the first question, we are interested in connections between geometry and design by considering the ways in which geometry knowledge is fundamental in addressing design challenges. With the second question, we are interested in identifying characteristics of design challenges that position students as designers. The third question advances the combination of design and geometry with a focus on design justice. Altogether, we are interested in answering the question of how we can best position students to use geometry to solve design challenges addressing social justice issues and leveraging on their cultural and community-based knowledge.

THEORETICAL UNDERPINNINGS

Human-Centered Design

Human-Centered Design (HCD) is a problem-solving approach where designers rely on tools and methods to address the needs of a population and develop meaningful and creative solutions through iteration (Brown, 2008). When designers use HCD they implement different processes that are guided by empathy and iteration (Brown, 2008). These processes include examining current designs related to the new design challenge, conducting user research, brainstorming design solutions, and creating prototypes (Zhang & Dong, 2008). Lawrence et al. (in press) developed an HCD taxonomy that incorporates and defines the HCD processes and practices in a way that can foster the teaching and learning of HCD in educational settings specifically for non-designers. The taxonomy includes 5 spaces: understand, synthesize, ideate, prototype, and implement. Each of the spaces include various processes such as observing, defining, proposing, evaluating, and executing. We integrate the HCD processes listed in the HCD taxonomy in geometry problems that focus on addressing social justice issues. The taxonomy allows us to apply design as an interdisciplinary perspective that can enrich the teaching and learning of geometry.

Design justice

One critique to design thinking perspectives is that the innovation cycle could continue to reinforce inequities in the world, perpetuate oppression, and erase the role of people who come up with ideas and solutions but are not named as designers (Benjamin, 2019). People who embrace a design justice approach are continuously working to overcome the barriers that designers may have in perpetuating inequities and seek to increase the agency and engagement of marginalized communities to achieve social justice through design (Costanza-Chock, 2020). New perspectives such as "design for the pluriverse" (Escobar, 2017) and "design justice" (Costanza-Chock, 2020) propose new accounts of the design process. Works like "The Black experience in design," (Berry et al., 2022) center Black designers in the design process to reclaim their contributions and extend opportunities that amplify what design is about. In our work we have used the ten principles by the Design Justice Network (n.d.) to conceive of ways in which geometry can integrate social justice issues.

OUR MODEL

Figure 1 shows our model for creating geometry lessons anchored in a design justice approach.

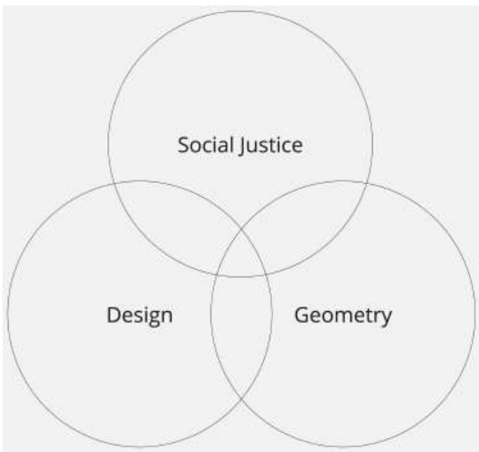

Figure 1: Model for Geometry Design Justice Lessons

In the model we consider the three fundamental perspectives in our curricular innovation. The geometry lessons must be anchored in curricular standards. In our case, we use the Common Core Standards for Mathematics, attending to the geometry standards (National Governors Association Center for Best Practices, 2010). We draw on recommendations by Goldman and Zielezinski (2022) who have adopted a four-stage process when applying design thinking to classrooms: exploring the problem space, empathizing, brainstorming, and prototyping cycles. In contrast with a problem-based approach which is often used in mathematics classrooms (see Boaler 1998; Chazan, 2000; Lampert, 2001), a design-thinking approach is more open-ended and involves students' investigation of a problem space and empathizing with those who will benefit from the design. As a result of their brainstorming, students propose possible solutions by creating prototypes that showcase a way to address the challenge. The prototypes are tangible deliverables typically the result of a collaborative process. For example, students may create a sketch, a display, or a diorama showing a prototype of their design challenge solution. In doing so, the students communicate with designers about how they addressed the challenge and use the feedback to improve their design.

Finally, the design justice approach allows us to identify problem spaces that are relevant to the students and their communities. This is important because in contrast with a typical problem-based lesson, the starting point for us in designing a lesson is not the geometry content, but the problem space. In identifying the problem space, we use the design justice principles as a starting point to brainstorm possible ways in which geometry can be applied to solve a design challenge.

PROTOTYPES OF LESSONS

Here we show examples of three prototypes of lessons that we have created. We will use the prototypes for teachers to create new lessons in collaboration with peers (González & Deal, 2019). There are some examples of design-based thinking lessons in mathematics (Bush et al., 2018), typically for middle grades. In our case, we aim that teachers construct lessons with substantial geometry content, typical of high school geometry instruction. Therefore, the prototypes can spur the teachers' creativity for seeking further connections with issues pertaining to their students and their communities.

Designing environmentally friendly and affordable house

<u>Problem</u>: Affordable housing is crucial to address important issues such as child poverty and homelessness. Event such as the 2008 mortgage crisis continue to affect affordable housing, and particularly affect underserved communities. Organizations such as Habitat for Humanity have advocated for safe and affordable housing. At the same time, there is a need for designing environmentally friendly buildings.

<u>Design challenge</u>: The state has launched a competition to design a 3-story environmentally friendly housing complex. The building should house 12 apartments and include features to be a carbon-negative home. That is, it should produce more energy than the one it consumes.

<u>Learning goals</u>: Students will produce a 3D scale model of the 2-story housing complex. Using geometric properties of solids and an understanding of cross-sections of 3D figures, students will apply their creativity to design affordable housing that suits the needs of their community. For example, the design must allocate spaces for play for youth as well as spaces for elderly folks to

interact. The design must also consider ways in which to minimize the effects on the environment such as including green spaces and solar panels.

Circular patterns and cultural experiences and identities

Problem: Geometric patterns have been used in different cultures for centuries and remain popular today. In many cultures, geometric patterns are used to decorate religious buildings, clothes, and other objects. Geometric patterns are also used primarily as decoration. For example, in art, geometric patterns are often used to express a certain emotion or mood. These patterns are also used to create illusions of movement and color in artworks. Geometric patterns can bring balance to a design by helping to create harmony between different elements. Moreover, geometric patterns can be used to express identity and values in relation to people's cultural wealth and experiences. Geometric patterns usually include many geometric figures such as circles, triangles, and trapezoids.

Design challenge: The Spurlock Museum in Urbana is excited to announce a community exhibit focusing on stories of immigration and identity. The Museum is seeking to borrow objects to display in this exhibit and help create a deeper, richer understanding of the cultural experiences and identities of community members. Objects, documents, and photographs can connect us to other people, no matter how far away in time or place. The Museum is interested in displaying objects that include circular patterns.

Learning goals: Students will design a low-fidelity prototype of an object that can be displayed in the exhibit. The object needs to have some circular patterns for visitors to empathize with the cultural experiences and identities of community members. The object needs to target a specific cultural experience or identity that is relevant to a specific community. To support students in completing these tasks, students will have access to supplementary materials about various cultural experiences and identities in their community. Students will sketch circular patterns with appropriate dimensions for the exhibit to be visible.

Designing a school garden

Problem: Designing a sustainable garden is an art. Gardens can be a form of expression, provide food security, and supply valuable nutrients to people's diets. Designing a school garden can be a viable solution to providing students with a fresh source of fruits and vegetables for their daily lunches. Designers' considerations about what is to be planted and how to arrange the plants in the garden can result in an incredible resource for students.

Design challenge: The Student Council has launched a competition to design a sustainable and nutritious school garden. The garden will increase access to food and gardening opportunities in your school community. The design must include quadrilaterals and circles for representing various garden plots. The design teams must present a scale model of the garden in the next meeting.

Learning goals: Students will collaborate in teams to explore spaces within their school for a viable garden location, research sustainable plant options, and crate a prototype garden using geometry software. Students will use ratios and proportions to create a scale model of their garden, apply coordinate geometry, and model with geometry throughout the prototyping process. Students will be empowered to bring their knowledge home to design their own gardens in their communities.

Characteristics of the lesson prototypes

The three lessons show various characteristics in our model. First, each lesson is anchored around one of the design justice principles. For example, Principle 1 states, "We Use Design to Sustain, Heal, and Empower our Communities, as Well as to Seek Liberation from Exploitative and Oppressive Systems" (Costanza-Chock, 2020, p. 190). This principle is important in the school garden problem as the food garden will result in providing new opportunities for students to connect with the Earth through faming and develop sustainable farming skills. Second, the problems intend to engage students in design-thinking processes such as building a prototype while also integrating activities where they empathize with the stakeholders who will be affected by the design. For example, in the affordable housing problem, students can interview people in their communities and learn about their needs to address their needs in their designs. Finally, the lessons require a thorough study of geometry, at times extending the geometric content in the standards. For example, in the circular patterns problems, students' examination of how circles have been used in various cultures can result in their deep examination of cultural practices (see Lawlor, 1982). We expect that students' motivation to propose solutions to the design challenges will drive their study of geometry as they will need to learn about geometric concepts to complete their designs. The more examples of lessons that we can create following this model, the better resources geometry teachers will have to engage in design justice challenges with their students.

FUTURE DIRECTIONS

The problems that we introduce here are meant to be prototypes for geometry teachers to create design challenges for their students. We are not seeking to create a geometry curriculum. Instead, we seek to empower geometry teachers to work with their students in becoming designers. Through this process, we expect students to become agents of change in their communities through applying geometry in identifying and addressing design challenges. Geometry provides unique opportunities for students to become designers as they can use geometry to achieve multiple goals such as solving authentic problems, creating beautiful designs, and appreciating how math is embedded in nature and culture. Our development of a framework for teachers to create design challenges for their students is meant to be a first step in addressing ways in which geometry can uniquely propel students' engagement in mathematics that would transform their educational experience. We believe that a multidisciplinary approach could help in making geometry relevant to address current social issues. Students who apply a design justice approach to geometry problem solving can be creators and develop a sense of ownership towards mathematics, two dimensions in the Rehumanizing Mathematics Framework (Gutiérrez, 2018). More importantly, students will be creators of new mathematics that will have a direct impact on their communities. Escobar (2017) uses the notion of "disoñar," or dream with design. We expect that when students are engaged in disoñar by using geometry they start imagining ways in which math is relevant for them and their communities to build a better future. Students' engagement in solving design problems with geometry can follow the ways in which geometry knowledge has been developed in the history of mathematics. Geometry knowledge was fundamental in real-world problems (Henderson, 2000; Joseph, 2010). Returning to the goal of situating geometry within human activities can re-engage students in appreciating and using geometry.

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VISUALIZATION AND SPATIAL VISUALIZATION IN GEOMETRY

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The disciplines of mathematics education and psychology have various definitions for visualization and spatial visualization with important implications for geometry. In this study we explored the influence of spatial visualization, as measured in psychology, and visualization on primary pre-proof geometry tasks. Results suggest that spatial visualization and visualization are related but distinct. Implications are discussed for incorporating psychological findings into geometry education.

INTRODUCTION

Ask mathematics educators what links geometry, visualization, and spatial reasoning and they might conclude that they are part of the same idea; that visualization is integral to geometry, geometry is inherently spatial, and spatial reasoning is geometry (Battista et al., 2018; Gutiérrez, 1996; Woolcott et al., 2020). Posing the same question to cognitive psychologists may initiate a discussion around the mechanisms that link geometry with spatial reasoning and visualization. Each of these discussions are underpinned by the definitions adopted in the respective fields (Resnick & Stieff, 2024).

In this paper, we conceptualize geometry in the context of primary-school curricula. From this perspective, space and geometry are woven together in curriculum guidelines (ACARA, 2023) and teaching practice (Lowrie & Logan, 2018). Primary, pre-proof geometry is fundamentally about exploring shape and space (Battista et al., 2018). These foundations may lead to the ability to visualize abstract geometric concepts when solving proofs (Arcavi, 2003) but primary school geometry largely involves imagining transformations, exploring representations, and engaging with concrete materials, all of which form the basis of *visualization* and are supported by spatial skills (Gutiérrez et al., 2018).

In parallel, cognitive psychologists find the strongest links to spatial skills within mathematics in geometry (Xie et al., 2020). However, geometry is rarely discussed in psychological literature because of the difficulty untangling geometric conventions from pure spatial skills (Newcombe, 2018). Spatial skills are clearly defined in psychology, yet they are examined in a vacuum and removed from their applications within classrooms (Resnick & Stieff, 2024). Although psychological research identifies specific spatial skills beneficial for geometry, there is minimal support for how those skills might be enacted in practice (Hawes et al., 2023). In this study we explore visualization and spatial visualization as they relate to geometry problem-solving, with the aim to understand the affordances of psychological spatial constructs for primary-school geometry.

Visualization

Visualization incorporates mental processes and the use of representational tools (such as diagrams) and has been considered synonymous with spatial reasoning (Gutiérrez, 1996). Psychological views of visualization are traditionally positioned more narrowly, for example in the process of restructuring, manipulation or transformation of objects (Ekstrom et al., 1976). Such definitions are by nature connected with mathematics content (e.g., imagining transformations on a co-ordinate grid; Battista et al., 2018). Yet often learners struggle to connect the notions of space and geometry in their mathematical practice that are intuitive to mathematics educators (Presmeg, 2014).

Seminal works in mathematics education conceptualize visualization as noun and verb, product and process; the generation of mental images and representations, and the integration of these in solving problems (Arcavi, 2003; Bishop, 1980). Underpinning these functions are the spatial skills that evolved out of psychology (Gutiérrez, 1996; Woolcott et al., 2020). However, the way psychologists conceptualize and measure spatial skills remains abstract (Resnick & Stieff, 2024). Although training spatial skills can improve mathematics achievement, it may be that these skills are so closely aligned to geometry that there is a transfer effect (Hawes et al., 2022, 2023) or that an intersection exists between the skills and their implementation within visualization that can be leveraged.

The intersection of visualization and spatial visualization

Spatial visualization, the ability to imagine and transform objects and relations (Battista et al., 2018), forms one component of Gutiérrez's (1996) visualization framework where visual or spatial elements are used to reason and problem-solve (Gutiérrez et al., 2018). Decades of research suggest that visualization is most effective when problem-solvers are supported by spatial skills and the ability to produce functional representations (Larkin & Simon, 1987; Presmeg, 2014).

Presmeg (1986) categorized visualization into types of imagery with different affordances for problem-solving and diverse solution pathways. The efficacy of these types of imagery would depend on the spatial skills of the problem-solver. For example, for pictorial imagery, strong spatial thinkers tend to construct mental images that facilitate problem-solving, while poor spatial thinkers tend to produce detailed mental images that do little to support problem-solving (Presmeg, 2006).

There is a pervasive discussion in the psychology community about the mechanisms linking spatial skills and mathematics (Resnick & Stieff, 2024). That is, whether a mechanistic or neural link exists between spatial and mathematics skills (Hawes et al., 2023) or is the link in the use of tools such as imagery, embodiment, and representation (Newcombe, 2018)? The former remains the focus of psychologists (Mix & Levine, 2018), the latter connects to the mathematical notion of visualization (Gutiérrez, 1996). Psychological research suggests that training isolated spatial skills is sufficient to improve geometry performance but proposes no reasons why or how to implement experimental findings into practice (Hawes et al., 2022, 2023). In an education framework we seek to understand how the skills or tools can be enacted in practice to develop content knowledge (Battista et al., 2018).

Context of this study

Our early scenario neglected to consider the expertise of mathematics educators. The hypothetical educators may "see through symbolic forms" (Arcavi, 2003, p. 223) when they describe all geometry as spatial. Although possessing strong spatial skills, many experts do not default to spatial strategies (Uttal & Cohen, 2012). Despite the inherent link between geometry and space, we cannot assume that students are intuitively making these connections in their mathematical practice (Presmeg, 2014). Psychological research has isolated spatial skills that support geometry understanding. For example, Hawes et al., (2022) found spatial visualization to be the most trained spatial skill with the greatest link to geometry performance. However, to date, there are few studies that explore the links between spatial visualization and implementation in geometry problem-solving using tools of visualization.

Research question: Does the function of visualization strategies differ depending on spatial visualization skills in primary, pre-proof, geometry problem-solving?

METHOD

Participants

Grades 3, 4, and 5 students ($N = 282$, age range 8 – 11 years) from two (non-government) schools in Canberra participated[22]. One school was girls-only, the other was a co-educational primary school. Consequently, the sample consisted of 81.9% females. All students obtained parental consent and gave consent to have their data included in the study.

Measures and procedure

Students completed a geometry test, a Mathematics Processing Instrument, and a spatial visualization test. All measures were administered on paper in the same order to class groups over three lessons.

Geometry Test (GeoT): The geometry items were sourced from Australia's National Assessment Program: Literacy and Numeracy (NAPLAN) for Grade 3 and 5 students. As there is no NAPLAN assessment for Grade 4, a combination of items from Grades 3 and 5 were used to provide a balanced assessment. Each grade had a unique, developmentally appropriate test (8 items for Grade 3, 10 items for Grade 4, and 13 items for Grade 5). Percentages were used for comparison purposes.

Mathematics Processing Instrument (MPI): After the GeoT, students completed an MPI (Suwarsono, 1982) which displayed the items and multiple-choice solution strategies based on previous work (Lowrie et al., 2016). Students were instructed to select the first strategy they used or use their own words. The strategies were categorized into analytical, mental imagery, and embodied. Analytical strategies involved calculation, counting or component matching. Mental imagery occurred when students reported imagining transformations in their mind. Embodied strategies included drawings or manipulating the question booklet. There were often multiple options within categories for each item. Initial inter-rater reliability was .80 and disputed ratings were discussed to reach agreement.

Spatial Visualization (SV): The items in the Paper Folding Test (16 items, 10 minutes; Lohman, 2012) displayed a series of paper folds. The folded paper was shown with holes and students had to pick the unfolded paper from multiple-choice options. This task is analogous to a psychological test (Ekstrom et al., 1976) but developed and normed on a primary school population (Cronbach's Alpha = .69).

RESULTS

Figure 1 contains descriptive statistics. Students were deemed Low ($n = 81$), Mid ($n = 105$), or High ($n = 80$) Spatial using frequencies on the SV measure to determine three relatively equal groupings. Students completed all measures above chance, with no violations to normality. Analysis of Variance revealed significant differences between the spatial groups on the GeoT, $F(2, 263) = 39.40$, $p < .001$ with High Spatial students outperforming Mid and Low Spatial students and Mid outperforming Low Spatial students. There were significant differences in mental imagery based on SV, $F(2, 265) = 5.15$, $p = .006$ but not analytical or embodied strategies, $F(2, 265) = .45$, $p = .64$ and $F(2, 265) = 2.40$, $p = .09$ respectively. High Spatial students used more mental imagery than Low and Mid students.

[22] This sample was taken from a larger Australian Research Council Discovery project [DP230100328]

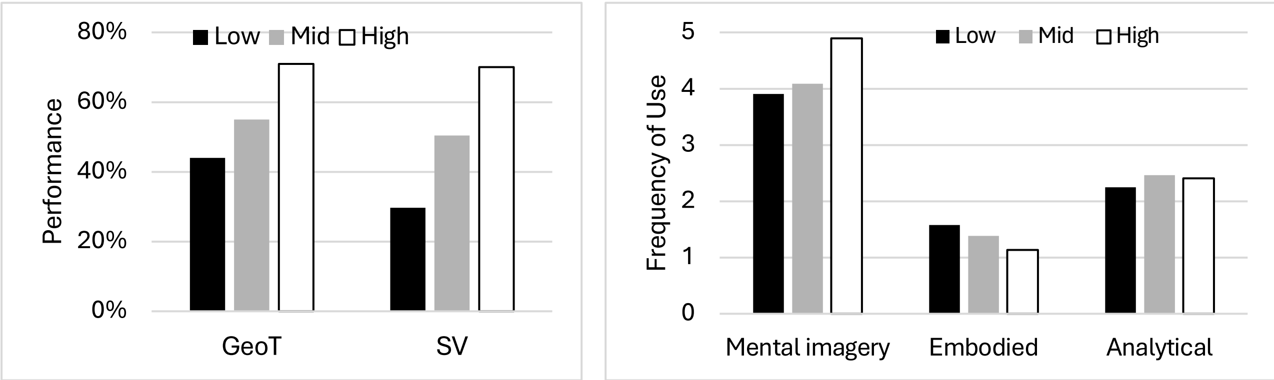


Figure 1: (a) Descriptive statistics (b) Frequency of strategy use

To determine the extent of association between GeoT, SV, and strategy use, correlations were calculated, including a strategy discrepancy variable (based on Battista, 1990) where higher scores relate to more mental imagery and lower scores relate to more analytical strategies (see Table 2).

	SV	Disc.	M.I.	E.	Analytical
Geometry (GeoT)	$.52^{**}$	$.19^{**}$	$.31^{**}$	$-.16^{**}$	0.04
Spatial Visualization (SV)	-	0.12	$.21^{**}$	$-.15^{*}$	0.06
Discrepancy		-	$.88^{**}$	$-.25^{**}$	$-.77^{**}$
Mental imagery (M.I.) strategies			-	$-.44^{**}$	$-.39^{**}$
Embodied (E.) strategies				-	-0.09

Table 2: Correlation table $*p < .05$, $**p < .01$

Performance on the GeoT had a strong, positive association with SV and the use of mental imagery but negative association with the use of embodied strategies. Spatial visualization was positively associated with mental imagery strategies and negatively associated with embodied strategies.

Figure 2 illustrates the impact of strategy dominance on GeoT as a function of SV. Low Spatial students who used mostly mental imagery tended to perform better on the GeoT compared with Low Spatial students who were more analytical. For Mid Spatial students, GeoT performance was fairly consistent, regardless of strategy dominance. For High Spatial students, those with a tendency for one dominant strategy performed better than those who used equal parts analytical and mental imagery.

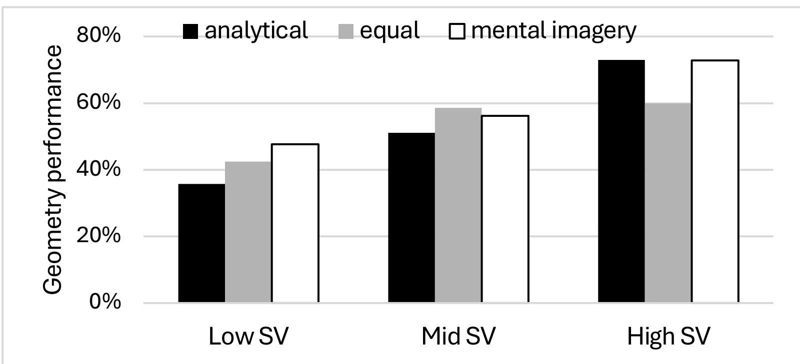


Figure 2: Geometry performance by spatial level and strategy dominance

Performance, visualization, and spatial visualization

Qualitative data was used to analyze strategy and performance on pairs of tasks (see Tables 3 and 4). Percentages relate to the proportion of students in each category[23] who correctly solved the problem.

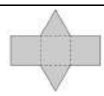
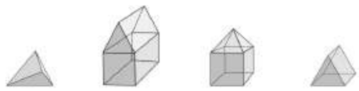
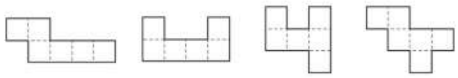
	Jack folded this piece of paper along the dotted lines to make a model.  Which of these models did Jack make? 			Which one of these is the net of a cube?  		
Correct	Imagery	Embodied	Analytical	Imagery	Embodied	Analytical
Overall	77.8%	85.7%	93.6%	41.0%	16.7%	-
Low	84.6%	66.7%	71.4%	23.1%	7.1%	-
Mid	93.2%	-	80.8%	43.9%	36.4%	-
High	98.4%	-	87.5%	58.8%	-	-

Table 3: Net items with different difficulty

Table 3 includes two net tasks but the first did not use any mathematical terminology while the second included the terms *net* and *cube*. The task with the added language convention was more difficult for all students. Spatial skills benefited students who used mental imagery. Even when students drew and cut out the shapes, they were not successful without knowledge of the mathematical language. In the task without the language convention, mental imagery was advantageous regardless of spatial skill. Success rates for analytical were still high, these involved matching components which, although methodical, still relied on visual processing, hence the greater success amongst High Spatial students.

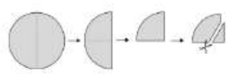
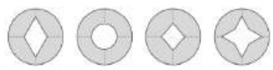
	This shoe print was found in the sand.  Which shoe made the print? 			Mike had a circular piece of paper. He folded it in half twice and cut a piece out as shown.  How will the piece of paper look when he unfolds it? 		
Correct	Imagery	Embodied	Analytical	Imagery	Embodied	Analytical
Overall	60.0%	46.2%	55.9%	50.6%	-	50.6%
Low	29.4%	-	29.4%	43.8%	-	54.2%
Mid	61%	33.3%	60.8%	40.7%	-	51.4%
High	86.1%	-	78.4%	70%	-	95.7%

Table 4: SV tasks with similar difficulty, different intent

Table 4 presented two items that required skills associated with spatial visualization (i.e., reflection and paper folding) and had similar difficulty (based on success) but the task objectives were different.

[23] Percentages were not reported when less than 5 students in a category used that strategy.

The reflection task had low success rates for Low Spatial students and those who used embodiment. Mid Spatial students were equally successful whether using mental imagery or analytical strategies. High Spatial students, although still successful, were better off with a mental imagery strategy.

Students had greater success on the paper folding and cutting task when they used an analytical strategy, despite analytical strategies on both tasks involving matching components. High Spatial students had a significant advantage on this task which was closely aligned to the spatial measure. It is possible that High Spatial students had efficient spatial strategies for both tasks. A slight advantage was evident for Low Spatial students on this geometry task compared with the pure spatial measure.

DISCUSSION

In this study we examined the intersection of visualization, in the form of strategies, and spatial visualization with geometry task performance amongst primary students. As expected, students with strong spatial skills performed better on the GeoT (Xie et al., 2020). Broadly, SV was more strongly associated with GeoT than with the use of visualization strategies. Previous work has considered these terms synonymous (Gutiérrez, 1996; Gutiérrez et al., 2018), but these results suggest that while visualization may be supported by SV skills, there is a degree of dissociation between the concepts. Overall, mental imagery strategies were the most frequently reported, followed by analytical and embodied. Strong spatial skills were associated with greater use of imagery and less embodiment.

Discrepancy scores revealed much greater (positive) scores for High Spatial students compared with Low and Mid, suggesting a tendency towards mental imagery over analytical strategies. Correlations revealed no significant relationship between SV and discrepancy. We interpret this to mean that although high SV scores and more mental imagery strategies were associated with better geometry performance, they were not inextricably linked for all students. That is, High Spatial students who tended to favor either mental imagery or analytic strategies were more successful on the GeoT than those who used the two strategy types equally. While for Low spatial students, the more mental imagery strategies they used, the greater their success on the GeoT. There was no relationship between analytic strategies and GeoT or SV. Literature would suggest that proficient students would resort to analytic strategies (Lowrie, 2020) but in this study, often the most effective and efficient strategies were mental imagery except for a small number of tasks and subset of High Spatial students.

Item-based comparisons

Item analysis revealed some important considerations often missed when focusing on global scores. Firstly, mathematical language and conventions are important. As predicted by psychologists, geometry conventions can inhibit students from demonstrating their transformations skills without sufficient content knowledge (Newcombe, 2018). This was evident in the net tasks. Students of all spatial levels were successful in performing the required transformation when the task did not include the interpretation of mathematical language. As a first step, students need opportunities to develop and demonstrate their spatial skills that are not predicated on their content knowledge. Subsequently, students need support to link their conceptual, spatial knowledge with their geometric understanding.

The largest proportion of strategies were visualization, and most often mental imagery. This is not surprising given the nature of pre-proof geometry tasks. Visualization requires the integration of multiple representations, either from the page or screen to a mental transformation or from one type of physical representation to another. Spatial skills can support the transfer and integration of this information but they are not sufficient (Presmeg, 1986, 2014). If the mental representation of the task is not functional, translating into a different representation may incorporate erroneous information and leave students without a useful reproduction (Larkin & Simon, 1987). For each of the analyzed tasks, spatial skills still underpinned the greatest variance in performance, but their influence was dissociated from visualization strategies.

Some geometry tasks, particularly in primary school, are closely aligned to spatial measures, like 2D to 3D conversions. These were the items in this study with the strongest evidence for the influence of spatial skills. However, in some of these tasks, Low Spatial students were outperforming their Mid spatial peers, even using spatial strategies. This shows the benefit and opportunities for incorporating the geometry context in spatial tasks (Resnick & Stieff, 2024).

What can we learn from this?

Psychology offers tools to explore and build student spatial visualization in ways that are abstract but powerful for supporting geometry achievement (Hawes et al., 2023). This study has demonstrated the strength of the relationship, on top of learned visualization strategies. Visualization is more common in primary schools before complex geometric proofs become part of standard practice. Therefore, it is an ideal time to build spatial skills that can support future geometry learning (Woolcott et al., 2020). However, we cannot assume that students are going to intuitively connect spatial skills with geometry content, or effectively use visualization strategies despite the inherent links with geometry without considerable support (Gutiérrez et al., 2018; Lowrie & Logan, 2018; Presmeg, 2014).

Limitations and future directions

There were few tasks that could be solved routinely with computations. For most items, analytic strategies involved conceptual knowledge of the structure of objects and inferences about what occurs after prescribed changes, both still required visual processing (Larkin & Simon, 1987). This clearly demonstrates visualization as “a necessary component for teaching and learning in geometry” (Gutiérrez, 1996, p. 1-4), and both a product and a process (Arcavi, 2003; Bishop, 1980). However, visualization is not a direct replacement for spatial skills. Future work could explore how these trends hold up across geometry tasks that require more computation.

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ICMI Study 26
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DISCOVERING GEOMETRY IN AFRICAN ETHNO ARTIFACTS

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The preschool experience opens an avenue for developing spatial reasoning and introducing geometrical concepts. This paper describes a design study, a curricular experiment – implemented at the University of Belgrade in Serbia with prospective preschool teachers. The research project aimed to document the development of students' math pedagogy when studying potentials for preschoolers' exploration of mathematical ideas in the context of the Museum of African Art. Through exploration of ethno artifacts students had the opportunity to learn to reflect on the culturally defined presence of geometrical ideas in them. Here we discuss findings related to students discovering geometry while studying ethno art. The findings are based on a sample of 46 prospective preschool teachers' interviews. The analysis reveals that the students show sensitivity to the presence of geometrical ideas in ethno artifacts from Africa and recognize opportunities for triggering the development of geometrical reasoning in preschool children.

INTRODUCTION

Geometry is all around us. This statement is one of the ideas of the Methodological class at the Faculty of Education for prospective preschool teachers in their final year of studying to become teachers for children aged 3 to 7. It directs students to search for the presence of geometrical ideas not only in nature but also in elements of culture (art, architecture, customs, home ethno products, etc.).

Research indicates that student achievement in the field of geometry and measurement is weaker than in other areas of mathematics represented at school (Milinković et al., 2017, Outhred & Mitchelmore, 2000, Tan-Sisman & Aksu, 2016). Understanding geometrical concepts is a complex construct that includes several elements that are not always simultaneously in focus (visualization, measurement, relations, or transformations (Sinclair et al., 2016). The main factor of limited achievement in learning geometry in primary grades is too much focus on procedural knowledge (e.g. learning formulas for calculating perimeter, area, or other quantitative characteristics of geometrical objects) instead of putting emphasis on developing conceptual understanding of basic geometrical concepts and attending to the development of spatial reasoning through observing similarities and differences between objects and their position in space (Tan-Sisman & Aksu, 2015). Some elements of geometrical thinking can be developed in preschool age. An early start of learning geometry in preschool is promising to support better achievements in school geometry.

The major objective of the preschool program in Serbia is to support the holistic development of young children situated in culture (MPNTR, 2018). Early intuitive development of spatial reasoning is indicated in research on early mathematics development. (Piaget,1976, Clements, 2003, Clements & Battista, 1992, Smith et al., 2011). The preschool curriculum is grounded in the same socio-constructivist position. Like other math domains, the development of initial geometrical concepts

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occurs in a shared socio-cultural environment. Ethno museum with cultural artifacts from Africa is a such context.

THEORETICAL BACKGROUND

Ethnomathematics is a science studying mathematical practices and transmission of mathematical knowledge of a culturally definable group (Fossa, 2023, D'Ambrosio, 1998). The last years have shown fast growth of ethnomathematics studies in geometry teaching yet the number is still modest; only 37 studies were conducted between 2011 and 2021 (Kyeremaeh et al., 2023). For example, Patricia Elena Ceballos Pimentel (2012) explored the possibility of teaching geometry based on the drawings of the Mapuche, a group of indigenous peoples from central Chile. In response, Fossa remarked that identifying triangles or calculating perimeters and areas in the drawings imposes academic mathematics on the native work. He points out that instead a Mapuche geometry or any other group geometry should be explored in the original cultural context. Our interest brought us to explore geometry within a collection of African artifacts. It provided a window into the significant presence of geometrical ideas in African culture.

The way prospective preschool teachers are educated reflects the way we expect them to work on the development of mathematics concepts for children at preschool. We operate from a social constructivist position (von Glaserfeld, 1990). The learning process is modeled by culture and evolves and changes in interactions of individuals with other people and cultural tools (Milinković & Bogavac, 2011). The preschool curriculum recommends the project approach as a dominant. A project is an in-depth investigation by children of a topic that is worthy of their time, attention, and energy (Katz & Chard, 2000). A project involves three phases selection, investigation, and reflection. Using cultural tools and products (such as what are symbols, objects, and customs...), as well as different strategies in building meaning, children learn in daily participation of the community's practices in which they are involved (family, kindergarten, immediate environment). The project approach provides opportunities for children to represent this learning through the construction of personally meaningful products. Teachers are expected to facilitate the process in the project including all phases of the project, planning, exploration, and culmination of the project, towards shared outcomes or the creation of artifacts, providing opportunities for children to explore in small groups, children are involved in investigating authentic objects and resources and reflecting on the outcomes of the project. Contextualized problems function as a source for the learning process. For example, the idea of searching for mathematical concepts in art was explored earlier (Grzegorczyk & Stylianou, 2006). The aim of such activities is for the preschool level to develop predispositions for searching for mathematics and recognizing mathematical ideas in their surroundings.

METHODOLOGY

During the summer semester of the final academic year, within the Methodology of Developing Basic Mathematical Concepts course for prospective preschool teachers, the topic of the experimental project-based program was African culture. Our students were already familiarized with the basics of the integrative approach and have designed preschool activities exploring geometric objects (line, point, surface), geometric shapes and solids, spatial relations (position, size, visual comparison of shapes, shapes of objects, symmetry, patterns). The aim of the program was for students to further develop skills to identify/create opportunities to recognize and connect mathematical ideas and grow awareness to different cultural contexts. The experimental program started with a guided tour of the Museum of African Art. The student's task was to come up with ideas for initiating research and playful activities with children based on what they observed and learned in the museum. The students created didactic tools to be used in game-like activities with young children. These didactical tools were displayed at the final exhibition.

A sample of 46 students was involved in the study.

The part of the experimental program reported here involved a visit to the Museum of African Art, situated in Belgrade. *The Museum of African Art – the Veda and Dr. Zdravko Pečar Collection* (https://www.mau.rs/en/) is exclusively dedicated to presenting and interpreting the art and culture of Africa and is the only specialized institution of its kind in Southeast Europe. The Museum exhibits a collection of cultural artifacts from different parts of Africa. The collection includes masks, sculptures in wood, bronze sculptures, bronze weights, ceramics, musical instruments, fabrics, jewelry, and items of everyday use. "Geometric representations exist in all cultural groups and their meaning may be different from one specific culture to another", as it was noted in the discussion document for this conference, and African cultural artifacts are no exception from this observation. The Museum guided tour included information about how the objects were originally used, stories related to the history of the African continent, and particular artifacts and different cultures living in Africa. For example, the tour guide remarked that it is visible that African masks incorporate "geometrics", the symbolism of what each mask represents, the fact that a person under a mask "loses personal identity", when they were used, who used them (tribal chiefs, sorcerers, or ritual dancers), the symbolism of animals on masks, and the mask's connecting the heavenly and the earthly.

Figure 1: Ntomo mask, Bambara mask, and Chi-wara mask, from left to right

The guided tour finished with a workshop on the traditional African game Mankala. The visit was followed by a wrap-up lesson at the Faculty. The final part of the project was designing a didactical tool activity ending with the exhibition titled *Discovering Mathematics on African Pathways*. Inspired by a visit to the Museum of African Art, the students came up with ideas for initiating research and playful activities with children. The didactical tools that were exhibited at this exhibition were used in activities with children. These tools, based not only on the museum artifacts but also on students' investigation of the topic, demonstrated the creativity and scope of the students' ideas.

Based on the goal of the research, the following tasks were formulated:

1. Determine the scope of geometrical ideas recognized in African cultural artifacts.
2. Analyze how much of and in what way geometrical concepts were present in students' didactical activities and products.

The museum visit provided context for the development of project activities inspired by the visit to the Museum and the implementation of the activity during in-preschool practice month. Major sources for researchers were 1) interviews, 2) students' practice diaries, and 3D didactical game models. The interviewer used an interview protocol consisting of nine focus questions. Respondents were asked to reflect on 1) their memories of the museum visit and 2) their project (plan of activities and results of the activities). All individual interviews were recorded and transcribed for qualitative analysis. In addition, students' practice diaries were used as evidence supporting claims made in interviews. The obtained data were processed by the method of statistical and descriptive qualitative analysis.

RESULTS

To begin, we will look at what students find remarkable at the exhibition and to what extent it is reflected in their choice of activity. The results show that Masks and the Mancala game earned the most attention but masks and fabrics were the two most explored objects in activities (Table 1).

Artifacts	Remembered	Used inactivity
Masks	27	13
Fabrics	11	12
Mancala	24	5
Jewelry	5	3
Wooden Sculptures (humans, animals)	10	2
Bronze sculpture and weights	0	1
Ceramics	1	1
Musical instruments	2	1
Items of everyday use	3	0

Table 1: Artifacts observed and used

An example of the way artifacts can be used in the project is a worksheet with tasks involving photographs of artifacts from the museum (Figure 2). Children were asked to recognize a pattern and count the number of appearances of the pattern. Others involved children in creating similar artwork such as a string of beads.

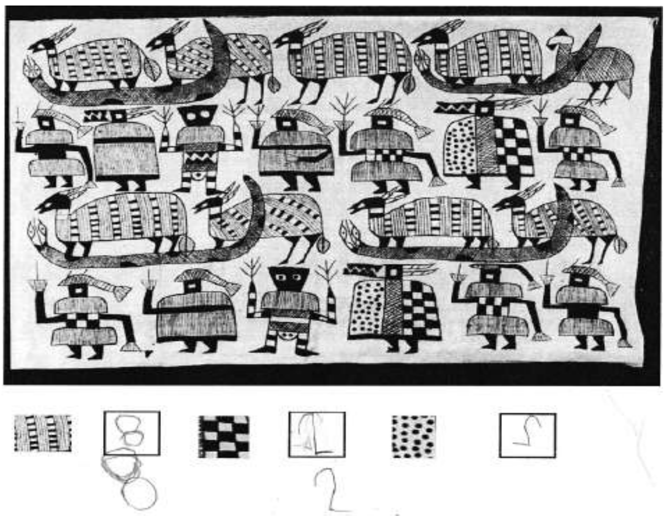

Figure 2: A pattern task from a worksheet

Another student used a bird fabric motif to explore the idea of a pattern. She created a drawing of birds with patterns of colors and discussed with children about these patterns and also the patterns of the birds seen at the museum. So, as the student wrote in her diary of practice the dialog with children included the following questions:

> 1. What kind of bird is this? 2. Did you notice anything on her body repeating itself? If yes, what? 4. How do colors repeat? In what order? 5. How are these two birds different? 6. What is the tail of another bird? Compare it with the first. 6. What is their difference, and what is their similarity? 7. How are colors repeated? 8. How is this bird different from the previous two birds? 9. What's her tail like? 10. What does the battle consist of? 11. How do they recur?

Children drew birds (Figure 3) with geometrical patterns and discussed the differences between birds. The majority of five-year-old children created simple two-color patterned bird's tails while African birds have had five colors. After finishing their drawing, young children described patterns, which colors were used, what was the order of colors, and how many elements the pattern consisted of. At the end, they have an exhibition of works and plays.

Figure 3: The African bird and children's productions

Our first research question was to investigate how much geometry students recognized in the objects displayed at the exhibition. Also, we investigated how many geometry-related ideas they attended to in the projects with young children. Several different ideas were recognized by a considerable number of students during the observation of the exhibition. Notable fewer of these ideas were discussed later, in projects with children (Table 2).

Geometrical concept	Observed concept in ethno artifacts	The concept incorporated into activities with children
Patterns (geometric pattern, number pattern)	46	35
Geometric shapes (geometric 2D-figures)	45	22
Symmetry	43	6
Classification	15	5
Line	24	4
Spatial reasoning (relations, position of objects)	10	4

Table 1: Observed and followed in projects' geometrical concepts

While some students mentioned only one math concept, some recognized mathematics in many places. For example, student Sonja reflected on many ideas brought about in her project with children.

Sonja: "We a) arranged rows of patterns can be observed on animals (on fabrics). Some animals can observe lines and circles alternate and create a pattern. Also, geometric concepts, such as circles and lines are present. b) On some fabrics the pattern is created by linearly changing a square, a square consisting of four triangles, and an octagon. Symmetry can also be seen in the painted shape below the pattern, c) symmetry can be observed because if we divide each of these wooden spoons into two halves longitudinally, both halves would be identical. An elliptical shape and a circle shape are observed, and d) symmetry in the form of a mask is observable; if we do not ignore the lack of shells on one side, i.e. half of the mask. The pattern is made by arranging the shells in an ordered sequence, linearly and cyclically, e) geometric terms (circle, triangle, lines, rectangle) are observable, patterns are observed on the bodies of animals, f) symmetry and pattern consisting of squares and rectangles are observed, e) symmetry is observed if we cross it, a pattern is observed, a repetition of the field."

For the second research question, we asked students to conceive of the possibility of extending the project and how would he/she proceed with the project. The majority of students responded positively that their project could be prolonged but few of them were thinking about further development of math concepts, and even fewer remembered geometry concepts in a perceived extension of the project (Table 2).

Choice of domain for extension of the project	Topics
Geometry	basic geometry concepts (line, curve, point) symmetry geometric shapes
Numbers	
Arts, Science, Language	
Not developing project	

Table 2: Question: In which direction would you further develop the project?

Ana envisioned moving from Africa to other cultures while Petra thought about ways to further explore artifacts from the Museum of African Art, primarily focusing on patterns.

Ana: "In addition to getting to know Africa, I would introduce children to other cultures.

Petra: "Another student said: I would take the children to the Museum of African Art, to look at the sculptures and masks together. They would still deal with African culture. They would observe them - make sculptures out of dough and clay - draw patterns, make a series and a pattern."

Tara: "In addition, we could continue to deal with Africa, their instruments, masks, clothes, etc. We could visit the museum with the children and see what interests them the most because that will be the easiest way to organize the activity."

Most of the prospective preschool teachers in the class, similarly to Petra remarked that they would go with children to the Museum, to identify what children find interesting to explore further.

The final project activity resulted in a collection of 33 models of didactical tools which were displayed in an exhibition at the Faculty building. For each product, a group of students who designed it explained what they meant to be the purpose of the tool. The analysis showed that students most frequently expected that playing with the tool would contribute to children's formation of concepts of geometrical shapes and symmetry. Their explanations however lack elaboration or complexity. Notable, all references to symmetry were made concerning mirror symmetry only. We show two examples of games students created inspired by Masks (Figure 4). In both, the expectation was that children would identify and name shapes. Similarly simple expectations repeatedly occurred in games involving patterns. All references to geometrical patterns referred only to linear patterns consisting of geometrical shapes (triangle, square, rectangle).

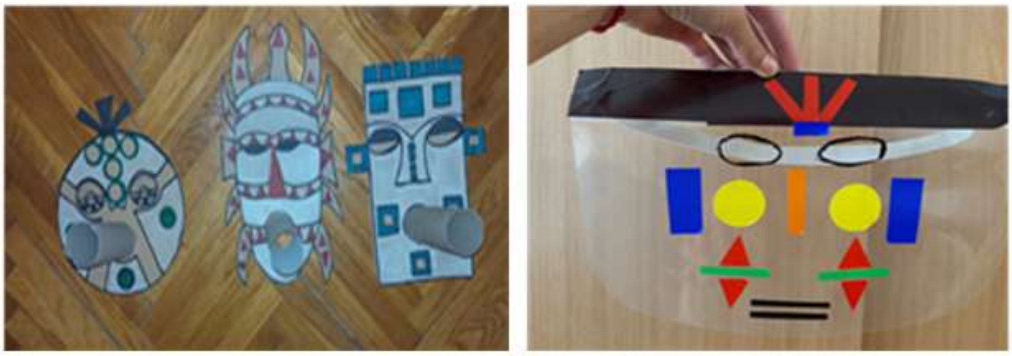

Figure 4: Masks as a didactical tool

DISCUSSION

While most students recognized mathematics in art pieces displayed in the Museum, some did not find a way to include directly any of the observed ideas in their plan of activity. This may be due to a lack of experience in project planning. Over 90 percent of students recognized patterns, geometric shapes, and symmetry in African exhibits. However, only the idea of the pattern was frequently explored in activities. Only about half of the students (47%) had activities involving geometric shapes and numbers. Even less, about 13 percent of students have designed activities involving symmetry.

CONCLUSIONS

The study explored the presence of geometrical concepts in projects inspired by African ethnic artifacts. Learning about African culture, not only raised their awareness of this culture but also helped them to get various creative ideas on how to incorporate elements of African culture in math activities with preschoolers. Students did recognize geometrical ideas in the artifacts (art pieces, games, legends, etc.). On the other side, the recognition of the potential for the development of geometrical concepts was limited in scope and depth. Inspired by a visit to the Museum of African Art, the students came up with ideas for initiating research and playful activities with children. The didactic tools that were exhibited at this exhibition were practically used in activities with children. These didactical tools based not only on the museum artifacts but also on students' further investigation of the topic, demonstrate the creativity and scope of the students' ideas.

"Geometry is all around us" is one of the basic premises on which preschool teachers should be working with young children. But it requires building up not only sensitivity to the culturally delineated world of geometry but also the geometry content knowledge of prospective teachers. We plan to pursue the experimental program in the future.

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INVESTIGATING GEOMETRIC KNOWLEDGE IN THE ART OF SISAL TAPESTRY IN A LOCAL COMMUNITY THROUGH ETHNOMODELLING

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Universidade Federal de Ouro Preto

This research was conducted in a Professional Master's Degree Program in Mathematics Education, at the Universidade Federal de Ouro Preto, in Brazil. The main objective was to develop an ethnomodelling based investigation on the art of sisal tapestry developed by the members of a local community of Cachoeira do Brumado, in the municipality of Mariana, in the state of Minas Gerais, Brazil. This study aimed to translate local geometric practices through the proposition of a pedagogical action from the perspective of mathematics teachers, for the development of geometric contents for Middle School students. The results of this investigation show that participants in this research (teachers and artisans) value and respect local know-how related to the art of tapestry and the making of sisal carpets, as they became aware of the relevance of their own experiences, which made it possible to understand the connection between etic (school) and emic (tapestry) knowledge in the teaching and learning process in mathematics.

INITIAL CONSIDERATIONS

There is a need to find alternative ways to attract students to learn through the use of pedagogical actions for the teaching and learning process in mathematics through the contextualization of problems and situations involving practices that occur in everyday life, which are permeated with the knowledge and practices developed by members of distinct cultural groups. At all times, these members are comparing, classifying, quantifying, measuring, explaining, generalizing, inferring, modelling, and, in some way, evaluating by using material and intellectual instruments that are specific to their own culture (D'Ambrosio, 2016).

It became important to plan a teaching and learning process in mathematics that encompasses cultural and everyday thinking, while at the same time, to take into account experiences lived by students in their own communities. Given that the experiences experienced by students in their own sociocultural context are often overlooked, this study was interested in how urban and rural communities, located close to the schools they serve, can positively influence the performance of teachers through interventions and pedagogical actions in the classroom.

In this regard, D'Ambrosio (1998) states that geometry education is related to a scientific field of knowledge production linked to the process of understanding geometric knowledge found in the daily life of the members of distinct cultures through the use of procedures, techniques, and practices developed locally. Similarly, Jones and Tzekaki (2016) affirm that geometry education is moving away from focusing on formal spaces and shapes toward a growing role for spatial reasoning and geometric problems encountered in daily life.

We investigated the process of ethnomodelling as the art of translating local geometric practices (emic) related to tapestry that is developed in a rural community in the state of Minas Gerais, Brazil

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through the development of pedagogical practices that enables students to develop their own ways of understanding geometric concepts. It is important to highlight that this approach aimed to develop local (emic), global (etic), and dialogical (glocal/cultural dynamism) ethnomodels, which were discussed with mathematics teachers in this community. The main objective was related to the performance of these teachers in developing a pedagogical action using ethnomodelling for the development of geometric content for their students emphasize the contents related to geometric knowledge that appear most prominently in the art of tapestry, mainly in relation to the making of sisal carpets.

In accordance with this context, the main objective of this study was based on ethnomodelling used to investigate and related to the art of tapestry developed in the local community by aiming to translate local geometric practices through a pedagogical action that is locally contextualized. It is important to highlight that the specific objectives of this study are to develop curricular activities based on the daily practices of a community of sisal carpet artisans by aiming to show how geometric knowledge can be influenced by actions carried out in daily activities.

Therefore, the problem statement proposed in this study was justified by how geometry education is full of ideas, procedures, techniques, and general strategies used to develop powerful and meaningful mathematics in an attractive and relevant way. It is of great value to discover the factors and characteristics that determine the way teachers act in the use of modelling, ethnomathematics, and ethnomodelling in the pedagogical action developed by teachers, with the aim of enabling the development of a teaching and learning process in mathematics that is innovative, attractive, useful, and efficient for all students and educators.

In this study, researchers proposed a different methodology for the teaching and learning process in mathematics, in relation to the dialogic approach of ethnomodelling. Thus, this research is based on the connection of a ethnomathematics perspective to the cultural aspects of mathematical modelling, which forms the primary assumption of ethnomodelling. Hence, researchers investigated how might possible contributions that ethnomathematics and modelling through the dialogical approach of ethnomodelling play in the teaching and learning process in relation to the making of sisal carpets by members of a local community. This particular pedagogical action is related, in particular, to academic geometric practices related to the school curriculum.

For example, the recognition of different types of polygons, angles, straight lines, symmetry, acts of both reflection, rotation and translation in addition to working with the units of measurement, ratio and proportion, through ethnomodelling, where the geometric contents of 7th grade middle school that made up this ethnomodelling pedagogical action and related to the recommendations of the *National Common Curricular Base* (Brasil, 2018), which is a document of a normative nature that defines the organic and progressive set of essential learning that all students must develop during their education.

It is noteworthy that the theme proposed in this study enabled the teacher-researcher to understand the need for, and to seek the data and information necessary for the development of a teaching and learning process that allowed him to contribute to the development of teaching practices linked to the proposed innovative pedagogies.

Thus, this study presented in an interactive way, the possibilities for bringing students closer to local and school geometric language, as well as its use in solving everyday problems. In this context, the activities developed in this research were relevant for mathematics teachers, participants in this study, to understand the relationship between the different contents studied in the school mathematics curriculum. These activities were also necessary to verify whether geometric contents have notions and concepts that can be applicable in daily tasks carried out in everyday life, such as, for example, making sisal carpets.

In this regard, Rosa and Orey (2017) state that research conducted in geometry education indicates that the geometric contents taught in schools, as well as the way in which geometry is taught, have not kept up with cultural and social developments. and technology that today's society craves. Therefore, conducting this study is also justified by the development of a contextualized pedagogical action that had meaning for students, as well it met the new demands of the teaching and learning process in mathematics.

In this context, the mathematical practices used in the daily lives of members of distinct cultural groups are associated with the development of their own techniques, strategies and procedures, which are the techniques developed by sisal carpet artisans, as there are different ways of solving problems that are inherent to the tapestry production process, which can be denoted by different ways of quantifying, classifying and measuring, which are related to the emic (local) knowledge developed by members of this specific cultural group.

According to Mendes (2008), geometric concepts are found in distinct cultures, as well as in art, sculpture, tapestry, and architecture. Similarly, geometric shapes appear in most carpets and ceramics made by nomadic populations that we know and whose patterns have been given new meanings in other societies for millennia. The varied designs related to the art of Persian tapestry, for example, indicate the origin and, often, even the age of the pieces, which highlights the main characteristics of this particular group of carpets. Experiences related to the connection between geometry and activities created through the use of crafts enable students to identify mathematical geometric concepts present in these cultural artifacts, such as embroidery, ceramic, and carpets.

Therefore, in accordance with the legal guidelines that guide Brazilian Basic Education, Rosa and Orey (2017b) emphasize that one of the main objectives of Ethnomodelling is to connect the cultural aspects of Mathematics with the school mathematical concepts to translate problems, situations, and phenomena present in the daily lives of members of distinct cultural groups to other mathematical knowledge systems, such as school and academia.

THEORETICAL FOUNDATION OF ETHNMODELLING

A search for innovative methodologies is necessary to record historical forms of geometric ideas, procedures, and techniques that occur in diverse sociocultural contexts to take advantage of an emerging *glocalization* process of mathematics and its cultural aspects by exploring and documenting geometric practices rooted in diverse cultural contexts, which recognizes the connection between geometric phenomena and culture.

In this regard, ethnomodelling differs from classical definitions of mathematics, which are often perceived as a form of universal knowledge that is applicable everywhere that is based on the theory of certainty and was imposed through capitalism and colonialism. Because it studies and investigates geometric knowledge developed locally and globally into a body of knowledge that is identified as glocal geometric practices.

In general, many scholars of geometric practices seek to understand traditions, ideologies, cosmologies, and beliefs of a particular culture by advocating the relevance of the complementarity between local (emic) and global (etic) geometric knowledge into a glocal (dialogic) approach through the development of cultural dynamism. In this process, the anthropological terms emic and etic are used as an analogy between *observers from the inside* (emic, local) and observers from the outside (etic, global) of a given culture (Orey & Rosa, 2021).

The global (etic) approach is related to extrinsic worldviews of the outside observers in relation to, in this case the geometric knowledge developed by the members of distinct cultural groups. It refers to an interpretation of the characteristics of other cultures from the application of analytical categories developed by external observers. This approach forms the external view made by observers, who are *looking* at a specific culture *from the outside perspective*, in a cross-cultural, comparative, and prescriptive stance, and which can be equated with objective explanations of sociocultural phenomena (Rosa & Orey, 2019).

Local (emic) approach aims to understand the characteristics of a given culture based on the intrinsic references developed by its members. It is related to their own worldview and cosmovisions regarding to the development of their geometric knowledge. This approach is considered as an internal view of members who are *looking* at their own culture *from the inside perspective*, in an intracultural, particular, and descriptive stance, which is identified with the understanding of the subjective experiences that they acquire, develop, accumulate, and diffuse through history (Orey & Rosa, 2021).

These approaches are related to geometric ideas, procedures, and practices that are linked to everyday phenomena, and are organized, interpreted, and evaluated through the elaboration of ethnomodels, which are representations of systems taken from the reality of members of distinct cultures. They enable these members to communicate, diffuse, and transmit their mathematization processes across generations by helping them in attributing meaning to the sociocultural context in which they perform their daily activities. Thus, ethnomodels are small units of information that link the development of geometric practices developed by these members who use their own sociocultural heritage to holistically understand their surroundings (Rosa & Orey 2019).

By applying ethnomodelling, researchers seek to value and respect ethnomathematical knowledge (local, emic), and in this case, the interpretations and contributions to geometric systematization through modelling (etic, global) in a glocal (dialogic) manner. In this context, Rosa & Orey (2021) state that ethnomodelling is an alternative methodological approach that is suited to different sociocultural realities, which proposes the rediscovery of geometric knowledge systems developed, accumulated, adopted, and adapted in other cultural contexts. Thus, this study focused on the glocal (dialogic) approach of ethnomodelling and how the interaction between local (emic) and global (etic) approaches promoted the understanding of cultural dynamism through the elaboration of ethnomodels.

METHDOLOGICAL PROCEDURES

Again, the study aimed to translate local geometric practices through the proposition of a pedagogical action for the development of geometric contents for students of the 7th year of Middle School, from the perspective of three mathematics teachers and two artisans who make sisal carpets. A pedagogical action based on ethnomodelling was developed that addressed geometric practices related to the school context, such as the recognition of different types of polygons, angles, lines, symmetry, reflection, rotation, and translation, in addition to working with the units of measurement related mainly to length, perimeter, area, volume, capacity, ratio, and proportion. Below is an example of the activity proposed in this pedagogical action.

> *After manufacturing the sisal carpet and the completion of the finishing process, the measurements are checked and the area of the carpet is calculated to determine the sales value of the craft. It is known that, currently, each sisal carpet is sold for R$ [24]65.00 for each square meter (m²).*

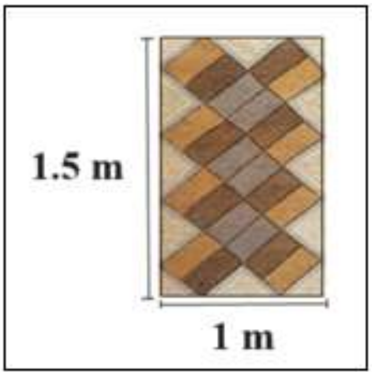

Figure 1: Sisal Carpet made by the local artisan (Source: Authors' personal file)

Answer the following questions:

a) What is the perimeter of the sisal carpet?
b) What the area of the sisal?
c) What is the final value of this carpet?
d) If every square meter of the carpet needs, on average, 2.5 kilograms of sisal, how many kilograms of this material, on average, did the artisan spend to make this carpet?
e) Regarding geometric shapes, how many parallelograms are there? How many rhombus? How many triangles? How many rectangles? How many polygons? How many 90º angles?
f) Regarding transformations in the plane, how many lines of symmetry are there? Are there rotation (tun), reflection (flip), and translation (slide).

This pedagogical action provided a differentiated educational opportunity for participating teachers. In this study the researcher used the local (emic) geometric knowledge and practices of the community of artisans of sisal carpets in connection with global (etic) school geometric knowledge for the elaboration of geometric activities that dialogue with the teaching and learning process in mathematics.

The problem statement of this study was defined by the research question: *How can ethnomodelling contribute to the proposition of a pedagogical action, to the development of geometric and geometric contents, based on the art of tapestry of a local community, from the perspective of mathematics teachers of 7th year Middle School?* Data collection was conducted through two questionnaires

[24] The Brazilian real or reais (R$)is the official currency of Brazil that is subdivided into 100 cents. In 01/09/2024, the currency exchange rate is U$ 1.00 = R$ 4.90.

(initial and final), three modules of activities, a focus group, notes in the researchers' field diary, and two semi-structured interviews.

This research had a qualitative approach by using the methodological design adapted from the Grounded Theory, which allowed the analysis of data collected through open coding, as well as the interpretation of results through axial coding, which enabled, respectively, the identification of preliminary codes and conceptual categories. In this adaptation, the selective coding, the identification of the central category and the writing of an emergent theory were not used because the researchers' main objective was to answer the research question proposed to this study.

RESULTS AND DISCUSSION

The main objective of the pedagogical action proposed in this study was to bring together the knowledge and practices originating from the tapestry weaver culture with the geometric knowledge developed in the school environment through the development of a pedagogical action in which the teachers reflected on the curricular activities proposed in this study. action. In this study, ethnomodelling is considered a pedagogical action that aroused the interest of the participating teachers, helping them to structure the way in which they analyzed the problem situations proposed in the activity blocks, as they expanded the geometric knowledge learned in the school system.

According to the interpretation of the results obtained in this study, it is inferred that the contextualization made it possible to raise awareness of the practical application of geometric and geometric concepts in problem situations related to the production of sisal carpets. Continuing with the interpretation of the results of this study, the blocks of activities carried out during this fieldwork contributed to the understanding of the applicability of geometric and geometric contents in the daily lives of tapestry artisans. This approach can minimize the fact that geometry is presented as an obstacle to students' learning because of the way it is taught in classrooms, given that its contents are memorized through the decontextualized application of exercises unrelated to everyday life. from the students.

In this way, the pedagogical action of ethnomodelling enabled the development of cultural dynamism between the students' school culture and the tapestry weaver culture by providing the opportunity for the contextualization of geometric and geometric concepts studied in the geometric curriculum. This demonstrates how through ethnomodelling, teachers and students develop essential tools so that they can reflect on reality, transforming it with the aim of achieving collective good and improving quality of life. of members of their communities (Rosa & Orey 2010).

According to the interpretation of the results obtained in this study, the participating teachers positively evaluated the work carried out while conducting the blocks of activities proposed for this study in relation to the importance of mathematics for carrying out daily tasks related to the activities of making sisal carpets.

For example, these participants evaluated the craft activities related to tapestry as useful and very good, as they witnessed a significant relationship between the problem situations proposed in the blocks and the geometric and geometric concepts present in the production of sisal carpets. However, the participants also highlighted how this craft is characteristic of this district and that it should be valued and recognized.

The findings support the need for teachers and students to be inserted in a learning environment that allows the use of geometric knowledge tacitly acquired in the school community itself through the elaboration of a pedagogical action based on complementarity between geometric knowledge (global, ethical) and everyday knowledge and practices related to tapestry (local, emic).

The participating teachers also responded that they did not encounter any difficulties in carrying out the activities proposed for this pedagogical action, as they are suitable for students in the 7th year of Elementary School. For example, participant *PF2* stated that these activities were “well explained, clearly and covered various geometric content”.

Similarly, participant *PM3* stated that “the activities varied between easy and average, requiring more refined reasoning to be able to respond correctly to what was being required” because “they addressed the crafts they know, in addition to clearly working on the predominant contents of the 7th year". Therefore, the connections between these two environments should be encouraged in schools, as they can reduce the gaps between theoretical and practical geometric knowledge.

For these participants, the most attractive geometric and geometric contents in the activities proposed in this pedagogical action, which are related to the sisal weaving, were the operations with natural numbers, the calculation of areas and perimeters of plane figures, the Cartesian plane, geometric figures, and the angles. For example, participant *PM3* argued that “students can determine areas and perimeters of the figures that are inside the mat using geometric formulas, counting the number of points that make up each figure”.

These participants stated that they previously used artisanal production to prepare mathematics lessons, such as the production of carpets to introduce basic geometric operations and area calculations, however, they highlighted that there is not enough time to carry out this type of pedagogical action. Ethnomodelling values and respects how geometric knowledge and practices specific to the ethnos of the members of this cultural group, which are in accordance with the sociocultural context itself. Consequently, ethnomodelling also provides an understanding of the curricular geometric activities proposed in this pedagogical action in the context of the tapestry.

FINAL CONSIDERATIONS

The results of this investigation showed that both teachers and artisans, participants in this research, value and respect local know-how related to the art of tapestry, in the making of sisal carpets, as they became aware of the relevance of their own experiences, which made it possible to understand the connection between etic (school) and emic (tapestry) knowledge in the teaching and learning process in mathematics.

This approach was developed through the analysis of a dialogical pedagogical action by the participating teachers, which contributed to the development of the process of mathematization of problem and situations experienced in the daily lives of community members, as well as made it possible to understand the curricular connections that can be conducted within and outside of schools and mathematics itself with the elaboration of ethnomodels.

At the end of the conduction of this investigation, an educational product was elaborated, in the form of a suggestion book, which, in addition to proposing the use of pedagogical activities for teachers, it also seeks to present the possibility of using everyday practices, developed by members of a certain cultural group, in this case, members of the culture of art in sisal tapestry, in the teaching and learning process in mathematics through ethnomodelling.

Consequently, the development of mathematical curricular activities proposed for the pedagogical action of ethnomodelling is in accordance with the emic (making sisal carpets) and ethical (school mathematical content) approaches, as it provided the dialogical development of ethnomodels, which became evident when the participating teachers. This study mathematized the ideas, procedures, and mathematical practices typical of everyday artisanal production, which had its natural encounter with activities carried out in the school environment.

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ICMI Study 26
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A PEDAGOGICAL ACTION BASED ON AN ETHNOMATHEMATICAL PERSPECTIVE FOR THE DEVELOPMENT OF GEOMETRIC CONTENT FOR VISUALLY IMPAIRED STUDENTS TO IMPROVE THE TEACHING PRACTICE OF A VISUALLY IMPAIRED MATHEMATICS TEACHER

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This qualitative study was conducted through a study based on the pedagogical action of ethnomathematics for the development of geometric contents for a visually impaired student, enrolled in the 9th grade of middle school, of a public school in Belo Horizonte, Minas Gerais, in the perspective of a visually impaired mathematics teacher, using manipulatives and concrete materials as mediators of this educational process. Its main objective was related to the following research question: "How can an ethnomathematics pedagogical action contribute to the development of geometric content for a visually impaired student to improve the teaching practice of a visually impaired mathematics teacher, based on ethnomathematics". Data were collected by using questionnaires (initial and final), semi-structured interviews, 10 modules of activities adapted for the visually impaired student with the help of his visually impaired mathematics teacher, and the researchers' field diary, which made up the theoretical sample of this investigation. Data analysis and the interpretation and discussion of the results were conducted by using an adaptation of Grounded Theory.

INITIAL CONSIDERATIONS

Results obtained in the last demographic census conducted in Brazil in 2022, by the *Brazilian Institute of Geography and Statistics* (Brasil 2022a), shows that the Brazilian population is 203.1 million inhabitants, in which 18.6 million inhabitants who are 2 years old or older have some type of disability. In the context of this study, there 6.5 million Brazilians who have severe visual impairment, 6 million who have great difficulty seeing, and 506 thousand Brazilians who are legally visually impaired. According to the 2022 Brazilian School Census (Brasil, 2022b), there are 80.429 students with low vision, 7.308 visually impaired students, and 628 deafblind students who enrolled in Brazilian basic education schools. This context allowed researchers to understand how a visually impaired student enrolled in a 9th year of Middle School with help of his visually impaired mathematics teacher can develop geometric content by using manipulatives and concrete materials in the classroom.

Thus, this research investigated the inclusion of a visually impaired student in the teaching and learning process in mathematics, from the perspective of a visually impaired mathematics teacher, which aimed to understand the geometric contents related to plane geometry and the Pythagorean Theorem by this student through the use of manipulate materials in an ethnomathematical perspective. In this study, inclusion is understood as the adaptation and transformation of society so that people with disabilities (including the visually impaired) have their needs and differences respected and valued, which provides opportunities for equality and equity in their coexistence in society (Brasil, 2015).

Given the complexity of the inclusive process, there is a need for both initial and continuing teacher training to develop teaching practices that are directed towards creating pedagogical actions contextualized for visually impaired students that seek an understanding of classroom geometric content from a ethnomathematical perspective. This includes educational policies that influence and support the development of inclusive practices in schools, and that establish the equal right of members of distinct cultural groups by supporting methodologies, educators, students and leadership for all learners (Unesco, 2015).

Teaching and learning processes of students with disabilities need to combine these two aspects, the professional and the intellectual, and for this it is necessary to develop the ability to re-elaborate knowledge (Pinheiro & Rosa, 2020). As well, other skills need to be worked on, such as elaboration, definition, and reinterpretation of curricula and programs that promote professionalization, appreciation, and teacher identification regarding students from minority groups. The pedagogical actions in this study are related to the use of a diversity of pedagogical resources, such as Geoboards, Multiplanes, and Cuisenaire manipulative materials. These were used to assist in identifying strategies t used to resolve problems and situations proposed as pedagogical action developed from an ethnomathematical perspective.

The pedagogical actions outlined in this study identified methods used by the visually impaired teacher in developing his teaching practice. As well, this study investigated strategies and techniques for visually impaired students in the context of geometric content based on the theoretical and methodological basis of an ethnomathematics program. This qualitative research was conducted through a study based on a pedagogical action of ethnomathematics for the development of geometric contents of a visually impaired student, enrolled in the 9th grade in a public middle school in Belo Horizonte, Minas Gerais, Brazil, in accordance with the perspective of a visually impaired mathematics teacher, through the use of manipulatives and concrete materials as mediators of this educational process.

For many visually impaired students, "manipulation of a concrete resource is essential so that, through touch, they can perceive the shape, size, textures, which will determine the characteristics of the manipulative element (Kaleff, 2016). These resources enable students to understand geometric concepts through tactile perception. Visually impaired students need to use manipulative teaching materials that have different textures, sizes, and shapes, as it is through their manipulation that these students can develop geometric concepts. Thus, the main purpose of this perspective is to make the process of teaching and learning mathematics contextualized in the daily lives of visually impaired students, as well as analyzing which pedagogical resources must be used to assist them in the development of geometric content.

INCLUSION, EQUITY, AND ETHNOMATHEMATICS

Inclusion is a process that helps students overcome barriers that limit their full participation and achievement in society as citizens. This pedagogical action includes the need to adapt schools for all students. For example, in the educational process as described here, inclusion valued and respected distinct identities, differences, and diversity, central to the elaboration of pedagogical materials and the development of curricular materials in order to address the unique educational needs of the visually impaired student (Silva & Rosa, 2023).

Given this context, it is necessary to highlight how inclusion and equity form comprehensive principles that guide educational policies, plans, and practices. These principles recognize how education is a basic human right and forms a foundation for the development of more inclusive, equitable and cohesive communities (Vitello & Mithaug, 1998). This ensures that all students have access to quality education by recognizing their intrinsic value and diversity by developing respect for their human dignity. In this regard, differences are perceived positively as a stimulus to foster learning and promote educational equality (Unesco, 2019). At the core, the concept of inclusion forms a principle that both supports and welcomes the cultural identity of all students.

Consequently, the main goal of inclusive education is to eliminate social exclusion that results from discriminatory attitudes related to race, social class, ethnicity, religion, gender, abilities, and disabilities. It is important to emphasize here that discussions about equity were only introduced in Brazil through the debate related to the document named: *Education Action Framework 2030*, which discusses the implications of education for social justice. The document places inclusive education as a priority, as it serves as a basic human right that lays a foundation for the pursuit of peace and the promotion of sustainable development (Unesco, 2017).

According to this perspective, the guidance for the development of inclusive and equitable education can serve as an opportunity for Brazilian educators to renew the discourse for public policies in favor of public education, particularly, in terms of its financial support. Therefore, the perspective of a more inclusive and fairer education in Brazil. This includes educational opportunities for poor students and those from minority groups, which are in accordance with the principles of the *2030 Educational Agenda*.

Hence, the search for improving the teaching and learning process in mathematics is still one of the main challenges faced by educators around the world in relation to the development of pedagogical actions that includes students, without distinction, in the educational process (Rosa, 2010). However, although there is recognition of the importance of inclusion for the success of students with disabilities in the school environment, there are still arguments that the inclusive process is restricted and that the search for inclusive pedagogical actions is very complex, requiring time and resources that are unavailable for educators can act in this direction (Rocha, 2017).

For this reason, educators have the duty to guarantee access to rich and diverse learning experiences, which contribute to the academic performance and success for all students. This includes the provision of opportunities for visually impaired students and mathematics teaching and learning experiences that promote the development of mathematical and sociocultural skills through the perspective of ethnomathematics (Pinheiro & Rosa, 2020).

Given this, D'Ambrosio (2001) stated that the importance of mathematics is indisputable in a global context. However, there is a need for this field of knowledge to interact and promote dialogue with other fields of knowledge and other forms of mathematics, enabling the development of a pedagogical action that is based on a sociocultural basis for an inclusive mathematical curriculum.

There is a need for pedagogical action in the classroom to focus on the generation, production, organization, transmission, and dissemination of mathematical knowledge developed and accumulated by members of different cultural groups (Rosa, 2010). In this regard, D'Ambrosio (2005) criticized the traditional mathematical curriculum practiced in schools, which was designed and detailed in an obsolete, uninteresting, and not very useful way, and that consists of a set of repetitive and abstract techniques that become uninteresting and unnecessary.

In this context, ethnomathematics includes inclusive forms of educational opportunities. These opportunities include an ethics that is fundamental to principles of respect, solidarity, cooperation, and a respectful dialogue with alterity. The pedagogical action proposed by ethnomathematics can be considered as a response to the demographic dynamics of society and social inequality, as this program values and respects the knowledge of members of the different groups that make up contemporary society (Rosa & Orey, 2017). In this regard, D'Ambrosio and Borba (2010) stated that ethnomathematics values and supports the production of knowledge developed by members of distinct cultural groups that have been marginalized in the process of globalization experienced by humanity.

From this perspective, Rosa (2010) states that ethnomathematics helps in the identification and interpretation of forms of mathematical knowledge developed by members of different cultural groups, as well as seeking to value mathematical knowledge and practices that have been historically neglected, such as those developed by visually impaired students. The development of the concept of culture for a group of visually impaired students can, according to Pinheiro and Rosa (2020), be based on an anthropological concepts of culture and the theoretical implications of ethnomathematics by envisioning the possibility of developing pedagogical actions towards the promotion of sociocultural inclusion.

METHDOLOGICAL PROCEDURES

In this qualitative study data were collected by using the following methodological instruments: questionnaires (initial and final), semi-structured interviews, and ten modules of activities adapted for both the visually impaired student and his visually impaired mathematics teacher. As well as the observations registered in the researchers' field diary were used to provide theoretical samples for this investigation. To conduct this investigation, researchers transcribed semi-structured interviews, coded and analyzed the answers given by the participants to the questionnaires and to the ten blocks of activities proposed in these data collection instruments (Silva & Rosa, 2023).

Data analysis was performed by using an adaptation of Grounded Theory in which *preliminary codes* (Table 1) were identified during the conduction of open coding and then they were grouped through their common concepts and characteristics in *conceptual categories* (Table 2) during the conduction of axial coding.

It is important to state here that in selective coding, the identification of a central category, and the writing of an emerging theory were not employed in this study by considering that our objective was to answer the research question.

The analytical phase of data and the interpretative phase of the results obtained in this study enabled the researchers to determine answers to the problem statement developed for this study. Two conceptual categories emerged from the analysis of the data: a) *Process of Teaching and Learning in Mathematics in the School Context* and b) *Ethnomathematics as a Pedagogical Action for an Inclusive Education*. During the interpretation and discussion of the results obtained in this study, the description of the categories allowed the quotes given by the participants, which were identified in this interpretative process, to be used to provide a holistic comprehension of its problem statement.

Collected Data	***Open Coding (Preliminary codes)***
Student: Manipulative materials helped me in learning geometric content (1), as geometry is related to the use of figures that are present in my daily life (2) in the environment that surrounds me (3). I would like to work more with manipulative materials in the classroom (4). ***Student***: Manipulative materials helped me in understanding geometric contents (1). It certainly helped me a lot (5) because it was much easier for me to measure figurers with them (6). ***Student***: I have experience with manipulative material (7): a) Geoplane (8) - it was very good because I managed to learn a lot from it (1). I found it much easier with it (4), b) Multiplane (8) – it was very different because if I can't do it in my head (5), I can do it with this material (4), and c) Cuisenaire rods (8) - it was good (4) because I could make several triangles (9) and do several problems with it (7) to solve the Theorem of Pythagoras (9). ***Teacher***: What is the formula for the area of a square? (9). ***Student***: The area of the square is like this! (10). The area of the square is side times side (9). ***Teacher***: This material contributed to his experience of abstract geometry (11) in terms of plane figures (9), so that he had experience to work with manipulative materials (7) to have tactile contact with geometry (12) and get to know the issue of the figures (9), so this helped a lot with his cognitive construction (11).	(1) Using manipulative material to learn/understand geometric content. (2) Presence of geometry in everyday life. (3) Sociocultural context. (4) Preference to use manipulative materials. (5) Importance of manipulative materials. (6) Manipulative materials as pedagogical tools. (7) Experience with Manipulative materials. (8) Types of Manipulative materials (9) Geometric content. (10) Concept of the aera of the square. (11) Contributions of manipulative materials. (12) Ethnomathematical technique.

Table 1: Prelimanry codes identified in the open coding process (Source: Authors' personal file)

Open Coding (Preliminary Codes)	***Axial Coding (Conceptual Categories)***
(1) Using manipulative material to learn/understand geometric content (4) Preference to use manipulative materials (5) Importance of manipulative materials (6) Manipulative materials as pedagogical tools (7) Experience with Manipulative materials (8) Types of Manipulative materials (9) Geometric content (10) Concept of the aera of the square (11) Contributions of manipulative materials	*Process of Teaching and Learning in Mathematics in the School Context*
(2) Presence of geometry in everyday life (3) Sociocultural context (12) Ethnomathematical technique	*Ethnomathematics as a Pedagogical Action for an Inclusive Education.*

Table 2: Conceptual categories identified in the axial coding process (Source: Authors' personal file)

RESULTS AND DISCUSSION

Results obtained in this study provided evidence that the development of differentiated pedagogical actions in the classroom stimulated the use of other senses of the participants, such as hearing, speech, and touch, by enabling to obtain positive results in the development of their cognitive process using a comprehensive communicative process. By using manipulative and concrete materials, the visually impaired mathematics teacher promoted the improvement of his teaching practice by promoting a pedagogical action for a more holistic teaching and learning process in mathematics, which enabled the development of mathematical and geometric content for the visually impaired student.

The adaptation of manipulatives and concrete materials such as the geoplane, the multiplane, and the cuisenaire rods, helped the student in exploration, investigation, and understanding of mathematical and geometric content proposed by the pedagogical actions conducted in the classroom. This was done by encouraging the student to associate aspects of the concrete to the abstract and included concepts of perimeter, area, and angles, as well as the characteristics and properties of geometric figures, and the identification and use of the Pythagorean Theorem.

These results, aimed at the improvement of an inclusive mathematics education, also showed the importance of the sociocultural differences that agreed with assumptions of ethnomathematics. The visually impaired student stated that "manipulative materials helped me in learning geometric content, as geometry is related to the use of figures that are present in my daily life in the environment that surrounds me. I would like to work more with manipulative materials in the classroom".

Thus, Costa (2018) highlights that it is necessary for the teaching dynamics to be organized with the objective of shifting the emphasis from the visual and/or spatial aspect of geometric objects to the tactile through appropriate materials by given that visually impaired students can grasp the information arising from mathematical propositions embodied in manipulative materials through tactile sensations and the communicative process.

In the final questionnaire, the visually impaired student argued that "manipulative materials helped me in understanding geometric contents. It certainly helped me a lot because it was much easier for me to measure figures with them". This pedagogical action enables the establishment of relations between the properties and characteristics of geometric objects through the mediation of manipulative materials. From this perspective, Moreira (2011) highlights that manipulative materials assume an organizing function in mathematical learning because they establish relations between different geometric figures and objects.

In the final questionnaire, the visually impaired student commented that: "I have experience with manipulative material: a) Geoplane - it was very good because I managed to learn a lot from it. I found it much easier with it, b) Multiplane – it was very different because if I can't do it in my head, I can do it with this material, and c) Cuisenaire rods - it was good because I could make several triangles and do several problems with it to solve the Theorem of Pythagoras". In this context, Moreira and Masini (2001) affirm that that teachers transfer the emphasis of learning from the visual aspect to the remaining senses, especially, the tactile one. Similarly, through the communicative process, students can associate information from teachers, the use of pedagogical resources and tacit knowledge existing in their cognitive structure.

In the final questionnaire, the visually impaired teacher explained about the development of the activity blocks proposed to the visually impaired student in the classroom by stating that: “This material contributed to his experience of abstract geometry in terms of plane figures, so that he could experience to work with manipulative materials to have tactile contact with geometry and get to know the issue of the figures, so this helped a lot with his cognitive construction”.

In one of the activity modules, the student identified a right-angled triangle made of a concrete material and, when feeling its sides with his fingers, he found that this geometric figure had the shape of a small chair and a toboggan. Thus, this student remembered and compared the right triangle with the little chair and its hypotenuse with the toboggan by commenting that it was “a 90º angle”. In this regard, the chair jargon was used by the teacher to explain the 90º angle and also to relate it to the right angle by referring his explanation to the notion of perpendicularity. Similarly, the jargon of toboggan was used by the teacher to tell the visually impaired student the side of this right-angled triangle with its hypotenuse.

The interpretation of the results obtained in this study showed that the visually impaired teacher asked the visually impaired student: “What is the formula for the area of a square?”. Thus, the visually impaired student took the teacher’s finger and placed it inside of the geometric figure that represented the square made of concrete material and then sliding it across the interior of the figure. This participant stated that: “The area of the square is like this!” by pointing his finger towards the internal region of this figure. As well as he slid his fingers down the sides of this figure and answered that: “The area of the square is side times side”.

The interpretation of these results shows that the student determined the area of squares by using his own techniques. From this perspective, Domingues (2010) comments on the need for teachers to promote specific learning opportunities for visually impaired students through the use of strategies and specific resources in order to improve their learning.

FINAL CONSIDERATIONS

The results obtained in this study show that the visually impaired student's active touch was triggered by the exploration and manipulation of manipulative or concrete materials that enable the provision of care targeted at the specificity of their disability. Therefore, it is noteworthy that this pedagogical approach is related to respect for the culture of visually impaired students, as well as the writing system itself and the values and rules of behavior in accordance with the specificities and needs of this school population.

This context enabled, according to Vergani (2007), the development of a conception of Ethnomathematics that also deals with the rational, psychic, emotional, social, and cultural integrity of members of distinct cultural groups through a stance that echoes from and to the different educational levels according to their different degrees of depth.

In the pedagogical action proposed for this study, the assumptions of ethnomathematics and inclusive education coincided, with the aim of respecting and valuing the social, cultural, and educational inclusion of the visually impaired student and the visually impaired mathematics teacher. According to the epistemological dimension of ethnomathematics, there is a need to implement a set of pedagogical actions aimed at developing an inclusive mathematical education that values and respects the differences by promoting cultural plurality in classrooms.

Therefore, the need for teachers to de-condition themselves from their academic classificatory systems in understanding the geometric knowledge and techniques developed by visually impaired students in their relationship with different sociocultural contexts stands out. Thus, in this study, the ethnomathematical perspective assumed the appreciation and respect for the culture of students from minority groups, whose geometric knowledge is despised by hegemonic and dominant cultures, given that it aims to include ideas, procedures, techniques and practices mathematics developed by visually impaired students in the school community itself.

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DISCUSION DOCUMENT

THE 26TH ICMI STUDY: ADVANCES IN GEOMETRY EDUCATION DISCUSSION DOCUMENT

INTRODUCTION AND BACKGROUND

This document announces a new *ICMI Study* being conducted by the International Commission on Mathematical Instruction (ICMI). The Study, the 26th led by ICMI, addresses the advancements and current challenges in the teaching and learning of geometry, with particular attention to how the landscape has changed in the past three decades since the ICMI Study 9 on "Perspectives on the Teaching of Geometry for the 21st Century" (Mammana & Villani, 1998). Fundamentally, the International Program Committee (IPC) considers that geometry is positioned as a powerful pathway to support the understanding of complex ideas within school mathematics and beyond. In fact, we boldly suggest that geometry education is at its most exciting and quickly advancing period of the past 100 years. This ICMI Study examines the nature and role of geometry in mathematics education and related fields with an eye to the increasing depth and range of geometry education research, knowledge, pedagogies, and approaches on an international scale.

AIMS AND RATIONALE

The primary aims of the Study are first to report the state of the field of mathematics education in the area of geometry education with respect to theory, research, practice, and policy; and, second, to suggest new directions of research that consider contextual, cultural, national, and political dimensions of practice.

Since there are different ways of understanding geometry education, the Study aims to include multiple theoretical perspectives and methodological approaches. Thus, the IPC encourages contributions that use a variety of methodological strategies including, among others, large-scale experimental and descriptive studies, case studies, and iterative or cyclical processes such as design research and action research. The IPC welcomes contributions from mathematics education researchers and from mathematics teachers, to ensure that teachers' voices are given prominence in accounts of their learning and teaching. Moreover, in this new Study, the IPC invites contributions from those working in the wider geometry space (such as cognitive scientists and educational psychologists). This interdisciplinary approach aims to provide innovative and collaborative opportunities not afforded by the current state of play that can leave work siloed within individual disciplines.

This Study aims to produce and share new knowledge about promoting effective pedagogy in the teaching and learning of geometry for all. It takes into account the large body of existing theory and research, socio-cultural diversity, cultural differences in curricular and institutional constraints, as well as progress the field. In particular, the IPC acknowledges the following specific goals for geometry education and anticipates that these will be developed by participants through the course of the ICMI Study:

- Bring together an expert reference group to analyze the state-of-the-art of research and practice in geometry education in order to contribute to better understanding of the challenges that geometry education faces in diverse contexts.
- Bring together communities of international scholars, with diverse representation within the ICMI community, and across regions and nationalities, to address research and practices in geometry education, ultimately resulting in the production of an ICMI Study volume.
- Emphasize the role of geometry as a facilitator of advances in logical reasoning, strategic thinking, and understanding of hierarchical relationships between mathematical (geometrical) objects.
- Analyze the influence of affective components of teaching and learning such as confidence, motivation, anxiety, efficacy, and others, on teachers' decisions and methodological choices and on students' behavior and related learning results.
- Facilitate advances in multi- and inter-disciplinary approaches (including cooperation with other bodies and scientific communities) to research and development in geometry education.
- Explore the role of Geometry as an arena or lever for developing children's and adolescents' mathematical creativity and flexible thinking by means of conjecturing, problem posing, and proving theorems.
- Disseminate scholarship in mathematics education —research, practices, methodologies, theories, findings and results, curricula design, etc.— in geometry teaching and learning.
- Identify and anticipate new research and development possibilities, challenges, and questions related to geometry education and research in geometry education.
- Produce sets of recommendations on effective resources for researchers, teachers, teacher educators, policy makers, curriculum developers, analysts, and the broad range of practitioners in mathematics and mathematics education.
- Act as an instigator of new research and innovation to be produced in the future.
- Promote and assist in the discussion of geometry education and related research in action at local and international levels.

STRUCTURE: TOPICS, SUB-TOPICS, AND QUESTIONS

The activity of the Study is organized based on four focused topics, aimed to provide complementary perspectives and approaches to the teaching and learning of geometry. Contributions to the topics will be organized around sets of related sub-topics, each sub-topic focusing on a specific issue and stating a set of questions aimed to lead discussions. The four topics are:

A. Theoretical perspectives

B. Curricular and methodological approaches

C. Resources for teaching and learning geometry

D. Multidisciplinary perspectives

TOPIC A: THEORETICAL PERSPECTIVES

Several theoretical and conceptual perspectives have been used to study geometry education —some have emerged within the field itself while others have been drawn from the education disciplines or applied from psychology or sociology. Theories specifically about geometry education that continue to be evident in research include the Van Hiele model (Van Hiele, 1986), the theory of figural concepts (Fischbein, 1993; Mariotti & Fischbein, 1997), and the theory of figural apprehension (Duval, 1998). Researchers are continuing to develop, refine and apply these theories. More recently, theories emerging in geometry education include embodied (Ng & Sinclair, 2015) and ecocultural perspectives (Owens, 2014), prototype phenomenon (Gal & Linchevski, 2010), and semiotic mediation (Bartolini & Mariotti, 2008).

In the recent ICMI Survey of geometry education, Sinclair et al. (2016) maintained that over the past decade, there has been an increased focus on theories related to visuospatial reasoning, the role of gestures and diagrams, and the use of digital technologies. As Sinclair and Bruce (2015) reinforce, post-Euclidian developments have moved beyond the emphasis of naming shapes and their properties to "a more active meaning-making orientation to geometry (including composing/decomposing, classifying, mapping and orienting, comparing and mentally manipulating two- and three-dimensional figures)" (p. 320).

Rather than revisiting the well-documented theoretical perspectives, this topic aims to dive more deeply into several emerging areas of research activity in geometry teaching and learning. Such emerging areas include new conceptions of the teaching and learning of proofs in geometry contexts, the role of spatial reasoning and the implications for geometry teaching and learning, and, in a related area, the role of visualization in geometry reasoning.

Sub-topic A1: Teaching and learning of proof in geometrical contexts

In the study of geometry, the concept of proof appears as a key element to support and validate mathematical statements. The IPC conceives of the proving process as a way to convince oneself or another person, through the presentation of different types of arguments in favor of the pertinence and validity of mathematical statements. In Euclidean geometry contexts, it is important to recognize that, depending on the education level of the students, the type of proofs to support conjectures vary. For instance, at elementary education, concrete examples could be sufficient to present and support mathematical relationships, but when students' progress or advance in their studies, they should refine their ideas and transit from empirical examples to present informal deductive proofs and, finally, formal proofs. In this context, teachers' educational programs should explicitly include the development of knowledge and experience in problem posing, formulating conjectures, and validating them. These are creative processes as much as they are logical thought processes.

It is also important to acknowledge that the consistent and systematic use of digital technologies opens different routes for teachers and students to represent and explore geometrical properties and also to support the truth of mathematical conjectures or statements.

Some overarching questions for contributions on the teaching and learning of mathematical proof in geometry include:

- How can the concept of mathematical proof in geometry be characterized at different educational levels? What types of geometry proofs should students learn during the different school levels? Are there paths or trajectories during the process of learning to prove geometric statements?
- What types of teaching sequences are helpful to promote the learning of geometric proofs?
- How might teacher and/or student uses of digital technologies provide a set of affordances to think of different types of arguments (visual, empirical, formal)?
- How do undergraduate students value the correctness of geometric proofs? Is this in line with their teachers' values?
- How do dynamic geometry environments affect conceptions of what is meant by geometric proof or what a geometric proof is?
- What affective components (e.g., anxiety, confidence, and efficacy) influence the learning of geometric proof? Are there gender differences in approaches to geometric proofs?
- In which ways can students be engaged in formulating conjectures and doing geometric proofs?

Sub-topic A2: The role of external visual inputs in geometric reasoning

Visualization is a broad construct that is important in mathematics. Simply put, it involves "the mind seeing things." Use of imagery is known to support geometric thinking and is instrumental in working in novel or complex situations. Along with many others, Mason's (2004) work on the "structure of attention" intersects with these notions of visualization. However, challenges prevail in mathematics education where canonical/prototypical and static inputs tend to dominate students' encoding of visual inputs. Consequently, it is important to consider how students can be offered a much wider variety of experiences (and inputs) as early as possible. Dynamic visualizations, where there is movement, such as in visualizations of compressions and stretches are one example of expanding these cases. Building on the work of Duval (1998), what is seen is quickly classified in the minds' eye as "triangle", for example; this is the iconic vision, and these canonical/prototypical constructs can be deconstructed. For instance, a square would need to be seen not only as a shape of square in an iconic way, but also as four lines intersecting orthogonally at equal distance and ultimately four points. New technologies offer interesting new affordances worthy of exploration. A burgeoning area of research related to visualization focuses on objects-to-think-with or implicit tools for use in a broad range of contexts. There is much to explore at this time in the arena of external visual inputs.

The IPC invites contributions that explore these key questions:

- In what ways are visualization and geometry linked, and how does visualization support geometric thinking?
- What affordances and challenges are engendered through technology uses for visualization?
- How might learners break away from canonical visual depictions of geometric figures? Why is this important?
- Are visual and analytic geometric processing distinct? How do they overlap and when should they overlap?

- How do objects-to-think-with, implicit structures, and/or forms of visualization operate as a tool for geometric thinking and supporting analytic reasoning?

Sub-topic A3: Spatial reasoning interventions and transfer to geometry

Spatial reasoning is integral to activities in daily life, such as packing a car for a trip or reading maps. At a broad level, spatial reasoning describes the ability to mentally represent, analyze, and transform objects and their relations, and is comprised of distinct, yet related, skills with strong ties to mathematics achievement. Spatial reasoning involves the location and movement of objects and ourselves, either mentally or physically, in space. It is not a single ability or process but actually refers to a considerable number of concepts, tools, and processes (National Research Council, 2006). The role of spatial reasoning within mathematics problem solving and performance have been noted for decades (Clements & Battista, 1992). In fact, a recent meta-analysis (Hawes et al., 2022) confirms that improving spatial reasoning skills causally supports improvements in mathematics performance.

Spatial reasoning is malleable, with improvements observed even in short periods of explicit instruction or exposure to spatial tools and playful experiences. Spatial reasoning is especially relevant to geometry education, with numerous (recent) studies demonstrating transfer to geometric understanding (Lowrie et al., 2019). What are less known are the mechanisms that support the transfer from spatial development to geometric understanding. This topic seeks advances in the field that describe interventions and learning programs that spatialize the geometry curriculum as well as spatial learning activities that support geometric understanding. Topics include the relationship between spatial reasoning and geometry in 2D and 3D transformations, composing and decomposing shape, perspective taking and activities involving symmetry and rotations of 2D shapes and 3D objects.

The IPC invites proposals that address some of the following questions:

- What are the mechanisms that support transfer from spatial reasoning to geometric understanding?
- What constructs or types of spatial reasoning should be included in school (Pre-K to 12) mathematics?
- Given the strong relationship between spatial reasoning and geometric understanding, what are promising ways of doing this?
- How is spatial reasoning positioned in school curricula and how does this vary from nation to nation? Is this serving students well? What is the impact?

TOPIC B: CURRICULAR AND METHODOLOGICAL APPROACHES

A core pillar of the 26th ICMI Study relates to geometry curricula and methodological approaches to teaching geometry, from the early years through to higher education. There are novel developments that are changing the landscape of geometry education with intensity of research activity in the early years and in higher education. When developments in curriculum for school geometry are considered, increased attention to spatial reasoning is apparent, alongside and/or within the geometry curriculum. In many nations, curriculum constitutes policy, which carries weight and encourages greater classroom time devoted to geometry learning. Recent research on professional learning focused on geometry teaching and learning has demonstrated that longer and more intensive professional training programs, which are situated in classrooms directly involving students, are proving to have lasting impact (Clements & Sarama, 2011). This includes attending to individual differences of learners and cultural and political contexts of learning.

Sub-topic B1: Geometric thinking across age levels (from preschool to higher education)

Despite decades of robust research in geometry education, there continues to be gaps in students' geometry experiences and learning across age levels from preschool to higher education. The gaps include limited opportunities to engage with real-world and mathematical objects that foster geometric modeling and reasoning, uneven quality of curriculum and teaching that students experience, limited connections made between spatial, geometric, and other mathematical concepts, and what students are asked to "do" (e.g., doing, describing, analyzing, proving, etc.). There is an urgent need to develop and/or refine ways of analyzing these gaps. In the last 30 years, there has been a flurry of research activity in the early years and in higher education that focuses on improving spatial sense and geometric reasoning. The literature on building spatial reasoning for young learners and for those pursuing careers in STEM education is developing at an accelerated pace.

Some of the important issues on which researchers and teachers alike could contribute are:

- What is the current status of geometry curricula across age levels and how do such curricula support student's conceptual understanding and development?
- What curricular and classroom-based innovations could help to understand, analyze, and fill the gaps among early, primary, secondary, and higher education? How can recent research efforts in the early years and in post-secondary education be built on?
- How do informal and formal geometry curricula seek to include the use of language, gestures, drawing, modelling, or representations to support student thinking? How can the effectiveness of these aspects of conveying meaning be analyzed?
- What are effective ways to investigate children's developing understanding of 2D and 3D shapes in early years? How are early years spatial-related experiences linked to later formal learning of geometry?
- What efforts have been made to ensure that formal geometry curricula build on earlier understandings?
- How are post-secondary programs in STEM areas supporting the geometric thinking required for professions such as engineering, medicine, chemistry (and beyond)?

Sub-topic B2: Individual differences, including students with learning difficulties or mathematical giftedness

The ability to study geometry successfully can be a consequence of learning opportunities provided to students or their innate capabilities, that may impede or advance the development of their geometry knowledge, skills, and interest in learning geometry.

Difficulties in processing geometric concepts can depend on a range of factors. For example, dyslexia or difficulties related to reading comprehension may affect students' capabilities to formulate, describe, and characterize geometrical object and their properties and read geometrical text, such as definitions and theorems. Students with learning difficulties, for example, may find visual-spatial supports that do not require language and number to provide new pathways into complex geometry. Dyscalculia can influence solving calculation problems in geometry whereas dysgraphia can be an obstacle for drawing geometric figures. Visual impairments also impact students' geometrical learning and processing. Students with learning disabilities need special programs and approaches to teaching geometry that allow them to study fluently and experience success in geometry lessons.

Moreover, students with high mathematical abilities who enjoy solving challenging geometry problems also need special attention. As Krutetskii (1976) demonstrated, mathematically gifted students are able to memorize mathematical objects, schemes, principles, and relationships and often reason both logically and spatially. Such mathematical processing at these high levels across different branches of mathematics, including geometry, provide opportunities for creativity-directed activities such as conjecturing, proving, or refuting geometry statements using different theorems and with different auxiliary contractions. *Mathematical cast of mind* (Krutetskii, 1976) reflects students' special capability to see mathematical objects within the real world as well as to be creative in mathematics. Mathematically gifted students can be exceptionally creative, and their creativity needs to be encouraged and promoted.

The IPC invites contributions that explore teaching and learning geometry as related to a continuum of mathematical capabilities. Submissions are encouraged of studies that include, but are not limited to, the following questions linked to (a) various student learning disabilities and (b) students with high mathematical abilities:

- How do student capabilities affect studying geometry?
- Are there types of geometry tasks that map to different capabilities?
- What teaching approaches are most effective for students with different characteristics?
- How can mathematics teachers be supported to work with students of varying capabilities?
- What type of assessment tasks are effective when teaching students with special educational needs?
- How can technological tools be used to support and challenge the teaching and learning of geometry by students with special learning needs, including gifted students?
- How can creativity-directed activities or creativity-promoting activities support learning geometry for students with learning disabilities or mathematically gifted students?

Sub-topic B3: Influence on the curriculum of different cultural and political traditions and contexts around the world

Curriculum is regarded as the knowledge, skills, values, and attitudes presented to the learners in order to change their behaviors to become active members of society. A school mathematics curriculum should reflect the political and cultural contexts of the students for which it is planned. Political context refers to the way power is achieved or used in a society. Cultural context refers to the shared values, attitudes, beliefs, and practices of a community. Geometry should not be regarded as a value-free curricular content area. In fact, some argue that Mathematics, including geometry, should equip students to deal with political and cultural issues; for example, closing the learning gap between informal and formal geometric knowledge.

The IPC invites proposals that address some of the following questions:

- What is the influence on the curriculum of different cultural traditions and contexts around the world regarding geometric concepts?

- What is the influence on the curriculum of different political traditions and contexts around the world regarding geometric concepts?
- How can political and cultural contexts of mathematics curricula help to recognize and respect the history, tradition, and mathematical thinking developed by members of distinct cultural groups through the development of geometric thinking?
- What is the significance of the influence on the curriculum of different political and cultural contexts for researchers, policy makers, and school teachers?

Sub-topic B4: Professional training and development in geometry education (pre-service and in-service)

There are a wide range of professional learning models related to geometry education both for pre-service and in-service educators. Early years teachers often receive mathematics professional training that focuses on playful approaches to exploring patterns (repeating patterns), informal volume and capacity activities (emptying and filling containers), size comparisons (ordering objects from smallest to largest), and in the area of geometry, naming and recognizing 2D shapes. Recently, programs that focus on mathematics for young children have intensified the attention on geometry and spatial reasoning (Moss, Bruce & Bobis, 2015).

In terms of educator challenges, some primary teachers have limited knowledge about geometry and do not necessarily gain further understanding in the pre-service years due to a typical focus on general mathematics that affects the amount of attention on geometry in the classroom. For example, there is evidence that teachers concentrate on the precision of geometrical drawings rather than on the validity of the geometrical constructions. Moreover, the integration of the spatial reasoning components in geometry learning are yet to be fully realized in elementary curricula. By contrast, some secondary teachers might find it challenging to present pedagogical practices that are engaging, especially if they lack an appreciation of the nature of geometry, the foundations of geometry and how to teach these principles and baseline concepts. Educators also face challenges when presented with professional training programs. These include a lack of time to engage in professional learning, a lack of resources to implement new strategies (to release teachers and provide support materials), fear of mathematics and a lack of geometry knowledge, and/or a devaluing of geometry as an important strand/area of mathematics.

Despite these challenges, some professional learning opportunities are showing promise in supporting geometry educators. For both pre-service and in-serving teaching, the opportunity to explore models of active teaching and use and design rich geometry tasks can help support the development of effective pedagogical practices. In-service training in geometry can be deeply supported by more intensive forms of professional learning such as lesson study and collaborative action research. Working directly and consistently with students and teachers in the classroom has also been shown to be effective in building effective practices.

Methodologically, many research-based professional learning programs are guided by a design research approach. This involves the collaborative and continual refinement of materials, tools and meaningful tasks that address learning needs which have been identified in the analysis of student thinking. Using this type of approach, classroom trials ultimately engender refinements that are increasingly scalable.

The IPC invites proposals that address some of the following questions:

- What models or approaches to professional learning with a focus on geometry are proving to be effective in building geometric reasoning, and helping teachers and students to develop understanding of fundamental ideas in geometry?
- What are some effective approaches to designing meaningful tasks (including tasks that involve technology use) that support and reveal geometric reasoning and/or proof building from intuitive ideas, and theoretical understanding of geometric principles? How can these tasks be leveraged to support pre-service and in-service teacher training? How can teachers themselves be supported in task design that promotes geometry reasoning?
- How can pre-service and in-service teachers be supported in understanding the importance of explicit and implicit use of instruments in the learning of geometry? How does this translate to classroom practice?
- To what extent do pre-service and in-service teachers value geometry and how can their epistemology of the subject be reinforced?

TOPIC C: RESOURCES FOR TEACHING AND LEARNING GEOMETRY

To teach and promote geometry understanding, teachers need to access resources and promote their use with students. These resources may be technological or non-technological in nature. Depending on the teaching context, these resources may operate as constraints or affordances, and be imposed, overabundant or limited. This topic explores questions about the impacts of various resources in supporting the teaching and learning of geometry. The inherent limitations of resources due to external factors are also considered.

Sub-topic C1: Digital technologies in learning and teaching geometry

Seven lenses through which to examine technological tools in teaching and learning of geometry, the development of spatial capabilities, and advancement of geometrical reasoning and creativity are as follows: (1) the types of technology: software vs. hardware; (2) the goals of uses of technology: teaching and learning vs. evaluation and analysis; (3) the level of openness of the tools: open vs. half-baked vs. closed; (4) the educational functions: exploration vs. verification vs. proving vs. demonstration; (5) the context: pure mathematics vs. real/practical situation vs. Virtual Reality; (6) the nature of space and shape —2D vs. 3D vs. … n-dimensional— and how it is represented; and (7) the types of geometry: Euclidean, Projective, Affine, Non-Euclidean geometries.

The tools under consideration could be, for example, dynamic geometry software, 3D spaces, virtual reality, intelligent tutors using AI, learning analytics, videos, programming tools and more. The hardware used in the studies can include, among others, computers, tablets, interactive whiteboards, 3D printers and robots. Information and communication technologies have the power of enhancing visual representations of shapes and space, readily conveying geometrical relations, and connecting multiple representations of concepts which are not possible by the tool of paper-and-pencil; mediating individual (teacher and student) and classroom discourse; broadening the scope of information provided by teachers and textbooks; and promoting students' multimodal, action-based, independent, and self-regulated learning.

Contributions can be about completed or in-progress research or innovation projects. Research can be empirical, experimental, or theoretical. The IPC invites proposals that address some of the following questions:

- What is the impact of using different types of technological tools in relation to one or more lenses proposed above?
- How does the impact of technological tools on geometry education depend on the types of tools, their openness, or their educational functions?
- How do the uses of different kinds of technologies by students, teachers, or teacher educators impact on their practices and thinking in geometry?
- What are some of the opportunities and challenges that are presented by the use of technology in geometry education?

Sub-topic C2: Manipulatives and visual tools in teaching and learning geometry

Visual tools and concrete manipulatives are often seen as important resources in promoting geometrical thinking. A review of the state of the art of research on the contribution of manipulatives to learning and teaching geometry is needed. Are manipulations and visualization sufficient to allow effective learning processes of geometry, including deductive reasoning, proving, and constructing in geometry? How is the use of manipulatives linked to modelling, abstracting, using language to describe geometrical objects, and proving?

Manipulatives and visual tools can be examined from different theoretical perspectives and regarding different goals of the use of manipulatives and visualization: creating experience, identifying critical properties, formulating conjectures, and proving or refuting the conjectures.

The questions posed can be about different types of visualization and objects such as material tools, photographs, video, and drawings. The contributions may address different uses of visualization and objects, depending on the ages of students and the specific difficulties they might have (e.g., blindness, developmental coordination disorder, etc.).

The IPC invites proposals that address some of the following questions:

- What are the links between the use of manipulatives, visual tools, and modelling activity?
- What is the difference between using the manipulatives or dynamic diagrams constructed by educational designers and constructing manipulatives or working in an open dynamic environment (e.g., GeoGebra)?
- What is the role of visual tools and manipulatives on geometrical proof processes?
- How can bridges be created between manipulation and visualization and mathematical thinking in geometry?
- How can students be helped to abstract from perceptive and concrete properties of physical objects?
- What is the place of language, signs, gestures in activities based on manipulatives?

Sub-topic C3: Learning geometry with resource constraints

The scope and focus of this sub-topic address issues relating to the teaching and learning of geometry in resource-constrained classrooms and under-resourced school communities. Resources, in this context, include physical resources (e.g., digital tools and non-digital manipulatives), non-tangible resources (e.g., the language of instruction), and human resources (e.g., specific languages used by visual or hearing-impaired students). Constraints refer to limitations in the use of resources. Three sources of resource constraints can be identified: (1) the resource itself: any physical resource has limitations in the scope of its use, and it is necessary for teachers to be aware of them, so they do not pose tasks to their pupils where the resources are an obstacle more than a help; (2) the users: some students may have difficulties in handling digital or manipulative resources due to some disability, and some teachers may not have adequate training in using resources, with consequences for their pupils' learning; (3) the socio-economic conditions: many schools do not have access to resources for teaching mathematics, so students cannot benefit from their use to help them develop their geometric conceptions.

The IPC aims at composing an overview of the current state of the difficulties encountered by students in resource-constrained classrooms in the learning of geometry by (1) documenting measures and attempts to address the constraints; (2) addressing theoretical issues relating to the teaching and learning of geometry in resource-constrained classrooms; and (3) suggesting ways forward in meeting the needs of students in learning geometry in resource-constrained classroom.

We invite proposals that address some of the following questions:

- What theoretical lenses on geometry learning in resources-constrained classrooms and communities are emerging, and how do these inform the understanding of students' learning of geometry in such classrooms?
- What practices have proved to be effective in resource-constrained geometry classrooms and under-resourced communities?
- How should instructional activities be designed in resource-constrained geometry lessons, in particular for students with disabilities?
- How can the teaching and learning of geometry be improved in resource-constrained contexts where use of software (e.g., dynamic geometry) and manipulatives is limited?
- How are local languages (e.g., indigenous verified mathematics) used as a resource for supporting mathematical access in teaching and learning geometry in multilingual countries?

TOPIC D: MULTIDISCIPLINARY PERSPECTIVES

The multidisciplinary perspectives can be seen as a polysemic interaction between different knowledge fields such as ethnomathematics. One of the perspectives can be related to how teaching and learning of geometry can contribute to the development of other disciplines regarding professional, social, and cultural activities and practices. Another perspective addresses the possible contributions of other sciences such as neuroscience and psychology disciplines, and professional needs, in the process of teaching and learning geometry. The third perspective relates to the roles that geometry may play within different disciplines and knowledge fields. The main goal is to further advance research and understanding in relation to geometric knowledge and its polysemic multidisciplinary connection with other knowledge fields and sociocultural contexts.

Sub-topic D1: Connections between geometry and professional contexts

Geometry has inherent importance. There are also practical implications in professional contexts where practitioners are using geometry consistently. Efforts have been made to examine the role of mathematics in Science, Technology, Engineering and Mathematics (STEM) professions (Van der Wal, Bakker & Drijvers, 2017). With respect to geometry, it is the case that compulsory schooling, initial professional training and workplace training differ in the nature and the emphasis on geometric learning. Some of these connections are obvious while others not. Interest in the way school geometry is taught in schools and colleges and the mathematics needed in a variety of work-related activities has been evident in the literature for some time (see Roth, 2014, for a description of bridging between formal mathematics and enacted practical geometric experiences). Carpenters for example, rely on Pythagoras' theorem (especially the 3-4-5 triangle) in the form of a set square for much of their daily activities. Construction professionals such as engineers, architects and builders use trigonometry to calculate angles and distances routinely in their drawings and interpretation of plans. Creative professionals including fashion designers use symmetry and tessellations to create patterns while town planners utilize Thales' theorem to level ground accurately over long distances. In a recent example, technological developments allow sonographers to measure the head circumference of an unborn baby in various static and dynamic images of 2D-3D space.

This topic seeks research links between school mathematics and applications for on-the-job training and applications in the real world. Such links raise several questions to be researched:

- Where might school geometry be needed in emerging professions and changing workplaces?
- How can school-based geometric tasks be designed to be meaningful for out-of-school professional applications?
- How can professional activities inform the design and development of innovative geometric experiences in the classroom?
- How does geometry needed in initial professional and on-the-job training differ from geometry taught in schools?

Sub-topic D2: Ethnomathematics and indigenous ways of understanding geometry

Cultural artifacts are objects created by members of distinct cultural groups that convey cultural meanings and information about their creators and users. Geometric ideas, procedures, techniques, and practices are related to the development of cultural artifacts that are socio-culturally situated as well as distributed among these members from generation to generation. This approach also includes embodied cognition in which cognitive activities use symbols as external resources that assist these members to develop their mental representations and manipulation of objects. Geometric representations exist in all cultural groups and their meaning may be different from one specific culture to another. It is important to emphasize that ethnomathematics is also a program that studies geometric concepts and theories developed locally as well as it is related to dynamic pedagogical actions that respond to the environmental, social, cultural, political, and economic needs of the students, which enables them to develop their imagination and creativity. Thus, there is a necessity to understand indigenous ways of understanding geometry through ethnomathematics.

The IPC invites proposals that address some of the following questions:

- What role does ethnomathematics play in the understanding of diverse indigenous ways of doing geometry? What are the sociocultural influences of the use of this knowledge in distinct cultural groups? What investigations have been done in this area to date? What has been learnt from these investigations?

- What cultural traits contribute to the construction of local geometric knowledge? Does this lead to the development of embodied cognition and situated learning regarding geometry? How might mainstream/traditional mathematics learn from these constructions of local geometric knowledge?
- In what ways does ethnomathematics contribute to and improve current formal geometry content and teaching in schools?
- How do members of distinct cultural groups extend their cognitive processes beyond the brain, the body, and their immediate environments in order to develop diverse ways of understanding geometry? Does this mean that cognition is not only embodied and situated, but also distributed as evidenced in their ways of creating, processing, accumulating, and diffusing information conjointly, including geometric knowledge?

Sub-topic D3: Contributions of psychology and neuroscience to research in mathematics education focusing on geometry

Cognitive scientists posit that student mathematical processing in general and geometrical processing in particular are determined by general cognitive traits and socio-emotional characteristics. Students' cognitive characteristics such as memory, attention, pattern recognition, speed, and types of information processing are proven to affect student mathematics learning and processing. In this sense, cognition research may inform mathematics education research on the role of general cognition and the role of affective factors in studying geometry.

In the field of Neuroscience, results from research using fMRI, EEG, and eye-tracking studies (as examples) suggest that solving mathematical problems, including geometry problems, are connected to student characteristics. The research also suggests that students with varied levels of mathematical performance exhibit different patterns of brain activity (Ansari, 2016; De Smedt & Grabner, 2016; Grabner, 2022).

Teacher efficacy, attitudes, beliefs, and knowledge in geometry is another related area of robust research activity. For example, in parallel work in Psychology and Mathematics Education, research on student self-efficacy in geometry is aiming to explicitly link confidence, motivation, persistence, anxiety, and comfort with the teaching and learning of geometry.

Beyond discrete disciplinary research, it is in the intersection of mind, brain, and education research where novel contributions are increasingly expanding conceptions of geometry and spatial reasoning learning. For example, the work on mental transformations of 2D and 3D figures has been jointly propelled by research activity in clinical and classroom settings and shows significant promise as a predictor of mathematics achievement but is particularly related to geometric reasoning (see Mix et al., 2016, Uttal et al., 2013). This includes research on transformations, mental mapping, coding, perspective taking, and first-person insertion technologies, for example, that may be subsequently linked to improving the teaching and learning of geometry.

The IPC invites proposals that address some of the following questions:

- What insights from the recent research on spatial reasoning (such as mental rotations) can be integrated into research on geometry thinking? What are the implications for teaching and the resultant learning?
- What affective constructs impact on geometry teaching and learning?
- How can students' cognitive characteristics be taken into account and what are the implications for geometry education?
- How can neuro-cognitive research inform geometry education about individual differences in geometric processing and in learning geometry?
- Is there room for transdisciplinary research and what should it focus on as it relates to geometry teaching and learning?

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THE 26TH ICMI STUDY PRODUCTS

The ICMI Studies are a major activity of ICMI. Their global aims are to contribute to a better understanding of the challenges faced by mathematics education in our multidisciplinary and culturally diverse world and to collaborate in advancing to their resolution. The first ICMI Study was launched in 1980. More detailed information can be found in the ICMI web page (https://www.mathunion.org/icmi/activities/icmi-studies-activities).

The main products of an ICMI Study are a conference and a volume of the ICMI Studies Series. The conference addresses a topic or issue of particular significance in contemporary mathematics education. It is aimed to gather together leading scholars and practitioners specialized in the field of inquiry of the Study, for them to engage in a productive interaction and collaboration to advance in the knowledge of the topics of the Study.

Based on the activity during and results from the conference, the final and most relevant product of an ICMI Study is a volume offering a coherent state-of-the-art view of the topic addressed by the Study. This volume will be part of the New ICMI Study Series.

THE ICMI STUDY CONFERENCE

The 26th ICMI Study focuses on *Advances in Geometry Education* and is planned to provide a platform for teachers, teacher educators, researchers, policy makers, and other stakeholders around the world to share research, practices, projects, and reports that advance the understanding of geometry. As in every ICMI Study, the 26th ICMI Study is built around an International Study Conference that is directed towards the preparation of a volume published in the New ICMI Studies Series. In the conference, substantial time will be allocated for collective work and discussion on significant problems within the four topics and the related sub-topics. Outcomes from the conference activity provide the foundations for the Study volume.

The Study Conference will be organized around working groups on the sub-topics presented above. The working groups will meet in parallel during the conference. It is the work of these groups that is captured as chapters in the ICMI Study volume.

Authors should nominate the topic (one of four) and sub-topic in which they would like their paper considered, by examining the content and questions outlined in the topic sections presented above. Contributions are encouraged that are analytical and innovative rather than solely descriptive in nature. It may be the case that interconnections between sub-topics emerge and warrant attention; consequently, papers may be re-allocated by the IPC if beneficial.

Participation

As is the usual practice for ICMI studies, participation in the Study Conference is by invitation only. Proposed papers will be reviewed, and a selection made according to the quality of the contribution, the potential to contribute to the advancement of the Study, with explicit links to the sub-topics proposed in this Discussion Document, and the need to ensure diversity among the perspectives and representation. The number of invited participants is limited to approximately 100 delegates. As part of the invitation process, potential participants will receive a registration code, which will be required to complete registration details. Unfortunately, an invitation to participate in the conference does not imply financial support from the organizers, and participants should be prepared to finance their own attendance at the conference.

THE ICMI STUDY VOLUME

The first product of the 26th ICMI Study is an electronic volume of the Conference proceedings, to be made available prior to the conference on the conference website and later the ICMI website. The proceedings will contain all the accepted papers as reviewed papers, so they can be cited as a refereed (peer-reviewed) publication with an ISBN number. They are published online only.

The second and main product is an edited volume published by Springer as part of the New ICMI Studies Series. The editing process and content of the volume will be the subject of discussion among the International Program Committee (IPC). It is expected that the structure of the volume will follow the organization and topics set out in this Discussion Document, although some changes might be introduced as a consequence of the discussions raised during the conference. The chapters in the volume will collectively and consensually integrate the outcomes of the discussions of the parallel working groups at the conference, informed by the papers presented. It must be appreciated that there is no guarantee that any of the papers accepted in the Study conference proceedings will appear in the volume. Furthermore, chapters in the volume may be an amalgamation of several presented papers.

A progress report on the Study will be presented at the ICME15 (Sydney, 7-14 July, 2024).

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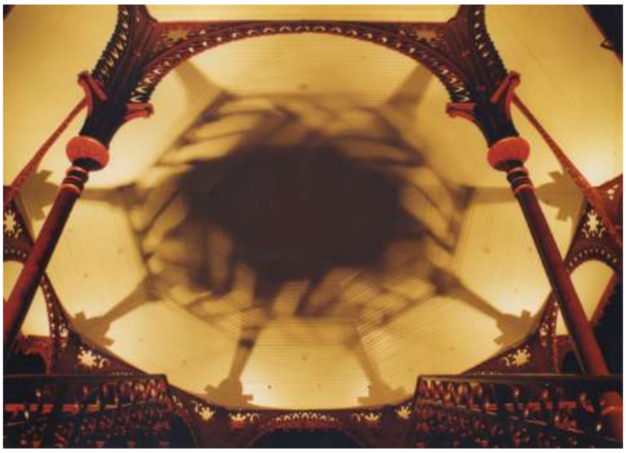

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